Aspects of gravitation in the light-cone gauge

A thesis
Submitted in partial fulfillment of the requirements
Of the degree of
Doctor of Philosophy

By

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December, 2016

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Certified that the work incorporated in the thesis entitled "*Aspects of gravitation in the light-cone gauge* ", submitted by *Mahendraprasad Mali* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institution.

Sudarshan A

Date 27/12/2016.

Dr. Sudarshan Ananth

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Acknowledgements

First and foremost, I would like to express my thanks and sincere gratitude to my supervisor Dr. Sudarshan Ananth for giving me the opportunity to work with him and I appreciate all his contributions of time and ideas to make my Ph.D. experience academically productive and stimulating. It was not a smooth ride but Dr. Ananth was very supportive at all times, he gave me the required moral support and freedom during my difficult times. He was a great mentor who always encouraged me and believed in me. His advice related to research and career were priceless.

I would also like to extend my gratitude towards my co-authors, Prof. Lars Brink, Sucheta Majumdar and Y.S. Akshay.
I would also like to thank members of my research advisory committee : Prof. Anil Gangal and Prof. S. Shankaranarayanan for their insightful suggestions and general encouragement.

I am grateful to the Department of Science and Technology, Government of India, for funding my research and making my Ph.D. work possible.
Last but not the least, I am grateful to all my friends. I thank my family for always being there for me and providing me with comfort and emotional support, in spite of being so far away. I thank Shreya Ray for being there always to help and for introducing me to Pune city by making surprising-spontaneous plans. I also thank my amazing friends - Kajari, Chaitanya, Shraddha, Wasim, Vimanshu, Kunal, Sneha and Amruta- who made my life in IISER so colourful and memorable. I am grateful to all the wonderful times spent with them – be it the spontaneous trips to the beautiful hills or the beaches, or the fun sleepless nights in the hostel. Thank you all for this priceless experience.

Mahendraprasad Mali

I dedicate this thesis to my parents for their immense love and support.

Abstract

In this thesis, we investigate gravitation in light-cone gauge. There are various motivations to study pure gravity, in four dimensions, as a field theory in light-cone gauge. These include recent results pertaining to scattering amplitudes, KLT relations and MHV Lagrangians. Further, recent analysis of the finiteness properties of $N = 8$ supergravity strongly suggests that we must revisit pure gravity as a field theory.

Although some work has gone into understanding light-cone gravity in flat spacetime, there are no findings as far as light-cone gravity on curved backgrounds is concerned. This is one of the gaps, in the literature, that the present thesis aims to address. Additionally, this thesis investigates some of the mathematical structures of the light-cone Hamiltonians describing gravity theories (both without and with supersymmetry). We find unique "quadratic form" structures in these Hamiltonians and comment on the possible significance of these structures.

Publications

Most of the results reported in this thesis have appeared in the following publications by myself (with collaborators).

1. "Gravitation and quadratic forms", Sudarshan Ananth, Lars Brink, Sucheta Majumdar,**Mahendra Mali**, Nabha Shah, *JHEP* **1703** (2017) 169.

2. "Yang-Mills theories and quadratic forms", Sudarshan Ananth, Lars Brink, **Mahendra Mali** , *JHEP* **1508** (2015) 153.

3. "Light-cone gravity in dS_4", Sudarshan Ananth, **Mahendra Mali** *Phys.Lett.*, **B 745** (2015) 48-51.

4. "Light-cone gravity in AdS_4 ", Y.S. Akshay, Sudarshan Ananth, **Mahendra Mali**, *Nucl.Phys.* **B 884** (2014) 66-73.

Contents

Chapter 1

Introduction

Complex processes in nature, ranging from planetary motion, photosynthesis and radioactivity to high energy experiments at the LHC, can be described by the four fundamental forces of nature. The fundamental forces are : the Strong force, the Weak force, the Electromagnetic force and the Gravitational force. By fundamental, we mean that any force we experience is of one of these four types (for example, friction is electromagnetic in origin). Our current understanding of these forces is based on quantum mechanics and general relativity. The former helps in studying the physics of particles and their interactions and plays an important role in describing short distance physics. The latter explains the motion of celestial objects (such as stars and galaxies) and is used as a tool to study the physics of large objects.

We appear to live in a world with three space dimensions and one time dimension. The group that describes the symmetries of our world is the Poincaré group. All content in the world can be classified into two types: matter and forces. Matter is made up of fermions (half integer spin particles) while forces are mediated by bosons (integer spin particles). The recently discovered Higgs particle which has spin 0 plays a crucial role in generating mass for various fermions and bosons.

Three of the four fundamental forces - the Strong, the Weak and the Electromagnetic - are mediated by spin-1 gauge bosons and described by quantum Yang-Mills theory [1]. We seem to understand fairly well the mechanism of these three forces (ie. how they operate). In fact, quantum Yang-Mills theories describe these three forces with great accuracy and the predictions of these theories have been verified experimentally to ten parts in a billion 10^{-8}.

Gravity, although discovered the earliest, is the force we understand least. The weakness of the gravitational force makes it almost impossible for us to learn more about this force from simple experiments. Unfortunately, all attempts to formulate a quantum theory of gravity have proven unsuccessful thus far. Such a theory is needed to explain the fundamental mechanism behind the gravitational force and also to understand what the detailed structure of spacetime is.

Classically, gravity is very successfully explained by the General Theory of Relativity. In fact, experimental checks have been so successful that a quantum theory of gravity may very likely be only an "extension" of General Relativity.

All of this motivates the need to further investigate our understanding of gravity and spacetime geometry through the General Theory of Relativity).

There has been over a century of study devoted to General Relativity. More recently, many decades have focused on understanding gravity as a (quantum) field theory, in the usual covariant gauges (a gauge choice is our freedom to choose some metric components to be zero based on the symmetries of the theory). However, there is not much literature on gravity, as a field theory, in light-cone gauge. The light-cone gauge has various advantages over covariant gauges and has led to some rather remarkable results (in string theory [2] and the proof of ultra-violet finiteness of the $\mathcal{N} = 4$ Yang-Mills theory [3, 4]). Key advantages in the light-cone gauge formulation of field theories [5] include: (a) only the physical degrees of freedom appear and this is achieved by eliminating redundant degrees of freedom using algebraic gauge fixing condition(s), thus avoiding the need to deal with ghost fields when quantizing the theory (b) on-shell results are easy to read off in this formalism, making it closely tied to physics and (c) convenient to look for hidden structures, making it an ideal formulation to look for new/hidden symmetries.

Gravitation, in light-cone gauge, has been studied to some extent on flat spacetime backgrounds [6–11]. However, it is very surprising that despite the considerable interest in anti-de Sitter spacetimes (in the context of string theory), and curved spacetime backgrounds in general, light-cone gravity has not been formulated in any background other than flat space. This is one of the gaps in the literature that this Ph.D. thesis aims to address.

This thesis is based primarily on my papers with collaborators: [12–15].

Having provided the big-picture motivation for investigating gravity in light-cone gauge, we highlight below, three of the most interesting recent developments in the field that have relevance to or motivate the work presented in this thesis.

(**i**) In the usual approach to calculating scattering amplitudes, we start with the covariant formalism and compute Feynman diagrams to arrive at the physical amplitudes. The issue with this approach is that each individual Feynman diagram is not gauge invariant (although the sum of diagrams is) and hence this approach does not use the power of symmetry entirely. Instead, working in light-cone gauge we find that only the physical degrees of freedom propagate. In this gauge, scattering amplitudes can be factorized and represented in very simple forms. For instance, the n-gluon tree-level amplitudes in QCD take the following simple form [16]:

$$A[1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+] = \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \ldots \langle n\,1\rangle}. \tag{1.1}$$

We define the angular brackets later in the thesis but the key point to note here is the simplicity of the answer/result for any number of gluons. This simplicity stems from the result depending only on the helicity states in the theory and since the light-cone formalism only works with these states, it is ideally suited for the study of scattering amplitudes.

(**ii**) Yang-Mills theory and gravity differ at the level of their building blocks (dynamical fields) and can show quite different dynamical behavior. Both the theories differ even at the level of Lagrangians: The Lagrangian of non-Abelian Yang-Mills theory contains only up to four-point interactions whereas gravity Lagrangian contains infinitely many interaction terms. Still, there are quite a few striking relations between Yang-Mills theory and gravity. One such relation stems from string perturbation theory, the Kawai-Lewellen-Tye (KLT) [17] relations show that at tree-level closed string amplitudes can be expressed as sums of products of open string amplitudes. The low energy limit of sting theory is quantum field theory. Therefore, the KLT relations imply that similar relations must exist between tree-level gravity amplitudes and tree-level amplitudes in Yang-Mills theory. We can heuristically write

$$(\text{Gravity}) \quad \sim \quad (\text{Yang-Mills}) \quad \times \quad (\text{Yang-Mills})$$

In field theory limit, the KLT relations for cubic and quartic interactions can be written as

$$\begin{aligned} M_3^{tree}(1,2,3) &= A_3^{tree}(1,2,3)\, A_3^{tree}(1,2,3) \\ M_4^{tree}(1,2,3,4) &= -i\, s_{12}\, A_4^{tree}(1,2,3,4)\, A_4^{tree}(1,2,3,4) \end{aligned} \tag{1.2}$$

where M_n is tree-level coupling-stripped gravity amplitude and A_n represents color-coupling-stripped tree-level amplitude of Yang-Mills theory $[s_{ij} \equiv -(p_i + p_j)^2]$.

The KLT relations can be exploited to obtain the quantum loops of from semi-classical tree-level amplitudes by using D-dimensional unitarity [18].

The KLT relations are semi-classical and have been proved to be true on four-dimensional flat Minkowski space-time background.

We know that our universe is not flat instead it is believed to be a four dimensional de Sitter space-time [19]. Therefore it will be very interesting to see if these KLT relations or KLT-like relations are valid even on curved spacetime backgrounds. This is one of the long-term aims of the work in this thesis: to have the perturbative Lagrangians for light-cone gravity up to cubic order on curved backgrounds such as AdS_4, dS_4 and other conformally flat backgrounds (we consider conformally flat backgrounds because of simplicity of light-cone formulation on these backgrounds). This is the primary motivation for chapters 3 and 4 in this thesis.

(iii) $\mathcal{N}=8$ supergravity and hidden symmetries.

Supersymmetry relates fermions to bosons. The $(\mathcal{N}=8, d=4)$ supergravity is the gravity theory in four dimensions with maximal supersymmetry. One of the most striking aspects of supersymmetry is how it improves the ultra-violet (high energy) behaviour in quantum field theories. Accordingly, the $\mathcal{N}=8$ theory has the best ultraviolet properties of any field theory of gravity. It has been found to be divergence free up to four loops [20,21] and it is believed to be finite up to seven loops [22] and perhaps beyond [23]. The presence of supersymmetry only partially explains this degree of finiteness. The question is "What else can account for this improved behavior?". It has been suggested repeatedly that the improved behavior at the loop levels is due to pure gravity itself (and such improved unexpected ultra-violet behavior usually points to a hidden symmetry). Another long-term goal for the work in this thesis, is to produce a bare-bones formalism of gravity (which the light-cone helps us achieve) wherein any such hidden symmetries or structures might appear.

Finally, having outlined three of the results that motivate our work, we comment on other aspect of this thesis: the aim to identify simple algebraic structures in theories of gravity. Simple structures usually indicate or stem from a symmetry principle.

The maximally supersymmetric $\mathcal{N}=4$ Yang-Mills theory and $\mathcal{N}=8$ supergravity Hamiltonians can be written as quadratic forms, involving the dynamical supersymmetry generator [24, 25]. In this thesis, we show that such structures (quadratic forms) extend even to the pure Yang-Mills and gravity theories. The exact physical significance of the quadratic form is still unclear to us but the fact that it appears only in the pure and maximally supersymmetric cases is clearly an important sign.

Overview

In chapter 2 we review light-cone coordinates and the Poincaré algebra in this language. The formalism developed in [6, 7] provides a closed form expression for the light-cone gravity Lagrangian which can then be used to obtain the perturbative Lagrangian order by order. Chapter 3 contains a review of Yang-Mills theory in light-cone gauge followed by derivation of the closed form expression of gravity. We have also derived perturbative expansion up to quartic interactions in field. Finally, we illustrate the KLT relation where cubic coupling-stripped helicity amplitude of gravity can be written as the "square" of cubic color-coupling stripped helicity amplitude of the Yang-Mills theory.

The major content of this thesis focuses on formulation of gravity on curved backgrounds in light-cone gauge. This is done is chapter 4 and chapter 5. In chapter 4 we take conformally flat Poincaré Patch of AdS_4 as background metric. After making suitable gauge choices, the constraint equations can be solved to fix redundant degrees of freedom. We have obtained a closed form expression of the action on AdS_4 . We have also derived perturbative expansion of the action up to first order in coupling constant in light-cone gauge [12]. Similarly in chapter 5, The conformally flat Poincaré Patch of dS_4 is taken as the dynamical background metric. We make similar gauge choices as in the case of AdS_4 . The conformal factor in this case is time dependent, therefore integrating factors are essential to solve constraint relations. We have obtained the closed form expression of action followed by perturbative expansion up to first order in coupling constant [13]. I would like to mention that the results in this thesis were derived by hand without help of any symbolic computation package.

Chapter 6 comprises of review on quadratic forms in supersymmetric and non-supersymmetric Yang-Mills theory and gravity.

Chapter 2

Light-cone coordinates and Poincaré algebra

We work in light-cone coordinates throughout this dissertation.

Light-cone coordinates

The light-cone coordinates were first introduced by Dirac in 1949 [5]. It was shown that the maximum number of Poincaré generators become independent of dynamics in the “front-form” or “light-cone ” formulation, including some of the Lorentz boosts. For example, in four dimensions, seven of the Poincaré generators are independent of dynamics and only three contain dynamical information. In this formulation, the Hamiltonian which usually satsifies an eigenvalue relation of the form

$$H\,|\psi\rangle = \sqrt{\overrightarrow{P}^{2} + M^{2}}\,|\psi\rangle$$

no longer contains square roots, implying that negative energy solutions can be avoided by suitable choices of dynamical variables.

We work in four dimensional Minkowski spacetime with coordinates (x^0, x^1, x^2, x^3) and metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

.

Consider a massless particle moving in the x^3-direction with momentum p_3 and 4-momentum $p_\mu = (p_0, 0, 0, p_3)$. For massless particles, we have $p_\mu p^\mu = 0$ implying that $p_0 = |p_3|$. The light-front (henceforth referred as light-cone) is a three-dimensional surface defined by $x^0 + x^3 =$ constant.

In high energy collision processes particle trajectories lie near a plane, which can be taken as $x^0 - x^3$ plane. The trajectories of right moving particles cluster about a light-like line $x^0 - x^3 = 0$ while trajectories of left moving particles cluster about the light-like line $x^0 + x^3 = 0$ in the $x^0 - x^3$ plane. Therefore, for studying high energy properties of amplitudes, the formulation of theories with these lines as the coordinate axes becomes extremely useful.

The light-cone coordinates are defined as

$$\begin{aligned} x^+ &= \frac{1}{\sqrt{2}}(x^0 + x^3) \\ x^- &= \frac{1}{\sqrt{2}}(x^0 - x^3) \end{aligned} \tag{2.1}$$

To write the theory in terms of fields which are in helicity eigenstates, the remaining two coordinate axes are combined to give

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}(x^1 + i\,x^2) \\ \bar{x} &= \frac{1}{\sqrt{2}}(x^1 - i\,x^2) \end{aligned} \tag{2.2}$$

With the corresponding derivatives defined by

$$
\begin{aligned}
\partial_+ &= \frac{\partial}{\partial x^+} = \frac{1}{\sqrt{2}}(\partial_0 + \partial_3) = -\partial^-, \\
\partial_- &= \frac{\partial}{\partial x^-} = \frac{1}{\sqrt{2}}(\partial_0 - \partial_3) = -\partial^+, \\
\partial &= \frac{\partial}{\partial x} = \frac{1}{\sqrt{2}}(\partial_1 + i\,\partial_2), \\
\bar{\partial} &= \frac{\partial}{\partial \bar{x}} = \frac{1}{\sqrt{2}}(\partial_1 - i\,\partial_2).
\end{aligned}
\tag{2.3}
$$

Conventionally, we choose x^+ as the time coordinate and the corresponding momentum $p_+ = -p^- = -i\frac{\partial}{\partial x^+}$ is the light-cone Hamiltonian. After coordinate transformations of (2.1) and (2.2), the Minkowski metric reads

$$
\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

The invariant interval becomes

$$
ds^2 = -2\,dx^+\,dx^- + 2\,dx\,d\bar{x} \tag{2.4}
$$

Poincaré algebra in light-cone gauge

A generic Lorentz transformation to leading order in parameter $(\omega_{\mu\nu})$ is

$$
\delta_J\,\phi = i\omega_{\mu\nu}\,J^{\mu\nu}\,\phi = i\,\omega_{\mu\nu}\,(M^{\mu\nu} + S^{\mu\nu})\phi \tag{2.5}
$$

where $M^{\mu\nu}$ are generators of Lorentz group and $S^{\mu\nu}$ are representation matrices.

The Poincaré generators in light-cone coordinates are obtained from the following covariant results

$$
\begin{aligned}
P_\mu = & \; -i\,\partial_\mu, & M_{\mu\nu} &= -i\,(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu), \\
[P_\mu, P_\nu] = & \; 0, & [M_{\mu\nu}, P_\rho] &= i\,(\eta_{\mu\rho}\,P_\nu - \eta_{\nu\rho}\,P_\mu), \\
& [M_{\mu\nu}, M_{\rho\sigma}] = i\,(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}\,M_{\mu\rho} - \eta_{\mu\sigma}\,M_{\nu\rho} - \eta_{\nu\rho}\,M_{\mu\sigma}). &&
\end{aligned} \tag{2.6}
$$

with $M_{\mu\nu} = -M_{\nu\mu}$. The light-cone form of the Poincaré generators is as follows

$$
\begin{aligned}
M^{+-} &= i\,(x^+\,\partial^- - x^-\,\partial^+), & M &= x\,\bar\partial - \bar x\partial, \\
M^+ &= i\,(x\,\partial^+ - x^+\,\partial), & \bar M^+ &= i\,(\bar x\partial^+ - x^+\bar\partial), \\
M^- &= i\,(x\,\partial^- - x^-\partial), & \bar M^- &= i\,(\bar x\,\partial^- - x^-\,\bar\partial).
\end{aligned} \tag{2.7}
$$

The generators P, $\bar P$, P_-, M, $\bar M$, M_{+-}, M^+ and $\bar M^+$ are non-dynamical whereas P_+, M^- and $\bar M^-$ are dynamical generators.

and the corresponding Poincaré algebra is given by

$$
\begin{aligned}
&[M^{+-}, P^+] = i\,P^+, && [M^{+-}, P^-] = -i\,P^-,\\
&[M^+, P^-] = -i\,P, && [\bar{M}^+, P^-] = -i\,P^-,\\
&[M^-, P^+] = -i\,P, && [\bar{M}^-, P^+] = -i\,P^-,\\
&[M^{+-}, M^+] = i\,M^+, && [M^{+-}, \bar{M}^+] = i\,\bar{M}^+,\\
&[M^{+-}, M^-] = -i\,M^-, && [M^{+-}, \bar{M}^-] = -i\,\bar{M}^-,\\
&[M^-, \bar{P}] = -i\,P^-, && [\bar{M}^-, P] = -i\,P^-\\
&[M^+, \bar{P}] = -i\,P^+, && [\bar{M}^+, P] = -i\,P^+.
\end{aligned}
\tag{2.8}
$$

Chapter 3

Gravity and Yang-Mills theory in light-cone gauge

This chapter is primarily a review of the work in [7].

In this chapter we formulate gravity in light-cone gauge. We present Yang-Mills theory as a toy model to illustrate our formalism.

Yang-Mills theory

Brief note on group theory

This section is not intended to be a review. Instead, we briefly highlight a few points and results from group theory that are relevant to the material presented in this thesis.

Special orthogonal group :$SO(N)$

The group formed by rotations in N-dimensional Euclidean space is: $SO(N)$. The group $SO(N)$ consists of $N \times N$ real matrices $\mathbf{R}$ which are orthogonal

$$\mathbf{R}^T \mathbf{R} = 1 \tag{3.1}$$

and have $\det \mathbf{R} = 1$. The fundamental representation of the group is given by N-dimensional vector $\vec{v} = \{v^i\} = \{v^1, v^2, v^3, \ldots, v^N\}$. Under the action of the group element $\mathbf{R}$ a vector transforms as

$$v^i \quad \rightarrow \quad v'^i = R^{ij} \, v^j \tag{3.2}$$

The members of $SO(N)$ are represented as $\mathbf{R} = \mathbf{e}^{\theta^{\mathbf{ij}} \mathbf{J}^{\mathbf{ij}}}$, where J^{ij} are $\frac{N(N-1)}{2}$ antisymmetric matrices. For example, rotations in two dimensions form $SO(2)$ which are generated by σ_2 - one of the Pauli matrices. The group of rotations (Lorentz group) in four space-time dimensions is $SO(3,1)$ which is locally isomorphic to $SO(3) \times SO(3)$.

Special unitary group: $SU(N)$

The special unitary group $SU(N)$ consists of all $N \times N$ matrices which are unitary

$$U^\dagger U = 1 \tag{3.3}$$

and have $\det U = 1$. The fundamental representation of $SU(N)$ consists of N objects $\psi^j, j = 1, 2, 3, \ldots, N$ which under the action of a group element transforms as

$$\psi^i \quad \rightarrow \quad \psi'^i = U^i_j \, \psi^j \tag{3.4}$$

Whereas in the adjoint representation the group element is represented by a traceless tensor ψ^i_j and it transforms as

$$\psi \quad \rightarrow \psi' = U \, \psi \, U^\dagger \tag{3.5}$$

The $SU(N)$ generators are $N \times N$ Hermitian and traceless matrices T^a, where ($a = 1, 2, \ldots, N^2 - 1$). Using this we write

$$U = e^{i\omega^a T^a} \tag{3.6}$$

where ω^a are real parameters. The commutation relations of generators

$$[T^a, T^b] = i f^{abc} T^c \tag{3.7}$$

form the Lie algebra of $SU(N)$, where f^{abc} is the structure constant. The Lie algebra of group $SO(3)$ is isomorphic to that of $SU(2)$, implying that $SO(3,1) \cong SU(2) \times SU(2)$. Therefore, we can associate a rotations with any given U. Quantum chromodynamics and electroweak interactions, for example, are governed by an $SU(3)$ and $SU(2) \times U(1)$ symmetry.

Review of Yang-Mills theory

Field theories which have been very successful in describing the real world are all gauge theories. These theories are based on the principle of gauge ("gauging" $\rightarrow$ "making local") invariance and in general refer to larger symmetry groups than the $U(1)$ group gauge invariance of quantum electrodynamics (QED). Yang-Mills theories are locally invariant under the internal symmetry transformations. A brief discussion on the non-Abelian gauge theories can be found in [26, 27].

Consider a classical action invariant under the local symmetry group G. We consider this to be non-Abelian, simple and compact, this allows us to choose $G = SU(N)$. Let ψ be the field with Lagrangian :

$$\mathcal{L} = \psi_a^\dagger \gamma^\mu \partial_\mu \psi_a. \tag{3.8}$$

where $a = 1, 2, 3...N$ is color index and the field ψ is a N component vector. ψ transforms as the fundamental representation, *viz.*

$$\psi \;\rightarrow\; \psi'(x) = exp(i\omega^A T^A)\,\psi \equiv U(x)\,\psi(x), \tag{3.9}$$

where, T^A are the $N^2 - 1$ traceless Hermitian matrices which generate N-dimensional representations of the $su(N)$ Lie algebra. The group structure function is defined through the commutator

$$[T^A, T^B] = i\,f^{ABC}\,T^C \tag{3.10}$$

To introduce invariance under local transformations in the Lagrangian (3.8), we need covariant derivatives $\mathcal{D}_\mu$ such that $\mathcal{D}_\mu\psi$ transforms in the same way as the field ψ in (3.9) *i.e.* obeys the following transformation law :

$$\mathcal{D}_\mu\psi(x) \;\rightarrow\; U(x)\,\mathcal{D}_\mu\psi(x) \tag{3.11}$$

or in operator form

$$\mathcal{D}_\mu \;\rightarrow\; \mathcal{D}'_\mu = U(x)\,\mathcal{D}_\mu\,U^\dagger(x) \tag{3.12}$$

The covariant derivative $\mathcal{D}_\mu$ is supposed to generalize the partial derivative ∂_μ so we define

$$\mathcal{D}_\mu\psi = (\partial_\mu - i\,g\,A_\mu)\psi. \tag{3.13}$$

where $A_\mu = A^C_\mu T^C$ is the gauge field (which has been introduced to ensure the correct transformation law) and g is a dimensionless coupling constant. Note that $\mathcal{D}_\mu$ is a $N \times N$ matrix. Explicitly, with all indices shown

$$[\mathcal{D}_\mu\psi]_a \;\rightarrow\; [U(x)]^b_a\,(\mathcal{D}_\mu)^c_b\,\psi_c(x) \tag{3.14}$$

Imposing the transformation law of $\mathcal{D}_\mu$ in (3.11) or (3.12) we obtain the transformation law for the gauge field.

$$A_\mu(x) \to A'_\mu(x) = U(x)\, A_\mu(x)\, U^\dagger(x) + \frac{i}{g}\, U(x)\, [\partial_\mu U^\dagger(x)] \tag{3.15}$$

The variation in the gauge field under infinitesimal gauge transformations can be found using (3.9) and (3.15).

$$\delta\, A^C_\mu = \frac{1}{g}\, \partial_\mu \omega^C(x) - f^{BDC}\, \omega^B(x)\, A^D_\mu(x) + \mathcal{O}(\omega^2) \tag{3.16}$$

where ω is the gauge parameter. It is remarkable that the gauge transformations are represented in a way that the transformations of gauge fields does not depend on the representation of fermion fields we started with. To understand the dynamics of the gauge fields we need an action which describes their dynamics and is invariant under the SU(N) transformations. This is achieved by introducing the field strength as

$$\begin{aligned} F_{\mu\nu} \quad &\equiv \quad \frac{i}{g}\, [\mathcal{D}_\mu, \mathcal{D}_\nu] \\ &= \quad \partial_\mu\, A_\nu - \partial_\nu\, A_\mu - i\, g\, [A_\mu, A_\nu]. \end{aligned} \tag{3.17}$$

where $F_{\mu\nu} = F^C_{\mu\nu}\, T^C$. From (3.12) and (3.17) we see that

$$F_{\mu\nu}(x) \to U(x)\, F_{\mu\nu}(x)\, U^\dagger(x) \tag{3.18}$$

Therefore, the trace of the quadratic Lorentz scalar of the field strength is invariant under the gauge transformation. The gauge invariant Lagrangian for the gauge field is

$$\mathcal{L} = -\frac{1}{4}\, F^{\mu\nu\, c}\, F^c_{\mu\nu}. \tag{3.19}$$

The above Lagrangian is invariant under the gauge transformation

$$A^{\mu\,a} \qquad \rightarrow \qquad A'^{\mu\,a} = A^{\mu\,a} + \mathcal{D}^{\mu}\,\Lambda^{a} \tag{3.20}$$

where $\Lambda^a(x^\mu)$ is a scalar function of spacetime. This implies that we can eliminate one degree of freedom from the gauge field, by simply choosing a suitable $\Lambda^a(x^\mu)$. The light-cone version of this gauge freedom will be discussed shortly.

The gauge invariant Lagrangian for the Dirac spinor ψ corresponding to the fundamental representation of $SU(N)$ and interacting with the gauge field A_μ can be written as

$$\mathcal{L} = -\frac{1}{4}\,F^{c\mu\nu}\,F^{c}_{\mu\nu} + \psi^{\dagger\,a}\,(\not{\mathcal{D}} - m)\psi^{a} \tag{3.21}$$

This would represent the Lagrangian for a gauge theory like QCD, with the gauge field referring to the gluons and the relevant group being $SU(3)$.

Yang-Mills theory in light-cone gauge

We study the gauge field action in four dimensional Minkowski space-time in light-cone coordinates. The action for just the gauge fields is

$$S = -\frac{1}{4}\int d^4x\,\mathrm{Tr}\,(F^{\mu\nu}\,F_{\mu\nu}) \tag{3.22}$$

with the corresponding Lagrangian defined in (3.19) and $\mu,\nu = 0,1,2,3$. $F_{\mu\nu}$ is the Yang-Mills generalization of the field strength (of electromagnetism) defined in (3.17). The extremum of the actionabove yields the equations of motion,

$$\partial_\mu\,F^{a\,\mu\nu} + g\,f^{abc}\,A^{b}_{\mu}\,F^{c\,\mu\nu} = 0 \tag{3.23}$$

where $a = 1,2,\ldots,N^2-1$, is the SU(N) index[1]. The gauge fields in the light-cone coordinates are defined as

[1] We have changed SU(N) index from "A" $\rightarrow$ "a" for calculation convenience.

$$
\begin{aligned}
A^{a+} &= \frac{1}{\sqrt{2}}(A^{a0} + A^{a3}) \\
A^{a-} &= \frac{1}{\sqrt{2}}(A^{a0} - A^{a3})
\end{aligned} \tag{3.24}
$$

We now choose to work in light-cone gauge by setting $A_0^a = A_3^a$ or :

$$
A^{a+} = \frac{1}{\sqrt{2}}(A^{a0} + A^{a3}) = 0 \;\Rightarrow\; A_-^a = 0 \tag{3.25}
$$

The gauge invariance of Yang-Mills theory allows us to make this one gauge choice. Any equations of motion (3.23) that do not explicitly contain time derivatives (meaning ∂_+) are constraint relations (since they must be valid at all values of light-cone time). We now examine the $\nu = +$ component of (3.23) while imposing our gauge choice (3.25). This yields

$$
A_+^a = \frac{\partial_i}{\partial_-} A_i^a + g\, f^{abc} \frac{1}{\partial_-^2}(A_i^b \partial_- A_i^c) \tag{3.26}
$$

The above equation shows us that the field component A_+^a can be expressed in terms of the transverse fields A_i^a, for $i = 1, 2$.

The operator $\frac{1}{\partial_-}$ is a non-local indefinite integration operator. It can be thought of as the Green function which satisfies

$$
\partial_- G(x^-, y^-) = \delta(x^- - y^-). \tag{3.27}
$$

With the help of suitable boundary conditions G can be determined up to a vanishing zero mode of the operator ∂_-. For additional details regarding $\frac{1}{\partial_-}$, we refer the reader to [3, 28].

The Yang-Mills Lagrangian (3.19) can now written entirely in light-cone gauge using (3.25) and (3.26). After dropping the boundary terms, we get

$$\mathcal{L} = -\frac{1}{2}A_i^a \Box A_i^a + 2\,g\,f^{abc}\,\frac{1}{\partial_-}A_i^a\,\partial_i\,A_j^b\,\partial_-\,A_j^c + \frac{1}{4}\,g^2\,f^{abc}\,f^{ade}\,A_i^b\,A_j^c\,A_i^d\,A_j^e$$

$$+\frac{1}{2}\,g^2\,f^{abc}\,f^{ade}\,\frac{1}{\partial_-}(A_i^b\,\partial_-\,A_i^c)\,\frac{1}{\partial_-}(A_j^d\,\partial_-\,A_j^e). \quad (3.28)$$

The above Lagrangian can now be rewritten in a helicity basis by the following change of variables in (3.28) :

$$A^a = \frac{1}{\sqrt{2}}\,(A_1^a + i\,A_2^a)$$

$$\bar{A}^a = \frac{1}{\sqrt{2}}\,(A_1^a - i\,A_2^a) \quad (3.29)$$

Under the Lorentz little group SO(2), A^a, ∂ have helicity of $+1$ while $\bar{A}^a, \bar{\partial}$ have helicity of -1. The helicity neutral Yang-Mills Lagrangian reads

$$\mathcal{L} = \bar{A}^a\,\Box\,A^a - 2\,g\,f^{abc}\left(\frac{\partial}{\partial_-}\,\bar{A}^a\,\partial_-\,A^b\,\bar{A}^c + \frac{\bar{\partial}}{\partial_-}\,A^a\partial_-\bar{A}^b\,A^c\right)$$

$$-2\,g^2\,f^{abc}\,f^{ade}\,\frac{1}{\partial_-}\left(\partial_-\,A^b\,\bar{A}^c\right)\frac{1}{\partial_-}\left(\partial_-\,\bar{A}^d\,A^e\right)\,. \quad (3.30)$$

The light-cone Lagrangian of Yang-Mills theory in (3.30) contains only the physical (transverse) fields. The price we pay here is that the Lagrangian lacks manifest covariance and locality.

Yang-Mills helicity vertex

On-shell amplitudes for multi-gluon processes can be written as remarkably simple expressions. The recently developed techniques to evaluate Feynman diagrams more elegantly has been discussed in a number of references [27, 29, 30]. This has been very helpful in providing new insights into gauge theories.

We calculate the cubic vertex of the Yang-Mills Lagrangian (3.30) in the spinor-helicity formalism. In this formalism a four vector is represented as a bi-spinor using the Pauli matrices as $p^{\dot{a}\,b} = p_\mu\,(\bar{\sigma}^\mu)^{\dot{a}\,b}$.

where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$ with $\sigma^i\,(i = 1, 2, 3)$ the usual Pauli-matrices.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We define the following spinor products :

$$\begin{aligned} \langle k\, l \rangle &= \sqrt{2}\,\frac{k\, l_- - l\, k_-}{\sqrt{k_-\, l_-}} \\ [k\, l] &= \sqrt{2}\,\frac{\bar{k}\, l_- - \bar{l}\, k_-}{\sqrt{k_-\, l_-}} \end{aligned} \tag{3.31}$$

where,

$$k = \frac{k_1 + i\,k_2}{\sqrt{2}}, \quad \bar{k} = \frac{k_1 - i\,k_2}{\sqrt{2}}, \quad k_+ = \frac{k_0 + k_3}{\sqrt{2}}, \quad k_- = \frac{k_0 - k_3}{\sqrt{2}} \tag{3.32}$$

We now start with the light-cone Lagrangian for Yang-Mills derived in the previous subsection. We first Fourier transform the Lagrangian to momentum space. We then examine the first cubic vertex which now reads

$$-2\,g\,f^{abc}\int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\,(2\,\pi)^4\,\delta^4(p+k+l)\,\frac{\langle k\,l\rangle^3}{\langle l\,p\rangle\langle p\,k\rangle}\,\bar{A}^a(p)\,A^b(k)\,\bar{A}^c(l)$$

To get the above expression we have used the spinor products that we introduced and explicitly anti-symmetrized the momentum factors. The color and coupling stripped vertex is then

$$\mathcal{A}(p,k,l) = \frac{\langle k\,l\rangle^3}{\langle l\,p\rangle\langle p\,k\rangle} \tag{3.33}$$

Similarly, the tree-level quartic amplitude can be calculated. There have been remarkable methods to calculate higher order scattering amplitudes efficiently. A tree-level n-gluon process in which exactly two external legs carry negative helicity *i.e,* $\mathcal{A}_n(-,-,++\,...+)$ is called a maximum helicity violating (MHV a.k.a. CSW) amplitude. In [31] Cachazo, Svrcek and Witten conjectured that tree level amplitudes of arbitrary helicities can be obtained by gluing together certain combination of these MHV amplitudes with propagators. A recursion relation for the n-gluon MHV tree-level amplitudes was given in [16]. An entirely MHV Lagrangian can then been obtained from the light-cone Yang-Mills Lagrangian using suitable field redefinitions [32, 33].

Another recent development in this regard is the work in [34]. It is an expansion of the color ordered diagrams involving both MHV and non-MHV amplitudes. This idea has been applied and extended at the tree-level in both, the Yang-Mills theory [35] and gravity [36]. Also, the prescription of BCFW and MHV amplitudes has been generalized to calculate one loop amplitudes [37].

Although very interesting, these ideas (CSW, MHV and BCFW) are, at present, outside the scope of this thesis.

General relativity

The action principle is our starting point for the study of General Relativity. It allows us to derive the equations of motion (field equations) and to identify conservation laws. The Lagrangian is invariant under symmetry transformations (like general coordinate transformations). In General Relativity, the metric is the dynamical variable and characterizes the geometric and gravitational properties of spacetime. The Einstein-Hilbert action reads

$$S_{EH} = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G}(\mathcal{R} - 2\Lambda) + \mathcal{L}_m \right]. \tag{3.34}$$

where

- G is Newton's constant.
- $\sqrt{-g}$ makes the volume element invariant under general coordinate transformations.
- $\mathcal{R}$ is the Ricci scalar (related to the Ricci tensor) and defined below.
- Λ is the cosmological constant corresponding to the spacetime metric.
- $\mathcal{L}_m$ is the Lagrangian for matter fields present in the theory.

The Ricci tensor is obtained by contracting two indices of Riemann tensor and is symmetric under the exchange of its two indices.

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^{\alpha}{}_{\mu\alpha\nu} \tag{3.35}$$

and Ricci scalar is the metric contracted Ricci tensor

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} \tag{3.36}$$

The Riemann tensor is given by

$$\mathcal{R}^{\alpha}_{\ \mu\nu\rho} \equiv \partial_\rho \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\rho} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\rho\sigma} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\alpha}_{\nu\sigma} \tag{3.37}$$

and its fully covariant form is $\mathcal{R}_{\lambda\mu\nu\rho} = g_{\lambda\alpha}\mathcal{R}^{\alpha}_{\ \mu\nu\rho}$. The Riemann tensor is *unique* in being the only tensor that can be constructed using the metric $g_{\mu\nu}$, its first and second derivatives and remaining linear in the derivatives. It has the following algebraic properties:

- **Symmetry:**

$$\mathcal{R}_{\lambda\mu\nu\rho} = \mathcal{R}_{\nu\rho\lambda\mu}$$

- **Anti-symmetry:**

$$\mathcal{R}_{\lambda\mu\nu\rho} = -\mathcal{R}_{\mu\lambda\nu\rho} = -\mathcal{R}_{\lambda\mu\rho\nu} = +\mathcal{R}_{\mu\lambda\rho\nu}$$

- **Cyclicity:**

$$\mathcal{R}_{\lambda\mu\nu\rho} + \mathcal{R}_{\lambda\rho\mu\nu} + R_{\lambda\nu\rho\mu} = 0$$

The first derivative structure present in the Riemann tensor is the Christoffel symbol or affine connection (which is not a tensor) defined by

$$\Gamma^{\alpha}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{3.38}$$

We can see that Christoffel symbol is symmetric in its covariant indices.

Einstein field equation and Bianchi identities

The equations of motion corresponding to (3.34) are the Einstein field equation and can be obtained using the variational principle.

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.39}$$

Here $T_{\mu\nu}$ is the stress-energy-momentum tensor and depends on the matter Lagrangian present in the Einstein-Hilbert action. Covariant differentials of the Riemann tensor satisfy a tensor equation, known as the Bianchi identity.

$$\nabla_\alpha \mathcal{R}_{\lambda\mu\nu\rho} + \nabla_\rho \mathcal{R}_{\lambda\mu\alpha\nu} + \nabla_\nu \mathcal{R}_{\lambda\mu\rho\alpha} = 0 \tag{3.40}$$

The doubly contracted Bianchi identity can be written as

$$\nabla^\mu G_{\mu\nu} = 0. \tag{3.41}$$

where $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}$ is the symmetric Einstein tensor. As we know the covariant derivative of the metric tensor vanishes i.e. $\nabla^\mu g_{\mu\nu} = 0$. Using this in (3.39) we get

$$\nabla^\mu T_{\mu\nu} = 0. \tag{3.42}$$

The above relation shows the local conservation of the stress-energy tensor.

Linearized gravity and diffeomorphism

Pure Einstein field equations in flat spacetimes where the cosmological constant vanishes are

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = 0 \tag{3.43}$$

A flat Minkowski spacetime implies the absence of a gravitational field. In the presence of a weak gravitational field the flat background remains "nearly" flat. A "small" deviation in the flat spactime metric is introduced

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad \text{where} \quad |h_{\mu\nu}| < 1. \tag{3.44}$$

The expansion of the equations of motion to linear order in $h_{\mu\nu}$ yields the linearized Einstein field equations. General Relativity is invariant under a symmetry group of coordinate transformations. Under a coordinate transformation of the type (called a diffeomorphism)

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \qquad\qquad \text{and} \quad |\partial_{\mu}\xi_{\nu}| << 1 \tag{3.45}$$

the fields transform as

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}) \tag{3.46}$$

This shows that we can find a coordinate system which is nearly flat even after adding an arbitrary vector $\xi^{\mu}(x)$. Therefore we can in principle choose components of ξ^{μ} to make the perturbative Einstein field equations as simple as possible. This process of choosing suitable coordinate transformations is known as gauge transformation.

Since ξ^{μ} has four components, we are allowed to make four gauge choices. In four dimensions, the symmetric field $h_{\mu\nu}$ has 10 independent components. The four gauge choices reduce this to six independent components. In the following, we will illustrate how the choice of light-cone gauge permits us to use four of the non-dynamical Einstein field equations as constraints to determine four more components of $h_{\mu\nu}$. Thus we are left with $10-4-4=2$ independent components. These are the two degrees of freedom or polarization states of the graviton in four spacetime dimensions.

Gravity in light-cone gauge

Having formulated Yang-Mills theory in light-cone gauge, we now turn our attention to Gravity. Before formulating pure gravity in light-cone gauge, we start with a brief review of the structure of linearized gravity in light-cone gauge.

Linearized Einstein equation in light-cone gauge

The linearized Einstein field equation (3.43) is obtained by expanding the vacuum Einstein equation in terms of the metric perturbation (3.44).

$$\mathcal{R}_{\mu\nu} = 0 \tag{3.47}$$

yields

$$\partial^{\sigma}\partial_{\mu}\, h_{\nu\sigma} + \partial^{\sigma}\partial_{\nu}\, h_{\mu\sigma} - \Box\, h_{\mu\nu} - \partial_{\mu}\partial_{\nu}\, h^{\sigma}{}_{\sigma} = 0 \tag{3.48}$$

where $\Box = \partial^{\mu}\,\partial_{\mu}$ is the flat space d' Alembert operator.

We see that the the above equation is invariant under the field transformations in (3.46) - invariance under infinitesimal diffeomorphisms.

Our aim is to formulate gravity in light-cone gauge. We start by recalling, as explained below (3.46), that we are allowed to make four gauge choices. We make the following four gauge choices:

$$h^{+\mu} = 0 \qquad \Rightarrow h_{-\mu} = 0 \qquad \mu = +,-,i, \qquad \text{and} \qquad i = 1,2. \tag{3.49}$$

Linearized equations can be divided into two groups, the ones which contain time derivatives provide dynamical information. The remaining equations do not contain time derivatives and are therefore constraint relations. These constraints may be solved to eliminate redundant degrees of freedom (in terms of physical degrees of freedom). This procedure generates higher order interaction terms containing only physical fields. The ten components of $\mathcal{R}_{\mu\nu}$ can be expanded in light-cone gauge.

$$\mathcal{R}_{--} = 0 \qquad \Rightarrow \qquad \partial_{-}^{2}\, h^{\rho}{}_{\rho} = 0 \tag{3.50}$$

Since $h^{\rho}{}_{\rho}$ is an arbitrary function of spacetime, therefore the above relation shows that $Tr(h_{\mu\nu}) = 0$. This shows that the last term in (3.48) vanishes. The other components of

$\mathcal{R}_{\mu\nu} = 0$ are

I. $\mathcal{R}_{-i} = 0 \qquad \Rightarrow \qquad h_{+i} = \frac{\partial^j}{\partial_-} h_{ij}$,

II. $\mathcal{R}_{+-} = 0 \qquad \Rightarrow \qquad h_{++} = \frac{\partial^i \partial^j}{\partial_-^2} h_{ij}$,

III. $\mathcal{R}_{++} = 0 \qquad \Rightarrow \qquad -\Box \frac{\partial^i \partial^j}{\partial_-^2} h_{ij} = 0$,

IV. $\mathcal{R}_{+i} = 0 \qquad \Rightarrow \qquad -\frac{\partial^j}{\partial_-} \Box h_{ij} = 0$,

V. $\mathcal{R}_{ij} = 0 \qquad \Rightarrow \qquad -\Box h_{ij} = 0$,

We see that relations I and II (constraints) allow us to express h_{+i} and h_{++} in terms of h_{ij}. The next three (dynamical) components III,IV and V are proportional to h_{ij}. Tracelessness and symmetry of the metric thus leaves only two components of h_{ij} independent and these are the two physical degrees of freedom.

Closed form expression

Having reviewed linearized gravity in light-cone gauge we now turn to the full theory of pure gravity in four spacetime dimensions. We essentially review how to derive a closed form expression for the Einstein-Hilbert action, in light-cone gauge, using the formalism of [6,7]. The closed form expression can then be used to extract interaction vertices/terms of the Lagrangian and here we present terms upto order κ^2 terms [7]. Using similar techniques, the quintic interaction terms have been calculated in [38].

The Einstein-Hilbert action (3.34) on a Minkowski background is

$$S_{EH} = \int d^4x\, \mathcal{L} = \frac{1}{2\,\kappa^2} \int d^4x \sqrt{-g}\,\mathcal{R}, \tag{3.51}$$

where $\kappa^2 = 8\pi G$ is the coupling constant. The equations of motion corresponding to the above action (3.51) are

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 0. \tag{3.52}$$

The dynamical variable of General Relativity is the metric '$g_{\mu\nu}$'. We start by making the following three gauge choices consistent with (3.49) [6,7]

$$g_{--} = g_{-i} = 0 \quad , \, i = 1,2 \; . \tag{3.53}$$

The above gauge choices reduce correctly to the flat spacetime metric components $\eta_{--} = \eta_{-i} = 0$ in absence of the perturbations. The fourth and final gauge choice will be made shortly (and will differ a little from the linearized case). The metric is parametrized as

$$\begin{aligned} g_{+-} &= -\, e^{\phi} \; , \\ g_{ij} &= e^{\psi} \gamma_{ij} \; . \end{aligned} \tag{3.54}$$

The parameters ϕ and ψ are real while γ^{ij} is 2×2 real, symmetric matrix with unit determinant. As in section 3.4.1 the components of the Einstein field equation which do not contain ∂_+ are constraint relations. The constraint $R_{--} = 0$ yields

$$2\,\partial_-\phi\,\partial_-\psi - 2\,\partial_-^2\psi - (\partial_-\psi)^2 + \frac{1}{2}\partial_-\gamma^{ij}\,\partial_-\gamma_{ij} = 0 \tag{3.55}$$

This equation becomes completely solvable if we make the fourth and last gauge choice to be

$$\phi = \frac{\psi}{2}. \tag{3.56}$$

This solves equation (3.55) and gives

$$\psi = \frac{1}{4}\frac{1}{\partial_-^2}\left(\partial_-\gamma^{ij}\,\partial_-\gamma_{ij}\right) \tag{3.57}$$

From the parameters in (3.54), we have thus successfully eliminated both ϕ and ψ.

The constraint $R_{-i} = 0$ can also be solved to obtain

$$\begin{aligned} g^{-i} \;&=\; \mathrm{e}^{-\phi}\,\frac{1}{\partial_-}\bigg[\gamma^{ij}\,\mathrm{e}^{\phi-2\psi}\,\frac{1}{\partial_-}\bigg\{\mathrm{e}^{\psi}\left(\frac{1}{2}\partial_-\gamma^{kl}\,\partial_j\,\gamma_{kl}-\partial_-\,\partial_j\,\phi\right.\\ &\qquad \left.-\,\partial_-\,\partial_j\,\psi+\partial_j\phi\,\partial_-\,\psi\right)+\,\partial_l\left(\mathrm{e}^{\psi}\,\gamma^{kl}\,\partial_-\,\gamma_{jk}\right)\bigg\}\bigg] \end{aligned} \tag{3.58}$$

Now using (3.53),(3.54),(3.57) and (3.58) We obtain the Einstein-Hilbert action (3.51) in light-cone gauge as a closed form

$$\begin{aligned} S \;&=\; \frac{1}{2\kappa^2}\int d^4x e^{\psi}\left(2\,\partial_+\partial_-\phi+\partial_+\partial_-\psi-\frac{1}{2}\,\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right)\\ &\quad -e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi+\frac{1}{2}\partial_i\phi\partial_j\phi-\partial_i\phi\partial_j\psi-\frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl}+\frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right)\\ &\quad -\frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j, \end{aligned} \tag{3.59}$$

where

$$R_i = e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk}-\partial_-\partial_i\phi-\partial_-\partial_i\psi+\partial_i\phi\partial_-\psi\right)+\partial_k(e^{\psi}\,\gamma^{jk}\partial_-\gamma_{ij})$$

Note the both ϕ and ψ have been retained in the expression above to make it easier to identify the origin of each term. While obtaining the result above, some total derivative terms have been dropped (since we assume the fields vanish at infinity).

Perturbative expansion

We will now expand the Lagrangian around flat spacetime to first order in the gravitational coupling constant κ. We now parametrize the matrix γ^{ij} as

$$\gamma_{ij} = (e^H)_{ij}, \tag{3.60}$$

where

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}, \tag{3.61}$$

$det(\gamma_{ij}) = 1$ implies that H is traceless *i.e.* $h_{22} = -h_{11}$. The two remaining components can be combined to form a complex field (as in Yang-Mills theory).

$$h = \frac{(h_{11} + i\,h_{12})}{\sqrt{2}}, \qquad \bar{h} = \frac{(h_{11} - i\,h_{12})}{\sqrt{2}} \tag{3.62}$$

Under the Lorentz little group SO(2), h have helicity of $+2$ while $\bar{h}$ have helicity of -2. In terms of these complex fields we find that the helicity neutral variable ψ reads

$$\psi = -\frac{1}{\partial_-^2}(\partial_- h\,\partial_- \bar{h}) + \mathcal{O}(h^4) \tag{3.63}$$

We redefine the field as

$$h \to \frac{h}{\kappa} \tag{3.64}$$

The Lagrangian is expanded around flat spacetime as

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \ldots$$

Note that the interaction vertices of arbitrarily higher order in coupling (κ) are present unlike in Yang-Mills theory. We are now ready to expand the closed form Lagrangian for gravity, in terms of complex fields defined above.

$$\mathcal{L}_2 = \frac{1}{2}\, h \Box \bar{h} \tag{3.65}$$

The Lagrangian at κ order is

$$\mathcal{L}_3 = 2\kappa\, \bar{h}\, \partial_-^2 \left[-h\, \frac{\bar{\partial}^2}{\partial_-^2} h + \frac{\bar{\partial}}{\partial_-} h\, \frac{\bar{\partial}}{\partial_-} h \right] + 2\kappa\, h\, \partial_-^2 \left[-\bar{h}\, \frac{\partial^2}{\partial_-^2} \bar{h} + \frac{\partial}{\partial_-} \bar{h}\, \frac{\partial}{\partial_-} \bar{h} \right] \tag{3.66}$$

The ∂_+ (which is time derivative) does appear in the quartic interaction vertex. However it can be removed by as suitable field redefinition [7] and we at κ^2 get

$$\begin{aligned} \mathcal{L}_4 \;\; = \;\; & 4\kappa^2 \Bigg\{ -2\, \frac{1}{\partial_-^2} \left(\frac{\bar{\partial}}{\partial_-}\, h\, \partial_-^3 \bar{h} - h\, \partial_-^2 \bar{\partial} \bar{h} \right) \frac{1}{\partial_-^2} \left(\frac{\partial}{\partial_-}\, \bar{h}\, \partial_-^3 h - \bar{h}\, \partial_-^2 \partial h \right) \\ & + \frac{1}{\partial_-^2} (\bar{\partial} h\, \partial_-^2 \bar{h} - \partial_- h\, \partial_- \bar{\partial} \bar{h})\, \frac{1}{\partial_-^2} (\partial \bar{h}\, \partial_-^2 h - \partial_- \bar{h}\, \partial_- \partial h) - 3\, \frac{1}{\partial_-} (\bar{\partial} h\, \partial_- \bar{h})\, \frac{1}{\partial_-} (\partial_- h\, \partial \bar{h}) \\ & + \frac{1}{\partial_-} (\bar{\partial} h\, \partial_- \bar{h} - \partial_- h\, \bar{\partial} \bar{h})\, \frac{1}{\partial_-} (\partial \bar{h}\, \partial_- h - \partial_- \bar{h}\, \partial h) + 3\, \frac{1}{\partial_-} (\partial_- h\, \partial_- \bar{h})\, \frac{1}{\partial_-} (\bar{\partial} h\, \partial \bar{h}) \\ & + \left[\frac{1}{\partial_-^2} (\partial_- h\, \partial_- \bar{h}) - h \bar{h} \right] (\bar{\partial} h\, \partial \bar{h} + \partial h\, \bar{\partial} \bar{h} - \partial_- h\, \frac{\partial \bar{\partial}}{\partial_-} \bar{h} - \partial_- h\, \frac{\partial \bar{\partial}}{\partial_-} \bar{h}) \Bigg\} \end{aligned} \tag{3.67}$$

KLT relations

The tree-level vertex of gravity can be calculated in a similar way to Yang-Mills theory. The amplitude at order κ in the coupling constant can be written in terms of spinor products defined in (3.31).
The tree-level three-point vertex of the cubic term in (3.66) can be obtained by symmetrizing the momentum factors.

$$+2\kappa \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4}\, (2\pi)^4\, \delta^4(p+k+l)\, \frac{\langle k\, l \rangle^6}{\langle l\, p \rangle^2 \langle p\, k \rangle^2}\, h(p)\, \bar{h}(k)\, \bar{h}(l) \tag{3.68}$$

The coupling-stripped vertex can be written as

$$\mathcal{M}(p,k,l) = \frac{\langle k\, l\rangle^6}{\langle l\, p\rangle^2 \langle p\, k\rangle^2} \tag{3.69}$$

We see that the coupling-stripped cubic vertex (3.69) of gravity can be written as square of the color-coupling-stripped cubic vertex of the Yang-Mills theory(3.33). A similar relation can be checked for higher order terms.

If we represent momentum states of external legs of tree-level diagram by numbers, the KLT relation in field theory for four-point amplitude is given by

$$\mathcal{M}_3(1,2,3,4) = -i\, s_{12}\, A_4(1,2,3,4)\, A_4(1,2,3,4) \tag{3.70}$$

where $s_{ij} \equiv -(p_i + p_j)^2$ and $\mathcal{M}_n$ represents the coupling-stripped gravity amplitude and $\mathcal{A}_n$ represents the color-coupling-stripped amplitude of the Yang-Mills theory. The generalization to higher order amplitudes can be seen in [39]. In a remarkable paper Berends, Giele and Kujif have exploited the KLT relation to calculate the gravity tree-level amplitudes from the gauge theory [40]. They obtained a formula for the n-graviton scattering amplitude using MHV amplitudes of the gauge theory.

The quantum loop correction in the theory are required to construct the complete scattering matrix. The KLT relations can be exploited to obtain quantum loop amplitudes directly from semi-classical tree-level amplitudes by using D-dimensional unitarity (this is outside the scope of this thesis and we refer the reader to [18]).

One of the major aim of the next two chapters is to check the existence of KLT relations on the curved backgrounds. For which we have developed perturbative Lagrangian of the gravity up to cubic order in field on various backgrounds such as AdS_4, dS_4.

Chapter 4

Light-cone gravity in AdS_4

This chapter is entirely based on [12].

In this chapter we start by reviewing the four dimensional Poincaré patch of the Anti-de Sitter hyper-surface embeded in five-dimensional flat spacetime. We derive a closed form expression for the Einstein-Hilbert action on AdS_4 using the same formalism described in chapter 3. We provide a perturbative expansion starting from the closed form Lagrangian, to order κ.

Anti-de Sitter Space-time

AdS_d is a maximally symmetric solution of the Einstein equations with negative curvature and a negative cosmological constant. By maximally symmetric we mean that AdS_d has the maximum number of spacetime symmetries, which is $\frac{1}{2}d(d+1)$. This is the same as the symmetries of $d+1$ dimensional flat spacetime, where we have d translations, $d-1$ number of boosts and $\frac{1}{2}(d-1)(d-2)$ rotations.

Anti-de Sitter spacetime is defined as the quadric hypersurface

$$-(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + ... + (\xi^{d-1})^2 + (\xi^d)^2 - (\xi^d)^2 = R^2 \tag{4.1}$$

embedded in $d+1$ dimensional flat spacetime with the metric

$$ds^2 = -(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + ... + (d\xi^d)^2 - (d\xi^d)^2 \tag{4.2}$$

We work in the Poincaré Patch (**PP**) which does not cover the entire AdS spacetime but makes the d-dimensional Poincaré subgroup of the conformal group manifest.
We consider a five-dimensional flat spacetime with metric signature $\eta_{MN} \equiv (-1,1,1,1,-1)$ and co-ordinates ξ^M, $M = 0\dots 4$. On this spacetime, AdS$_4$ is defined by the hypersurface

$$-(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2 = R^2 \ , \tag{4.3}$$

with radius R. We now introduce Poincaré coordinates $x^\mu \equiv (x^0, x^1, z, x^3)$ on AdS$_4$

$$\xi^0 = \frac{R}{z}x^0 \qquad \xi^1 = \frac{R}{z}x^1 \qquad \xi^3 = \frac{R}{z}x^3 \ , \tag{4.4}$$

$$\xi^2 = \frac{1}{2z}\left[R^2 - \{-(x^0)^2 + (x^1)^2 + (x^3)^2 - z^2\}\right] \ , \tag{4.5}$$

$$\xi^4 = \frac{1}{2z}\left[R^2 + \{-(x^0)^2 + (x^1)^2 + (x^3)^2 - z^2\}\right] \ . \tag{4.6}$$

These coordinates satisfy (4.3) where z plays the role of a radial coordinate and divides the spacetime into two regions. These regions are $0 < z < +\infty$ and $0 > z > -\infty$. We work in the 'patch' $z > 0$ where $z = 0$ is part of the AdS boundary. The following metric is induced on this spacetime

$$g^{(0)}_{\mu\nu} = \partial_\mu \xi^M \partial_\nu \xi^N \eta_{MN} = \frac{R^2}{z^2}\eta_{\mu\nu} \ , \tag{4.7}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. We now introduce light-cone coordinates as $x^\mu \equiv (x^+, x^-, x^1, z)$. The cosmological constant for AdS$_4$ is

$$\Lambda = -\frac{3}{R^2} \ . \tag{4.8}$$

Preliminaries

The Einstein-Hilbert action reads

$$S_{EH} = \int d^4x\, L = \frac{1}{2\,\kappa^2} \int d^4x \sqrt{-g}\,(\,\mathcal{R} - 2\,\Lambda\,)\ , \tag{4.9}$$

The Einstein-Hilbert action on manifold M with boundary ∂M can be divided in two parts: contributions from the bulk and contributions from the boundary. In general the form of the action on such manifolds can be written as

$$S_{EH} = \int_M d^4x\, \mathcal{L}_M + \int_{\partial M} d^3x\, \mathcal{L}_{\partial M} \tag{4.10}$$

As boundary terms do not contribute to the equations of motion, we focus only on the bulk contributions and drop the boundary terms [41]. Though the boundary terms can affect the quantum theory, we do not address such issues as they are outside the scope of this thesis.

Einstein-Hilbert action on AdS$_4$

Our aim is to study pertubative gravity on the AdS$_4$ background in light-cone gauge. The metric $g_{\mu\nu}$ must reduce to the background metric $g^{(0)}_{\mu\nu}$ in absence of perturbations. We start by making the following three gauge choices

$$g_{--} = g_{-i} = 0 \quad , \, i = 1, z\ . \tag{4.11}$$

Note that these gauge choices are consistent with background metric in absence of fluctuations, since in light-cone coordinates, $\eta_{--} = \eta_{-i} = 0$. We will make fourth gauge choice shortly, The metric is parametrized as

$$
\begin{aligned}
g_{+-} &= -e^{\phi} \ , \\
g_{ij} &= e^{\psi}\,\gamma_{ij} \ .
\end{aligned}
\tag{4.12}
$$

Closed form expression

The equations of motion corresponding to the Einstein-Hilbert action read

$$
\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\,\mathcal{R} = -\Lambda\, g_{\mu\nu} \ . \tag{4.13}
$$

As mentioned before, the equations of motion which contain time derivative ∂_+ possess dynamical information and should not be touched. The ones which do not contain time derivatives ∂_+ are the constraint equations [11].

The first constraint equation $\mathcal{R}_{--} = 0$ reads

$$
2\,\partial_-\,\phi\,\partial_-\,\psi - 2\,{\partial_-}^2\,\psi - (\partial_-\,\psi)^2 + \frac{1}{2}\,\partial_-\,\gamma^{kl}\,\partial_-\,\gamma_{kl} = 0 \ . \tag{4.14}
$$

This constraint equation may be solved by making our fourth gauge choice as

$$
\phi = \frac{1}{2}\,\psi \ . \tag{4.15}
$$

This permits us to solve equation (4.14) and we obtain

$$
\psi = \frac{1}{4}\frac{1}{{\partial_-}^2}\left[\partial_-\gamma^{ij}\partial_-\gamma_{ij}\right] + 2\,ln\,\frac{R^2}{z^2} \ . \tag{4.16}
$$

The second term, in ψ, is required to make sure that the metric components g_{ij} and g_{+-} reduce correctly to $g^{(0)}_{ij}$ and $g^{(0)}_{+-}$ respectively. Notice that this result differs from that in [7] where this term does not appear.

We now calculate the determinant of γ_{ij} by using the second relation in (4.12), which implies that

$$\det g^{(0)}_{ij} = \left(\frac{R^2}{z^2}\right)^4 \det \gamma^{(0)}_{ij} \ , \tag{4.17}$$

Here $\{\}^{(0)}$ superscripts implies the absence of fluctuations and in that case the metric must reduce to the background metric. In this limit, the AdS$_4$ metric reduces to $\frac{R^2}{z^2}$ times the Minkowski metric so the L.H.S of (4.17) is $(\frac{R^2}{z^2})^2$ this constrains the determinant of γ_{ij}.

$$\det \gamma^{(0)}_{ij} = (\frac{z^2}{R^2})^2 \ . \tag{4.18}$$

Note that the above result is distinct from the flat background result. On flat background γ_{ij} is unimodular [6, 7]. Here we will choose the full γ_{ij} (including the fluctuations) to have the same determinant as in (4.18). Since this choice will ensure that our fluctuation-field (introduced in next section) in γ_{ij} remain traceless thereby making computations simpler.

The second constraint relation $\mathcal{R}_{-i} = 0$ yields

$$\begin{aligned} g^{-i} = \mathrm{e}^{-\phi}\,\frac{1}{\partial_-}\bigg[\,\gamma^{ij}\,\mathrm{e}^{\phi-2\psi}\,\frac{1}{\partial_-}\Big\{\mathrm{e}^{\psi}\Big(\frac{1}{2}\partial_-\,\gamma^{kl}\,\partial_j\,\gamma_{kl} - \partial_-\,\partial_j\,\phi \\ - \partial_-\,\partial_j\,\psi + \partial_j\phi\partial_-\,\psi\Big) + \partial_l\Big(\mathrm{e}^{\psi}\,\gamma^{kl}\,\partial_-\,\gamma_{jk}\Big)\Big\}\bigg] \ . \end{aligned} \tag{4.19}$$

The Einstein-Hilbert action in light-cone gauge is

$$S=\int d^4x\mathcal{L}=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(2g^{+-}R_{+-}+g^{ij}R_{ij}-2\Lambda\right)\ . \tag{4.20}$$

We now calculate components of Ricci tensor in above expression with the help of results listed above. We derive the closed form expression for the action in AdS_4 in terms of the dimensionless parameters of metric, ϕ,ψ ans γ_{ij}.

$$\begin{aligned}S\ &=\frac{1}{2\kappa^2}\int d^3x\int dz\Bigg\{\frac{z^2}{R^2}e^{\psi}\left(2\partial_+\partial_-\phi+\partial_+\partial_-\psi-\frac{1}{2}\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right)\\&-\frac{z^2}{R^2}e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi+\frac{1}{2}\partial_i\phi\partial_j\phi-\partial_i\phi\partial_j\psi-\frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl}+\frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right)\\&-\frac{z^2}{2R^2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j+\frac{2}{R^2}e^{\phi}\gamma^{zz}-2\,\frac{z^2}{R^2}\,e^{\psi}\,e^{\phi}\,\Lambda\Bigg\}\ ,\end{aligned} \tag{4.21}$$

with

$$\begin{aligned}R_i\ &=\ \mathrm{e}^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk}-\partial_-\partial_i\phi-\partial_i\partial_-\psi+\partial_i\phi\partial_-\psi\right)\\&\quad+\partial_k\left(\mathrm{e}^{\psi}\gamma^{jk}\partial_-\gamma_{ij}\right)\ .\end{aligned}$$

While obtaining the above result, we have dropped several boundary terms because of the arguments outlined in Section 4.2. This then is the entire action for pure gravity in AdS_4 written purely in terms of the physical degrees of freedom.

Comparison with flat space result

There three main differences between our result (4.21) and the result obtained on flat background [6, 7]: the presence of conformal factor of $\frac{z^2}{R^2}$, the penultimate term proportional to γ^{zz} and a term proportional to the cosmological constant.

Perturbative expansion

In this section we do the perturbative expansion of the action in (4.21) to cubic order in the fields. We do this by choosing the following infinite series expansion.

$$\begin{aligned} \gamma_{ij} &= \frac{z^2}{R^2}\left(\mathrm{e}^{H}\right)_{ij} \;, \\ H &= \begin{pmatrix} h_{11} & h_{1z} \\ h_{1z} & h_{zz} \end{pmatrix} \;. \end{aligned} \tag{4.22}$$

where $h_{zz} = -h_{11}$ can be inferred from the explanation given below equation (4.18). In terms of these perturbation fields equation (4.16) becomes

$$\psi = -\frac{1}{4}\frac{1}{\partial_-{}^2}\left[\partial_- h_{ij}\partial_- h_{ij}\right] + 2\,ln\,\frac{R^2}{z^2} + \mathcal{O}(h^4) \;. \tag{4.23}$$

We now redefine the field as

$$h \to \frac{1}{\sqrt{2}\,\kappa}\,h \tag{4.24}$$

In terms of these fields the action at $\mathcal{O}(h^2)$ reads

$$S_2 = \int d^3x \int dz\,\mathcal{L}_2 \;, \tag{4.25}$$

with

$$\mathcal{L}_2 = +\frac{R^2}{2z^2}\partial_+ h_{ij}\partial_- h_{ij} - \frac{R^2}{4z^2}\partial_i h_{kl}\partial_i h_{kl} - 2\frac{R^2}{z^3}h_{ik}\partial_k h_{iz} + \frac{R^2}{z^4}h_{zk}h_{kz} \;.$$

In the above equation we have used cosmological constant (4.8) and fact that $\mathrm{Tr}(h=0)$. Note that from (4.21) we can deduce that contributions from the cosmological term appears only in the interaction vertices involving even number of fields.
The computation of action at $\mathcal{O}(h^3)$ reads

$$S_3 = \int d^3x \int dz \, \frac{1}{\sqrt{2}} \, \mathcal{L}_3 \ , \tag{4.26}$$

where

$$\begin{aligned}
\mathcal{L}_3 = \ \kappa \bigg\{ & -\frac{1}{2}\frac{R^2}{z^2}\partial_k h_{ik}\partial_l h_{ij} h_{jl} + 2\frac{R^2}{z^3} h_{iz}\partial_k h_{ij} h_{jk} + \frac{R^2}{4z^2} h_{ij}\partial_i h_{kl}\partial_j h_{kl} \\
& -\frac{R^2}{2z^2} h_{ij}\partial_i h_{kl}\partial_k h_{jl} - 2\frac{R^2}{z^3} h_{iz}\frac{1}{\partial_-}(\partial_- h_{lm}\partial_i h_{lm}) \\
& +4\frac{R^2}{z^4} h_{iz}\frac{1}{\partial_-}(h_{lz}\partial_- h_{il}) + 4\frac{R^2}{z^3}\partial_k h_{iz}\frac{1}{\partial_-}(h_{lk}\partial_- h_{il}) \\
& +\frac{1}{2}\frac{R^2}{z^2}\partial_k h_{ik}\frac{1}{\partial_-}(\partial_- h_{lm}\partial_i h_{lm}) - 4\frac{R^2}{z^3}\partial_k h_{ik}\frac{1}{\partial_-}(h_{lz}\partial_- h_{il}) \\
& +\frac{R^2}{z^2}\partial_k h_{ik}\frac{\partial_l}{\partial_-}(h_{ml}\partial_- h_{im}) \\
& -2\frac{R^2}{z^2} h_{ij}\partial_i\partial_j B - 6\frac{R^2}{z^3} h_{iz}\partial_i B - 4\frac{R^2}{z^4} h_{zz} B \bigg\} \ ,
\end{aligned}$$

with

$$B = -\frac{1}{8}\frac{1}{\partial_-{}^2}\left[\partial_- h_{ij}\partial_- h_{ij}\right] \ . \tag{4.27}$$

We see that the kinetic and cubic vertices in AdS_4 are more complicated than their flat background counterparts. The unique feature of the KLT relations will most likely change as we move away from flat spacetime (this is because open and closed string correlation functions are likely to change differently on curved backgrounds). The Lagrangian derived above is an ideal starting point to verify the KLT relations on AdS_4.

The following is a list of the formulae used extensively in our computations.

$$
\begin{aligned}
g^{+-} &= -e^{-\phi} \\
g^{ij} &= e^{-\psi}\gamma^{ij} \\
\gamma^{ij} &= \frac{R^2}{z^2}(e^{-H})_{ij} \\
g^{\mu\nu}g_{\mu\rho} &= \delta^{\nu}_{\rho} \implies g^{++} = g^{+i} = 0 \\
g_{+i} &= -g_{+-}g_{ij}g^{-j} \\
g_{++} &= -e^{\psi}g^{--} + e^{\phi}g^{-i}g_{+i} \\
\gamma^{ij}\gamma_{ij} &= 2 \\
\gamma^{ij}\partial_k\gamma_{ij} &= \frac{4}{z}\delta_{kz} \\
\sqrt{-g} &= \frac{z^2}{R^2}e^{\psi}e^{\phi} \\
det(\gamma_{ij}) &= \frac{z^4}{R^4} \implies Tr(H) = 0 \ .
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
\Gamma^{+}_{++} &= \frac{1}{2}g^{+-}[2\partial_{+}g_{+-}-\partial_{-}g_{++}] \\
\Gamma^{+}_{+-} &= 0 \\
\Gamma^{+}_{--} &= 0 \\
\Gamma^{+}_{i-} &= 0 \\
\Gamma^{+}_{i+} &= \frac{1}{2}g^{+-}[\partial_{i}g_{+-}-\partial_{-}g_{i+}] \\
\Gamma^{+}_{ij} &= -\frac{1}{2}g^{+-}\partial_{-}g_{ij} \\
\Gamma^{-}_{--} &= g^{+-}\partial_{-}g_{+-} \\
\Gamma^{-}_{+-} &= \frac{1}{2}\{g^{+-}\partial_{-}g_{++}+g^{-i}[\partial_{-}g_{i+}-\partial_{i}g_{+-}]\} \\
\Gamma^{-}_{++} &= \frac{1}{2}\{g^{+-}\partial_{+}g_{++}+g^{--}[2\partial_{+}g_{+-}-\partial_{-}g_{++}] \\
&\quad +g^{-i}[2\partial_{+}g_{i+}-\partial_{i}g_{++}]\} \\
\Gamma^{-}_{+i} &= \frac{1}{2}\{g^{+-}\partial_{i}g_{++}+g^{--}[\partial_{i}g_{+-}-\partial_{-}g_{i+}] \\
&\quad +g^{-j}[\partial_{i}g_{+j}+\partial_{+}g_{ij}-\partial_{j}g_{+i}]\} \\
\Gamma^{-}_{-i} &= \frac{1}{2}\{g^{+-}[\partial_{i}g_{+-}+\partial_{-}g_{+i}]+g^{-j}\partial_{-}g_{ij}\} \\
\Gamma^{-}_{ij} &= \frac{1}{2}\{g^{+-}[\partial_{i}g_{+j}+\partial_{j}g_{+i}-\partial_{+}g_{ij}]-g^{--}\partial_{-}g_{ij} \\
&\quad +g^{-k}[\partial_{i}g_{kj}+\partial_{j}g_{ik}-\partial_{k}g_{ij}]\} \\
\Gamma^{i}_{jk} &= \frac{1}{2}\{-g^{-i}\partial_{-}g_{jk}+g^{im}[\partial_{j}g_{mk}+\partial_{k}g_{mj}-\partial_{m}g_{jk}]\} \\
\Gamma^{i}_{-j} &= \frac{1}{2}g^{ik}\partial_{-}g_{kj} \\
\Gamma^{i}_{+-} &= \frac{1}{2}g^{ij}[\partial_{-}g_{j+}-\partial_{j}g_{+-}]
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
\Gamma^i_{+j} &= \frac{1}{2}\{g^{-i}[\partial_j g_{+-} - \partial_- g_{+j}] + g^{ik}[\partial_j g_{+k} + \partial_+ g_{kj} - \partial_k g_{+j}]\} \\
\Gamma^i_{++} &= \frac{1}{2}\{g^{-i}[2\partial_+ g_{+-} - \partial_- g_{++}] + g^{ij}[2\partial_+ g_{+j} - \partial_j g_{++}]\} \\
\Gamma^i_{--} &= 0 \\
\Gamma^j_{ij} &= \frac{1}{2}\{-g^{-j}\partial_- g_{ij} + g^{jl}[\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}]\}
\end{aligned}
$$

Chapter 5

Light-cone gravity in dS_4

This chapter is based on [13].

Here, we formulate pure gravity on a four dimensional conformally flat de Sitter background using light-cone gauge.

de Sitter Space

The de Sitter spacetime is the maximally symmetric Lorentzian space with positive (constant) scalar curvature. d-dimensional de Sitter space is a vacuum solution of the Einstein field equations with positive cosmological constant Λ [42].

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{(d-1)(d-2)}{2} H^2 g_{\mu\nu} \tag{5.1}$$

where H is the Hubble constant. It appears in the above expression through the definition of the cosmological constant

$$\Lambda = \frac{(d-1)(d-2)}{2} H^2 . \tag{5.2}$$

d-dimensional de Sitter space is defined as a hyperboloid

$$-(\xi^0)^2 + (\xi^1)^2 + ... + (\xi^{d-2})^2 + (\xi^{d-1})^2 + (\xi^d)^2 \equiv g_{AB}\,\xi^A \xi^B = l^2 = H^{-2} \qquad (5.3)$$

embedded in $(d+1)$-dimensional ambient space which in this case is the Minkowski background with signature $\eta_{AB} = (-,+,+,...,+)$ and metric $ds^2 = g_{AB}\,d\xi^A\,d\xi^B$,
$A, B = 0, 1, ...d$. The radius of the de Sitter space, l , is related to the Hubble constant by $H = l^{-1}$.

Global coordinate of de Sitter Metric

The dS metric induced from ambient spacetime can be defined by solving (5.3). One of the solutions is

$$\xi^0 = \frac{\sinh(Ht)}{H}, \;\; \xi^i = \frac{a_i \cosh(Ht)}{H}, \quad i = 1, ..., d. \qquad (5.4)$$

where a_i is d-dimensional unit vector with $a_i^2 = 1$. Now, one can choose

$$\begin{aligned}
a_1 &= \cos\theta_1, \quad -\frac{\pi}{2} \le \theta_1 \le \frac{\pi}{2} \\
a_2 &= \sin\theta_1 \cos\theta_2, \quad -\frac{\pi}{2} \le \theta_1 \le \frac{\pi}{2} \\
&... \\
a_{d-2} &= \sin\theta_1 \sin\theta_2 ... \sin\theta_{d-3} \cos\theta_{d-2}, \quad -\frac{\pi}{2} \le \theta_{d-2} \le \frac{\pi}{2} \\
a_{d-1} &= \sin\theta_1 \sin\theta_2 ... \sin\theta_{d-2} \cos\theta_{d-1}, \quad -\pi \le \theta_{d-1} \le \pi \\
a_d &= \sin\theta_1 \sin\theta_2 ... \sin\theta_{d-2} \sin\theta_{d-1}.
\end{aligned} \qquad (5.5)$$

Using the above coordinate system (5.5), the metric becomes

$$ds^2 = -dt^2 + \frac{\cosh^2(Ht)}{H^2} d\Omega^2_{d-1}, \tag{5.6}$$

where

$$d\Omega^2_{d-1} = \sum_{j=1}^{d-1} \left(\prod_{i=1}^{j-1} \sin^2 \theta_i \right) d\theta_j^2 \tag{5.7}$$

is the line element of the $(d-1)$-dimensional sphere. This metric in (5.6) covers the full de Sitter spacetime and is termed as global.

Note that the topology of de Sitter spacetime is $\mathbf{R} \times \mathbf{S^d}$. Its spatial slices are compact d-dimensional spheres which evolve in time. From the embedding definition in (5.3) we see that the time evolution proceeds as follows : in the past the d-sphere is large and its radius diverges as $\xi^0 \to -\infty$. The radius shrinks as time evolves forward and reaches a minimum radius of H^{-1} at $\xi^o = 0$. This shrinking is followed by an expanding phase which again leads to the diverging radius of the d-sphere at $\xi^0 \to +\infty$.

Poincaré patches

Another solution of (5.3) can be found by choosing,

$$\begin{aligned} -(H\,\xi^0)^2 + (H\,\xi^d)^2 &= 1 - (H\,x^i)^2\, e^{2Ht} \\ (H\,\xi^1)^2 + ... + (H\,\xi^{d-1})^2 &= (H\,x^i)^2\, e^{2Ht} \end{aligned} \tag{5.8}$$

One can see that the above equations are satisfied by by the following coordinate choices:

$$
\begin{aligned}
H\,\xi^0 &= \sinh(H\,t) + \frac{(H\,x^i)^2}{2} e^{Ht} \\
H\,\xi^i &= H\,x^i\,e^{H\,t} \quad i = 1,2,...(d-1) \\
H\,\xi^d &= -\cosh(H\,t) + \frac{(H\,x^i)^2}{2} e^{Ht}
\end{aligned} \tag{5.9}
$$

With these coordinates the induced metric becomes

$$
ds^2 = -dt^2 + e^{2Ht}(dx^i)^2 \tag{5.10}
$$

Note that the coordinate choice in (5.9) imposes the following restriction :

$$
-\xi^0 + \xi^d = -\frac{1}{H} e^{Ht} \leq 0 \tag{5.11}
$$

which suggests that this coordinate system covers only half, $\xi^0 \geq \xi^d$, of the entire dS spacetime. It is called the Expanding Poincaré patch (EPP) of de Sitter space. The formulation in this chapter has been done on the expanding Poincré patch of dS_4 .

The other half, $\xi^0 \leq \xi^d$, of the dS spacetime is referred to as Contracting Poincaré patch (CPP) and is covered by the metric

$$
ds^2 = -dt^2 + e^{-2Ht}(dx^i)^2 \tag{5.12}
$$

Expanding Poincaré patch in four dimensions

The four dimensional de Sitter spacetime is a hyperboloid embedded in five dimensional Minkowski spacetime. Consider a five-dimensional flat spacetime with metric

$\eta_{MN} \equiv diag(-1,1,1,1,1)$ then the invariant interval reads

$$ds^2 = -(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 + (d\xi^4)^2 \tag{5.13}$$

with $\xi^M \in (-\infty, +\infty), M = 0 \ldots 4$. de Sitter spacetime is defined by the hypersurface,

$$-(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2 = H^{-2} \tag{5.14}$$

The above equation (5.14) has a similar solution to that in (5.9).

$$\begin{aligned} -(H\xi^0)^2 + (H\xi^4)^2 &= 1 - (Hx^i)^2 e^{2Ht} \\ (H\xi^1)^2 + (H\xi^2)^2 + (H\xi^3)^2 &= (Hx^i)^2 e^{2Ht} \end{aligned} \tag{5.15}$$

For which, we can define

$$\begin{aligned} H\xi^0 &= \sinh(Ht) + \frac{(Hx^i)^2}{2} e^{Ht} \\ H\xi^i &= Hx^i e^{Ht} \\ H\xi^4 &= -\cosh(Ht) + \frac{(Hx^i)^2}{2} e^{Ht} \end{aligned} \tag{5.16}$$

Here, $x^i \in (-\infty, +\infty), i = 1,2,3$ and $t \in (-\infty, +\infty)$. Using these coordinates, induced metric can be written as

$$ds^2 = -(dt)^2 + e^{2Ht}\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\} \tag{5.17}$$

As discussed in the section 5.1.2, the choice of coordinates in (5.16) imposes following constraint:

$$-\xi^0 + \xi^4 = -\frac{1}{H} e^{Ht} \leq 0 \implies \xi^0 \geq \xi^4 \tag{5.18}$$

Therefore, metric in (5.17) covers only half of the de Sitter space-time: the expanding Poincaré patch. In this patch, we define conformal time by

$$H\eta = e^{-Ht}, \tag{5.19}$$

which modifies the metric (5.17) to

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2) \tag{5.20}$$

In the expanding Poincaré patch, conformal time changes from $\eta = +\infty$ (which is at past infinity i.e. $t = -\infty$) to $\eta = 0$ (which is at future infinity i.e. $t = \infty$).

Gravity on dS_4

Preliminaries

The Einstein-Hilbert action of pure gravity on curved background reads

$$S_{EH} = \int d^4x\, L = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\,(\mathcal{R} - 2\Lambda)\ , \tag{5.21}$$

In this chapter, we derive the pure gravity Lagrangian, in light-cone gauge, on a dS_4 spacetime characterized by cosmological constant Λ.

Consider metric of expanding Poincaré patch in (5.20),

$$g^{(0)}_{\mu\nu} = \frac{1}{H^2\eta^2}\eta_{\mu\nu} \tag{5.22}$$

where $\eta_{\mu\nu} = (-1,1,1,1,)$ is Minkowski metric in four dimensions . The light cone coordinates are introduced as $x^\mu \equiv (x^+, x^-, x^i)$ where

$$x^\pm = \frac{\eta \pm x^3}{\sqrt{2}} \ . \tag{5.23}$$

We define

$$X = x^+ + x^-, \tag{5.24}$$

Metric now becomes

$$g^{(0)}_{\mu\nu} = \frac{2}{H^2X^2}\eta^{L.C.}_{\mu\nu} \tag{5.25}$$

The cosmological constant(positive) of dS_4 is

$$\Lambda = 3H^2 \tag{5.26}$$

Closed form expression

We now proceed to derive closed form expression of the gravity action in light-cone gauge by making the following gauge choices.

$$g_{--} = g_{-i} = 0 \quad , i = 1{,}2 \ . \tag{5.27}$$

Again, these choices are consistent with $g^{(0)}_{\mu\nu}$. The fourth (and last) gauge choice will be made shortly. Other components of the metric are parametrized as

$$\begin{aligned} g_{+-} &= -\frac{2}{H^2X^2}e^{\phi} \ , \\ g_{ij} &= \frac{2}{H^2X^2}e^{\psi}\gamma_{ij} \ . \end{aligned} \tag{5.28}$$

The key difference in solving constraint relations in dS_4 stems from the fact that X depends on∂_-, as opposed to the case of both flat spacetime and AdS_4. Therefore we will need integrating factors to solve these constraint equations. The first constraint relation $\mathcal{R}_{--} = 0$ with the use of (5.28) reads

$$\partial_-\phi\partial_-\psi - \partial_-{}^2\psi - \frac{1}{2}(\partial_-\psi)^2 - \frac{2}{X}\partial_-\phi + \frac{1}{4}\partial_-\gamma^{kl}\partial_-\gamma_{kl} = 0 \tag{5.29}$$

This constraint equation is solvable if we make the fourth and the last gauge choice as

$$\phi = \frac{1}{2}\psi \tag{5.30}$$

This reduces (5.29) to

$$\frac{1}{4}\partial_-\gamma^{kl}\partial_-\gamma_{kl} - \partial_-{}^2\psi - \frac{2}{X}\partial_-\phi = 0 \tag{5.31}$$

which after multiplication with integrating factor (X) yields

$$\psi = \frac{1}{4}\frac{1}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}\left(X\partial_-\gamma^{kl}\partial_-\gamma_{kl}\right)\right] \tag{5.32}$$

Note that the structure of ψ in (5.32) is different from the analogous result on flat space and AdS_4 [7, 12]. The second constraint relation independent of ∂_+ is $\mathcal{R}_{-i} = 0$. Which with the use of (5.27) and (5.28) yields

$$\begin{aligned} g^{-i} = \; & H^2X^2 e^{-\phi}\,\tfrac{1}{\partial_-}\Bigg[X^2\gamma^{ij}\,e^{\phi-2\psi}\,\tfrac{1}{\partial_-}\Bigg\{\tfrac{1}{X^2}\,e^{\psi}\left(\tfrac{1}{4}\partial_-\gamma^{kl}\,\partial_j\gamma_{kl} - \tfrac{1}{2}\partial_-\,\partial_j\,\phi\right. \\ & \left. -\tfrac{1}{2}\partial_-\,\partial_j\,\psi + \tfrac{1}{2}\partial_j\phi\,\partial_-\,\psi - \tfrac{2}{X}\partial_j\,\phi\right) + \tfrac{1}{2X^2}\,\partial_l\left(e^{\psi}\,\gamma^{kl}\,\partial_-\,\gamma_{jk}\right)\Bigg\}\Bigg] \end{aligned} \tag{5.33}$$

To get the above expression and solve for g^{-i}, an integrating factor, $\frac{1}{X^2}$ has been multiplied. Having determined these components, now we can proceed to calculate Einstein-Hilbert action

$$S = \int d^4x\mathcal{L} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(2g^{+-}R_{+-} + g^{ij}R_{ij} - 2\Lambda\right) \tag{5.34}$$

which explicitly exactly yields

$$\begin{aligned} S \;\; = \;\; & \int d^4x\frac{1}{H^2X^2}e^{\psi}\left(\frac{24}{X^2} + 4\partial_+\partial_-\phi - 2\partial_+\psi\partial_-\psi - \partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) \\ & - \frac{1}{H^2X^2}e^{\phi}\gamma^{ij}\left(2\partial_i\partial_j\phi + \partial_i\phi\partial_j\phi - 2\partial_i\phi\partial_j\psi - \frac{1}{2}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \partial_i\gamma^{kl}\partial_k\gamma_{jl}\right) \\ & - \frac{4}{H^2X^2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j - \frac{8}{H^4X^4}\,e^{\psi}\,e^{\phi}\,\Lambda \end{aligned} \tag{5.35}$$

where

$$
\begin{aligned}
R_i &= \frac{1}{X^2}\,\mathrm{e}^{\psi}\left(\frac{1}{4}\partial_-\gamma^{kl}\,\partial_i\gamma_{kl} - \frac{1}{2}\partial_-\,\partial_i\,\phi - \frac{1}{2}\partial_-\,\partial_i\,\psi + \frac{1}{2}\partial_i\phi\,\partial_-\,\psi - \frac{2}{X}\partial_i\,\phi\right) \\
&\quad + \frac{1}{2X^2}\,\partial_l\left(\mathrm{e}^{\psi}\,\gamma^{kl}\,\partial_-\,\gamma_{ik}\right)
\end{aligned}
$$

While obtaining this expression some boundary terms have been dropped. This closed form of action (5.35) is valid on both patches of de Sitter.

Perturbative expansion

Now we obtain perturbative expansion of action in (5.35) to first order in gravitational coupling constant (κ). We parametrize γ_{ij} as

$$\gamma_{ij} = (e^H)_{ij}$$

with

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}, \tag{5.36}$$

$h_{22} = -h_{11}$ makes the above matrix traceless. ψ in terms of fluctuation fields is expanded as

$$\psi = -\frac{1}{4}\,\frac{1}{\partial_-}\left[\frac{1}{X}\,\frac{1}{\partial_-}(X\,\partial_- h_{ij}\,\partial_- h_{ij})\right] + \mathcal{O}(h^4) \tag{5.37}$$

We now rescale the field h as

$$h \to \frac{1}{\sqrt{2}\kappa} h \tag{5.38}$$

Now kinetic and cubic interaction vertices of action (5.35) can be expanded.

$$S_2 = \int d^4x \, \mathcal{L}_2 \tag{5.39}$$

where

$$\begin{aligned}
\mathcal{L}_2 \;=\;& \frac{1}{2H^2X^4}\frac{1}{\partial_-}(X\partial_- h_{ij}\partial_- h_{ij}) - \frac{1}{2H^2X^3}\frac{\partial_+}{\partial_-}(X\partial_- h_{ij}\partial_- h_{ij}) \\
&+ \frac{1}{H^2X^2}\partial_+ h_{ij}\partial_- h_{ij} + \frac{1}{2H^2X^2}\frac{\partial_i\partial_i}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}(X\partial_- h_{jk}\partial_- h_{jk})\right] \\
&- \frac{1}{2H^2X^2}\partial_i h_{jk}\partial_i h_{jk} + \frac{1}{H^2X^2}\partial_i h_{jk}\partial_j h_{ik} \\
&+ \frac{3}{H^2X^4}\frac{1}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}(X\partial_- h_{ij}\partial_- h_{ij})\right] \\
&- \frac{1}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_j\partial_- h_{ij}\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_k\partial_- h_{ik}\right)
\end{aligned} \tag{5.40}$$

Note that (5.30) and the last term of (5.35) implies that the cosmological constant is always accompanied by ψ. Therefore from (5.37) we can infer that the term containing cosmological term contributes to interaction vertices involving even number of fields.

Now, action at $\mathcal{O}(h^3)$ is

$$S_3 = \int d^4x \, \frac{1}{\sqrt{2}} \mathcal{L}_3 \tag{5.41}$$

with

$$
\begin{aligned}
\mathcal{L}_3 \;=\; \kappa\Bigg\{ & \frac{1}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_-h_{jk}\partial_ih_{jk}\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right) \\
& -\frac{3}{H^2X^2}\frac{1}{\partial_-}\left[\frac{1}{X}\frac{\partial_i}{\partial_-}\left(X\partial_-h_{jk}\partial_-h_{jk}\right)\right]\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right) \\
& -\frac{1}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X}\frac{\partial_i}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}(X\partial_-h_{jk}\partial_-h_{jk})\right]\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right) \\
& +\frac{2}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_jh_{jk}\partial_-h_{ik}\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right) \\
& -\frac{1}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_j\partial_-h_{ij}^2\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right) \\
& +\frac{1}{H^2X^2}h_{ij}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_k\partial_-h_{ik}\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right)\Bigg\}
\end{aligned}
\tag{5.42}
$$

As expected, closed form and eventually interaction terms of perturbative expansion are very much tangled compared to their counterparts on flat spacetime [7] and four dimensional Anti-de Sitter background [12]. The differences can be seen in terms of coordinate dependent conformal like factors along with additional terms appearing in closed form expression.

The following is a list of frequently used identities in this chapter.

$$\begin{aligned}
g^{+-} &= -\frac{H^2X^2}{2}e^{-\phi} \\
g^{ij} &= \frac{H^2X^2}{2}e^{-\psi}\gamma^{ij} \\
\gamma^{ij} &= (e^{-H})_{ij} \\
g^{\mu\nu}g_{\mu\rho} &= \delta^\nu_\rho \implies g^{++} = g^{+i} = 0 \\
g_{+i} &= -g_{+-}g_{ij}g^{-j} \\
g_{++} &= -\frac{4}{H^4X^4}e^{\psi}g^{--} + \frac{2}{H^2X^2}e^{\phi}g^{-i}g_{+i} \\
\gamma^{ij}\gamma_{ij} &= 2 \\
\gamma^{ij}\partial_k\gamma_{ij} &= \gamma^{ij}\partial_-\gamma_{ij} = 0 \\
\sqrt{-g} &= \frac{4}{H^4X^4}e^{\psi}e^{\phi}
\end{aligned} \tag{5.43}$$

Chapter 6

Quadratic forms in light-cone gauge

This chapter is a review based on the series of publications [14, 15, 24, 25, 43] in which the authors have investigated the algebraic properties of both the pure and the maximally supersymmetric theories of gravity and Yang-Mills theory.

Thus far, in this thesis, we have shown how to formulate pure gravity, in light-cone gauge, on a variety of four-dimensional backgrounds. In each of these cases, we obtained a closed form expression for the light-cone Lagrangian. From this, we performed a perturbative expansion and thus derived a range of light-cone interaction vertices. In this chapter, we will discuss the structure of the Hamiltonian in pure gravity.

However, before we discuss pure gravity, we will take a detour to discuss the occurence of quadratic forms (QF) in light-cone field theories. The best way to illustrate these structures is again using Yang-Mills theories.

Pure Yang-Mills theory

In the following, we prove that the Hamiltonian of pure Yang-Mills theory, in light-cone gauge, can be written as a quadratic form [14]. The key point of interest will be that this simple and elegant structure only appears in the pure and maximally supersymmetric Yang-Mills theories. We start our study of these quadratic forms by showing that pure Yang-Mills theory may be written as a quadratic form.

Four spacetime dimensions

The Lagrangian for pure Yang-Mills theory was derived in the third chapter - equation (3.30)

$$
\begin{aligned}
\mathcal{L} \;\; &= \;\; \bar{A}^a \,\Box\, A^a - 2\,g\,f^{abc}\Big(\frac{\bar{\partial}}{\partial_-}\,A^a\partial_-\bar{A}^b\,A^c + \frac{\partial}{\partial_-}\,\bar{A}^a\,\partial_-\,A^b\,\bar{A}^c\Big) \\
&\quad -2\,g^2\,f^{abc}\,f^{ade}\,\frac{1}{\partial_-}\Big(\partial_-\,A^b\,\bar{A}^c\Big)\,\frac{1}{\partial_-}(\partial_-\,\bar{A}^d\,A^e)\;. \qquad (6.1)
\end{aligned}
$$

The Hamiltonian corresponding to (3.30) reads

$$
\begin{aligned}
\mathcal{H} \;\; &= \;\; \bar{A}^a\,\bar{\partial}\partial\,A^a + g\,f^{abc}\Big(\frac{\bar{\partial}}{\partial_-}\,A^a\partial_-\bar{A}^b\,A^c + \frac{\partial}{\partial_-}\,\bar{A}^a\,\partial_-\,A^b\,\bar{A}^c\Big) \\
&\quad +g^2\,f^{abc}\,f^{ade}\,\frac{1}{\partial_-}\Big(\partial_-\,A^b\,\bar{A}^c\Big)\,\frac{1}{\partial_-}(\partial_-\,\bar{A}^d\,A^e)\;. \qquad (6.2)
\end{aligned}
$$

We now define the following operator

$$
\bar{\mathcal{D}}\,A^a \equiv \bar{\partial}A^a - g\,f^{abc}\,\frac{1}{\partial_-}\Big(\bar{A}^b\partial_-\,A^c\Big)\;, \qquad (6.3)
$$

using which we can write the Hamiltonian (6.2) as

$$
\mathcal{H} = -\int d^3x \mathcal{D}\bar{A}^a\,\bar{\mathcal{D}}A^a\;. \qquad (6.4)
$$

Thus providing a quadratic form for pure Yang-Mills theory in four dimensions.

A note on gauge invariance

It is know that even after making gauge choices in non-covariant gauges a residual gauge invariance is left. We can then ask if the form (6.4) is due to presence of some such residual gauge symmetry.

As discussed earlier in chapter 3, Yang-Mills theory remains invariant under the transformation $A^{\mu\,a} \to A^{\mu\,a} + \mathcal{D}^{\mu}\Lambda^a$. Therefore in light-cone gauge, our choice $A^{+\,a} = 0$ implies that there ought to be a remaining gauge invariance with gauge parameter satisfying $\partial^+\Lambda^a = 0$. So there is still an infinitesimal symmetry that satisfies $\partial^+\Lambda^a = 0$ as well as $\bar{\partial}\partial\Lambda^a = 0$ (necessary for invariance of the Yang-Mills Lagrangian). These facts allow us to determine that a derivative structure, if introduced as in (6.3), must be of the form

$$\bar{\mathcal{D}}\,A^a \equiv \bar{\partial}A^a - g\,f^{abc}\,\frac{1}{\partial_-^n}(\bar{A}^b\partial_-^n\,A^c), \tag{6.5}$$

We can determine that $n = 1$ in the above Ansatz by requiring Poincaré invariance. In fact the operator $\bar{\mathcal{D}}$ may be regarded as a covariant derivative and we see that the expression for the Hamiltonian is invariant under the remaining gauge invariance. There can various forms of second term in operator $\bar{\mathcal{D}}A^a$ which may appear different from (6.3), however we can prove that all such terms converge to the second terms of (6.3). Therefore we can say that $\bar{\mathcal{D}}A^a$ defined in (6.3) is unique.

Non-helicity basis

All the work in this thesis has been done in a helicity basis. However, we could have equally chosen to work in a non-helicity basis. For example, we could start with the Lagrangian from (3.28)).

$$\begin{aligned}\mathcal{L} \;&=\; \frac{1}{2}A_i^a\Box A_i^a - 2\,g\,f^{abc}\,\frac{1}{\partial_-}A_i^a\,\partial_i\,A_j^b\,\partial_-\,A_j^c - \frac{1}{4}\,g^2\,f^{abc}\,f^{ade}\,A_i^b\,A_j^c\,A_i^d\,A_j^e \\ &\quad - \frac{1}{2}\,g^2\,f^{abc}\,f^{ade}\,\frac{1}{\partial_-}\Big(A_i^b\,\partial_-\,A_i^c\Big)\,\frac{1}{\partial_-}\Big(A_j^d\,\partial_-\,A_j^e\Big)\end{aligned} \tag{6.6}$$

If we were to introduce an suitable covariant derivative, we could start with

$$\bar{\mathcal{D}}\,A^a = \frac{1}{2}(\partial_i A_i{}^a - g f^{abc}\frac{1}{\partial_-}\,(A_i{}^b\partial_- A_i{}^c)) + \frac{i}{4}\epsilon^{ij}F_{ij}{}^a \tag{6.7}$$

where

$$F_{ij}{}^{a} = \partial_i A_j{}^a - \partial_j A_i{}^a - g f^{abc} \frac{1}{\partial_-{}^n} (A_i{}^b \partial_-{}^n A_j{}^c) + g f^{abc} \frac{1}{\partial_-{}^n} (A_j{}^b \partial_-{}^n A_i{}^c) \tag{6.8}$$

However, it is easy to verify that the Hamiltonian cannot be expressed as the square of such a covariant derivative (as we did when in a helicity basis). This is because we encounter additional terms that do not cancel when the Hamiltonian is written in terms of F^a_{ij}. Hence it is only in a helicity base that the Hamiltonian is a quadratic form.

Ten spacetime dimensions

Although seemingly outside the purview of this thesis, we will discuss the maximally supersymmetric $\mathcal{N}=4$ Yang-Mills theory in the next section. This supersymmetric theory may be obtained from its ten-dimensional parent theory, $\mathcal{N}=1$ Yang-Mills. Thus we briefly discuss quadratic forms in higher dimensions as well, in the present chapter.

In ten dimensions, there is no helicity basis. The structure of the Lagrangian (and the Hamiltonian) of pure Yang-Mills theory in $d=10$ is the same as that of the $d=4$ theory (6.6) with $i,j=1...8$. The arguments presented above are therefore valid in $d=10$ as well and show that the pure Yang-Mills Hamiltonian in $d=10$ cannot be expressed as a quadratic form.

($\mathcal{N}=4, d=4$) Maximally supersymmetric Yang-Mills theory

The $\mathcal{N}=1$ super Yang-Mills theory was constructed in ten dimensions and its four dimensional $\mathcal{N}=4$ theory was obtained by dimensional reduction in light-cone superspace [44]. In reference [45] the authors obtained the action for the $\mathcal{N}=4$ theory in terms of a highly constrained scalar superfield which captures all the degrees of freedom. The action was obtained by requiring SuperPoincaré invariance in four dimensions. A brief review of results in [44,45] is presented below since it is a prerequisite for the material in this chapter.

We start with describing $\mathcal{N} = 4$ Yang-Mills light-cone superspace action followed by a derivation of its Hamiltonian as a quadratic form.

The constituents of the $\mathcal{N} = 4$ theory are: one complex bosonic field (the gauge field), four complex Grassmann fields and six scalar fields. The form of the theory described here can be obtained in two ways. The first one involves making the usual light-cone gauge choice as $A^+ = 0$ and then eliminating the unphysical field A^- by solving the constraint equation. The other way is to start with the superfield (described below) and verify that the superPoincaré algebra closes on it. The non-linear contribution to dynamical generators will introduce interaction terms in the Hamiltonian (as it is one of the generators itself: P_-. Thus closing the superPoincaré algebra will allow us to determine the full Hamiltonian. To write all physical fields of $\mathcal{N} = 4$ in a compact form we introduce anticommuting Grassmann variables θ^m and $\bar{\theta}_m$,

$$\{\theta^m, \theta^n\} = \{\theta^m, \bar{\theta}_n\} = \{\bar{\theta}_m, \bar{\theta}_n\} = 0 \tag{6.9}$$

where $m, n, \ldots = 1, 2, 3, 4$, denote $SU(4)$ spinor indices. With their derivatives defined as

$$\bar{\partial}_m \equiv \frac{\partial}{\partial\,\theta^m}; \qquad \partial^m \equiv \frac{\partial}{\partial\,\bar{\theta}_m}, \tag{6.10}$$

These satisfy

$$\{\partial^m, \bar{\theta}_n\} = \delta^m{}_n; \qquad \{\bar{\partial}_m, \theta^n\} = \delta_m{}^n. \tag{6.11}$$

All the physical degrees of freedom of $\mathcal{N} = 4$ super Yang-Mills theory can be captured in a single complex superfield as described in [44].

$$\phi\,(y) \;\; = \;\; -\frac{1}{\partial_-}\,A\,(y) + \frac{i}{\sqrt{2}}\,\theta^m\,\theta^n\,\bar{C}_{mn}\,(y) - \frac{1}{12}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\epsilon_{mnpq}\,\partial_-\,\bar{A}\,(y)$$

$$- \frac{i}{\partial_-}\,\theta^m\,\bar\chi_m(y) + \frac{\sqrt{2}}{6}\theta^m\,\theta^n\,\theta^p\,\epsilon_{mnpq}\,\chi^q(y)\ . \qquad (6.12)$$

In the above superfield, all fields carry adjoint indices (not shown here) and are local in the following modified light-cone coordinates

$$y \;=\; (\,x, \bar x, x^+, y^- \equiv x^- - \frac{i}{\sqrt{2}}\theta^m\,\bar\theta_m\,)\ . \qquad (6.13)$$

The fields A and $\bar A$ are the two gauge fields defined in (3.29). The six scalar fields are written as antisymmetric $SU(4)$ bi-spinors

$$C^{m4} \;=\; \frac{1}{\sqrt{2}}(A_{m+3} + i\,A_{m+6}); \qquad \bar C^{m4} \;=\; \frac{1}{\sqrt{2}}(A_{m+3} - i\,A_{m+6}), \qquad (6.14)$$

for $m \neq 4$; and satisfy

$$\bar C_{mn} \;=\; \frac{1}{2}\,\epsilon_{mnpq}\,C^{pq}\ . \qquad (6.15)$$

The fermion fields are χ^m and $\bar\chi_m$.
We introduce the chiral derivatives

$$d^m = -\partial^m + \frac{i}{\sqrt{2}}\,\theta^m\,\partial_-\ ; \qquad \bar d_n = \;\bar\partial_n - \frac{i}{\sqrt{2}}\,\bar\theta_n\,\partial_-\ , \qquad (6.16)$$

which satisfy

$$\{\,d^m\,,\,\bar d_n\,\} \;=\; -i\,\sqrt{2}\,\delta^m{}_n\,\partial_+\ . \qquad (6.17)$$

The superfield ϕ and its complex conjugate $\bar{\phi}$ satisfy chiral constraints

$$d^m\,\phi = 0\ ; \qquad \bar{d}_m\,\bar{\phi} = 0\ , \tag{6.18}$$

and the "inside-out" constraints

$$\begin{aligned} \bar{d}_m\,\bar{d}_n\,\phi &= \frac{1}{2}\,\epsilon_{mnpq}\,d^p\,d^q\,\bar{\phi}\ , \\ d^m\,d^n\,\bar{\phi} &= \frac{1}{2}\,\epsilon^{mnpq}\,\bar{d}_p\,\bar{d}_q\,\phi\ . \end{aligned} \tag{6.19}$$

The $(\mathcal{N}=4, d=4)$ action in light-cone superspace is

$$\int d^4x \int d^4\theta\, d^4\bar{\theta}\,\mathcal{L}\ , \tag{6.20}$$

where Lagrangian in terms of superfield is written as

$$\begin{aligned} \mathcal{L} &= -\bar{\phi}\,\frac{\Box}{\partial_-^2}\,\phi\ - \frac{4g}{3}\,f^{abc}\left(\frac{1}{\partial_-}\,\bar{\phi}^a\,\phi^b\,\bar{\partial}\,\phi^c + \text{complex conjugate}\right) \\ &\quad -g^2\,f^{abc}\,f^{ade}\left(\frac{1}{\partial_-}(\phi^b\,\partial^+\,\phi^c)\,\frac{1}{\partial_-}\,(\bar{\phi}^d\,\partial_-\,\bar{\phi}^e) + \frac{1}{2}\,\phi^b\bar{\phi}^c\,\phi^d\,\bar{\phi}^e\right)\ . \end{aligned} \tag{6.21}$$

Grassmann integrations are normalized: $\int d^4\theta\,\theta^1\theta^2\theta^3\theta^4 = 1.$ and f^{abc} are structure functions of the Lie algebra.

The kinematical supersymmetries are

$$q_+^m = -\partial^m - \frac{i}{\sqrt{2}}\,\theta^m\,\partial_-\ ; \qquad \bar{q}_{+n} = \ \bar{\partial}_n + \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial_-\ , \tag{6.22}$$

and they satisfy

$$\{\, q_+^m \,,\, \bar{q}_{+n} \,\} = i\,\sqrt{2}\,\delta^m{}_n\,\partial_+ \ , \tag{6.23}$$

The dynamical boosts are

$$j^- \;=\; -i\,x\,\frac{\partial\bar{\partial}}{\partial_-} - i\,x^-\,\partial - i\Big(\theta^p\,\bar{\partial}_p - \frac{i}{4\sqrt{2}\,\partial_-}(d^p\,\bar{d}_P - \bar{d}_p\,d^p)\Big)\,\frac{\partial}{\partial_-} \tag{6.24}$$

$$\bar{j}^- \;=\; -i\,\bar{x}\,\frac{\partial\bar{\partial}}{\partial_-} - i\,x^-\,\bar{\partial} - i\Big(\bar{\theta}_p\,\partial^p - \frac{i}{4\sqrt{2}\,\partial_-}(d^p\,\bar{d}_P - \bar{d}_p\,d^p)\Big)\,\frac{\bar{\partial}}{\partial_-} \tag{6.25}$$

Dynamical supersymmetries are obtained from the kinematical supersymmetries

$$q_-^m \;\equiv\; i\,[\,\bar{j}^- \,,\, q_+^m\,] \;=\; -\,\frac{\partial}{\partial_-}\,q_+^m \ , \qquad \bar{q}_{-m} \;\equiv\; i\,[\,j^- \,,\, \bar{q}_{+m}\,] \;=\; -\,\frac{\bar{\partial}}{\partial_-}\,\bar{q}_{+m} \ , \tag{6.26}$$

and satisfy the free $N=4$ supersymmetry algebra

$$\{\, q_-^m \,,\, \bar{q}_{-n} \,\} \;=\; -\,i\,\sqrt{2}\,\delta^m{}_n\,\frac{\partial\bar{\partial}}{\partial_-}. \tag{6.27}$$

In the free (linear) theory the generators act straight on the superfield.

$$\delta_O\phi = O\phi. \tag{6.28}$$

The dynamical generators transform the system forward in time. The action of these generators on fields will produce non-linear terms since their canonical expressions are modified by interaction terms. Therefore, we have to find the non-linear terms such that the algebra closes. For the dynamical supersymmetry the result is

$$\delta_{\bar{q}_{-m}}\,\phi^a \;=\; -\,\frac{1}{\partial_-}\,\Big\{(\bar{\partial}\,\delta^{ab} + g f^{abc}\partial_-\,\phi^c\,)\,\delta_{\bar{q}_{+m}}\,\phi^b\Big\}, \tag{6.29}$$

Consider now the Hamiltonian that we get from (6.20)

$$H = \int d^3x\, d^4\theta\, d^4\bar\theta \left\{ \bar\phi^a \frac{2\partial\bar\partial}{\partial_-^2} \phi^a \;+ \frac{4}{3} g\, f^{abc} \left(\frac{1}{\partial_-} \bar\phi^a \phi^b \bar\partial \phi^c + \frac{1}{\partial_-} \phi^a \bar\phi^b \partial \bar\phi^c \right) \right.$$

$$\left. + g^2 f^{abc} f^{ade} \left(\frac{1}{\partial_-} (\phi^b \partial_- \phi^c) \frac{1}{\partial_-} (\bar\phi^d \partial_- \bar\phi^e) + \frac{1}{2} \phi^b \bar\phi^c \phi^d \bar\phi^e \right) \right\} , \tag{6.30}$$

With the help of the form (6.29) it can be written as [24]

$$H = -\frac{i}{\sqrt{2}} \int d^3x\, d^4\theta\, d^4\bar\theta\, \delta_{q^m} \bar\phi^a \frac{1}{\partial_-} \delta_{\bar q_m} \phi^a. \tag{6.31}$$

This result was obtained by making use of the "inside-out constraint" (6.19). Note that, this point is important since it implies that other supersymmetric Yang-Mills theories cannot be written as such simple quadratic forms since for those theories there is no equivalent constraint.

($\mathcal{N} = 1, d = 10$) Maximally supersymmetric Yang-Mills theory

This theory is the “parent” theory of ($\mathcal{N} = 4, d = 4$) Yang-Mills theory. In [43] it was shown that the action for the $\mathcal{N} = 1$ theory in $d = 10$ could be obtained by simply ‘oxidizing’ the action for the $\mathcal{N} = 4$ theory (6.20).
This can be done in three steps. Initially, two of the transverse light-cone directions of the four dimensional theory are generalized to eight. This is achieved by introducing six extra coordinates and their derivatives without changing the superfield (6.12) of the ($\mathcal{N} = 4$, $d = 4$) theory . These extra coordinates are represented as antisymmetric bi-spinors

$$x^{m4} = \frac{1}{\sqrt{2}} (x_{m+3} + i\, x_{m+6}) , \qquad \partial^{m4} = \frac{1}{\sqrt{2}} (\partial_{m+3} + i\, \partial_{m+6}) , \tag{6.32}$$

for $m \neq 4$, and their complex conjugates defined as

$$\bar{x}_{pq} = \frac{1}{2}\,\epsilon_{pqmn}\,x^{mn}\ ; \qquad \bar{\partial}_{pq} = \frac{1}{2}\,\epsilon_{pqmn}\,\partial^{mn}, \tag{6.33}$$

Their derivatives satisfy

$$\bar{\partial}_{mn}\,x^{pq} = (\delta_m^{\ p}\,\delta_n^{\ q} - \delta_m^{\ q}\,\delta_n^{\ p}); \qquad \partial^{mn}\,\bar{x}_{pq} = (\delta^m_{\ p}\,\delta^n_{\ q} - \delta^m_{\ q}\,\delta^n_{\ p}), \tag{6.34}$$

and

$$\partial^{mn}\,x^{pq} = \frac{1}{2}\,\epsilon^{pqrs}\,\partial^{mn}\,\bar{x}_{rs} = \epsilon^{mnpq}. \tag{6.35}$$

Second, all fields are made dependent on the extra coordinates

$$\phi(y) = \phi(x,\bar{x},x^{mn},\bar{x}_{mn},x^+,y^-) \tag{6.36}$$

The four supersymmetries in four dimensions become one supersymmetry in ten dimensions. The kinematical supersymmetries q_+^n and $\bar{q}_{+n}$, are fabricated into one $SO(8)$ spinor. The dynamical supersymmetries are obtained by boosting the kinematical ones.

$$i\,[\,\bar{J}^-\,,\,q_+^m\,]\ \equiv\ \mathcal{Q}^m\ ; \qquad i\,[\,J^-\,,\,\bar{q}_{+m}\,]\ \equiv\ \bar{\mathcal{Q}}_m\ , \tag{6.37}$$

where the linear part of the dynamical boosts are

$$\begin{aligned} J^- \ &=\ -i\,x\,\frac{\partial\bar{\partial} + \frac{1}{4}\,\bar{\partial}_{pq}\,\partial^{pq}}{\partial_-}\ -\ i\,x^-\,\partial - i\,\frac{\partial}{\partial_-}\left\{\theta^m\,\bar{\partial}_m\ -\ \frac{i}{4\sqrt{2}\,\partial_-}\,(d^p\,\bar{d}_p - \bar{d}_p\,d^p)\right\} - \\ &+\ \frac{1}{2}\,\frac{\bar{\partial}_{pq}}{\partial_-}\left\{-\frac{\partial_-}{\sqrt{2}}\,\theta^p\,\theta^q + \frac{\sqrt{2}}{\partial_-}\,\partial^p\,\partial^q - \frac{1}{\sqrt{2}\partial_-}\,d^p\,d^q\right\}\ , \end{aligned} \tag{6.38}$$

and its conjugate [43]. These yield the linear parts of the dynamical supersymmetry generators.

$$\mathcal{Q}^m \;=\; -\frac{\bar\partial}{\partial_-}\, q_+{}^m - \frac{\partial^{mn}}{\partial_-}\, \bar q_{+\,n} \ ,$$

$$\bar{\mathcal{Q}}_m \;=\; -\frac{\partial}{\partial_-}\, \bar q_{+\,m} - \frac{\bar\partial_{mn}}{\partial_-}\, q_+^{\;n} \ , \tag{6.39}$$

which satisfy

$$\{\, \mathcal{Q}^m \,,\, \bar{\mathcal{Q}}_n \,\} \;=\; -\,i\,\sqrt{2}\,\frac{1}{\partial_-}\left(\delta^m_{\;n}\,\partial\,\bar\partial + \bar\partial_{mp}\,\partial^{np}\right) \ , \tag{6.40}$$

Third, the introduction of a 'generalized' derivative

$$\bar\nabla \;\equiv\; \bar\partial - \frac{i\,\alpha}{4\,\sqrt{2}\,\partial_-}\,\bar d_p\,\bar d_q\,\partial^{pq} \ , \tag{6.41}$$

and its conjugate.
The action constructed using the above operators in ten dimensions is invariant under $SO(8)$ group [43].
The take home message of [43] was that the covariance in ten dimensions can be achieved by simply replacing the transverse derivatives of four dimensions ∂ and $\bar\partial$ by ∇ and $\bar\nabla$, respectively in ten dimensions. Therefore, the action of $(\mathcal{N}=1, d=10)$ super Yang-Mills can be written as

$$\int d^{10}x \int d^4\theta\, d^4\bar\theta\, \mathcal{L}_{10} \ , \tag{6.42}$$

with Lagrangian

$$\mathcal{L}_{10} \;=\; -\bar\phi\,\frac{\Box_{10}}{\partial_-^2}\,\phi \;-\; \frac{4g}{3}\, f^{abc}\left(\frac{1}{\partial_-}\,\bar\phi^a\,\phi^b\,\bar\nabla\,\phi^c + \text{complex conjugate}\right)$$

$$-g^2 f^{abc}\, f^{ade}\Big(\frac{1}{\partial_-}(\phi^b\,\partial_-\phi^c)\frac{1}{\partial_-}\,(\bar\phi^d\,\partial_-\,\bar\phi^e)+\frac{1}{2}\,\phi^b\bar\phi^c\,\phi^d\,\bar\phi^e\Big)\ . \tag{6.43}$$

where $\Box_{10}$ is d'Alembertian operator in ten dimensions.

Quadratic forms

We have seen that the $(\mathcal{N}=4, d=4)$ Hamiltonian is a quadratic form (6.31). In this section we will show that the Hamiltonian for the ten-dimensional theory described above is also a quadratic form.

The kinetic term

We start from the free dynamical supersymmetry generators in (6.39) and after adding the non-linear term in (6.39) the resulting full non-linear dynamical supersymmetry generators are

$$\begin{aligned}\delta_{q_-{}^m}\bar\phi^a &= Q^m\,\bar\phi^a - g f^{abc}\frac{1}{\partial_-}(q_+^m\,\bar\phi^b\partial_-\bar\phi^c)\ ,\\ \delta_{\bar q_{-m}}\phi^a &= \bar Q_m\,\phi^a - g f^{abc}\frac{1}{\partial_-}(\bar q_{+m}\,\phi^b\partial_-\phi^c)\ ,\end{aligned} \tag{6.44}$$

Note that this superfield depends on all ten coordinates.
We claim that the ten-dimensional Hamiltonian is again simply

$$H\,=\,-\frac{i}{\sqrt{2}}\int d^9x\,d^4\theta\,d^4\bar\theta\,\delta_{q^m}\bar\phi^a\,\frac{1}{\partial_-}\,\delta_{\bar q_m}\phi^a. \tag{6.45}$$

Now we verify this claim, at the free level

$$\begin{aligned}
\delta_{q^m}\bar{\phi}^a\,\frac{1}{\partial_-}\,\delta_{\bar{q}_m}\phi^a. \;&=\; \{(\frac{\bar{\partial}}{\partial_-}q_+^m+\frac{\partial^{mn}}{\partial_-}\,\bar{q}_{+n})\,\bar{\phi}^a\,\frac{1}{\partial_-}(\frac{\partial}{\partial_-}\bar{q}_{+m}+\frac{\bar{\partial}_{mp}}{\partial_-}q_+^p)\,\phi^a\} \\
&=\; -\{\frac{\bar{\partial}}{\partial_-}q_+^m\,\bar{\phi}^a\,\frac{\partial}{\partial_-^2}\bar{q}_{+m}\,\phi^a+\frac{\partial^{mn}}{\partial_-}\bar{q}_{+n}\bar{\phi}^a\frac{\partial}{\partial_-^2}\bar{q}_{+m}\phi^a \\
&\quad +\frac{\bar{\partial}}{\partial_-}q_+^m\,\bar{\phi}^a\,\frac{\bar{\partial}_{mp}}{\partial_-^2}q_+^p\,\phi^a+\frac{\partial^{mn}}{\partial_-}\bar{q}_{+n}\,\bar{\phi}^a\,\frac{\bar{\partial}_{mp}}{\partial_-^2}q_+^p\,\phi^a\} \\
&=\; \{\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}\} \qquad (6.46)
\end{aligned}$$

At first we focus on term $\mathcal{B}$ and after some integration by parts and use of the 'inside-out' constraint in (6.19) yields

$$\mathcal{B}=\frac{1}{2}\bar{\phi}^a\,\frac{\partial^{mn}\partial}{\partial_-^3}\,\{\bar{q}_{+n},\bar{q}_{+m}\}\,\phi^a=0 \qquad (6.47)$$

The term $\mathcal{C}$ vanishes in the similar way. The terms $\mathcal{A}$ and $\mathcal{D}$ are non-zero. After similar simplification the term $\mathcal{A}$ becomes

$$\mathcal{A}=\frac{1}{2}\;\bar{\phi}^a\;\frac{\partial\bar{\partial}}{\partial_-^3}\;\{q_+^m,\bar{q}_{+m}\}\;\phi^a\;=\;-\,i2\,\sqrt{2}\,\bar{\phi}^a\;\frac{\partial\bar{\partial}}{\partial_-^2}\;\phi^a\;, \qquad (6.48)$$

while $\mathcal{D}$ reads

$$\mathcal{D}\;=\;-\,\frac{i}{\sqrt{2}}\;\bar{\phi}^a\;\frac{\partial^{mn}\bar{\partial}_{mn}}{\partial_-^2}\;\phi^a \qquad (6.49)$$

Therefore, the free ten-dimensional Hamiltonian reads

$$H\;=\;2\;\bar{\phi}^a\;\frac{\partial\bar{\partial}}{\partial_-^2}\;\phi^a+\frac{1}{2}\;\bar{\phi}^a\;\frac{\partial^{mn}\bar{\partial}_{mn}}{\partial_-^2}\;\phi^a\;, \qquad (6.50)$$

which is consistent with (6.42).

The cubic interaction vertex

After proving above that the free Hamiltonian is a quadratic form, we will now examine the cubic interaction vertex. We take a relevant piece from (6.45)

$$\delta_{q^m}\bar{\phi}^a\,\frac{1}{\partial_-}\,\delta_{\bar{q}_m}\phi^a|_g = -\,f^{abc}\frac{1}{\partial_-}(q_+^m\bar{\phi}^b\partial_-\bar{\phi}^c)\,\frac{1}{\partial_-}\bar{Q}_m\,\phi^a\ . \tag{6.51}$$

Now we only need to focus on the term involving the new transverse derivatives. This can be written as

$$\begin{aligned} & f^{abc}\int\frac{\bar{\partial}_{mn}}{\partial_-^2}q_+^n\phi^a\frac{1}{\partial_-}(q_+^m\bar{\phi}^b\partial_-\bar{\phi}^c) \\ & =\frac{1}{3}f^{abc}\int\left(\frac{1}{\partial_-}\phi^a\bar{\phi}^b\frac{d^m d^n\bar{\partial}_{mn}}{\partial_-}\bar{\phi}^c-\frac{1}{2}\phi^a\bar{\phi}^b\frac{d^m d^n\bar{\partial}_{mn}}{\partial_-^2}\bar{\phi}^c\right) \end{aligned} \tag{6.52}$$

We are going to briefly review the derivation of the above result. We use explicit form of q_+ to write the L.H.S of (6.52) as

$$+i\sqrt{2}f^{abc}\int\frac{\bar{\partial}_{mn}}{\partial_-^2}\phi^a\theta^m\partial^n\bar{\phi}^b\partial_-\bar{\phi}^c \tag{6.53}$$

The partial integration with respect to ∂^n (f^{abc} and the integral sign are suppressed) gives

$$-i\sqrt{2}\,\theta^m\partial^n\frac{\bar{\partial}_{mn}}{\partial_-^2}\phi^a\bar{\phi}^b\partial_-\bar{\phi}^c-i\sqrt{2}\,\frac{\bar{\partial}_{mn}}{\partial_-^2}\phi^a\bar{\phi}^b\partial_-\theta^m\partial^n\bar{\phi}^c\equiv I+II \tag{6.54}$$

By making use of 'inside out' constraint and partial integration of the $\bar{d}'$s, the first term of (6.54) gives

$$
\begin{aligned}
I &= -\frac{i\sqrt{2}}{2\cdot 4!}(\epsilon^{ijkl}\bar{d}_i\bar{d}_j\bar{d}_k\bar{d}_l)\theta^m\partial^n\frac{\bar{\partial}_{mn}}{\partial_-^2}\phi^a\bar{\phi}^b\frac{1}{\partial_-}\phi^c \\
&= -\frac{1}{2}\frac{d^m d^n\bar{\partial}_{mn}}{\partial_-}\bar{\phi}^a\bar{\phi}^b\frac{1}{\partial_-}\phi^c - i\sqrt{2}\,\theta^m\partial^n\bar{\partial}_{mn}\bar{\phi}^a\bar{\phi}^b\frac{1}{\partial_-}\phi^c
\end{aligned} \tag{6.55}
$$

The similar manipulation of the second term in (6.54) yields

$$
II = i\sqrt{2}\,\frac{\bar{\partial}_{mn}}{\partial_-}\phi^a\bar{\phi}^b\theta^m\partial^n\bar{\phi}^c + i\sqrt{2}\,\frac{\bar{\partial}_{mn}}{\partial_-^2}\phi^a\partial_-\bar{\phi}^b\theta^m\partial^n\bar{\phi}^c \tag{6.56}
$$

Integrating by parts with respect to $\bar{\partial}_{mn}$, first term of (6.56) gives

$$
-i\sqrt{2}\,\frac{1}{\partial_-}\phi^a\bar{\partial}^{mn}\bar{\phi}^b\theta^m\partial^n\bar{\phi}^c - i\sqrt{2}\,\frac{1}{\partial_-}\phi^a\bar{\phi}^b\bar{\partial}_{mn}\theta^m\partial^n\bar{\phi}^c \tag{6.57}
$$

We see that the second term of (6.57) cancels with second term of (6.55). The first term of (6.57) after using 'inside out' constraint on $\bar{\phi}^b$, becomes

$$
-i\sqrt{2}\,\partial_-\bar{\phi}^a\frac{\bar{\partial}_{mn}}{\partial_-^2}\phi^b\theta^m\partial^n\bar{\phi}^c + \frac{1}{2}\phi^a\bar{\phi}^c\frac{\bar{\partial}_{mn}}{\partial_-^2}d^m d^n\bar{\phi}^b - \frac{1}{2}\frac{1}{\partial_-}\phi^a\bar{\phi}^c\frac{\bar{\partial}_{mn}}{\partial_-}d^m d^n\bar{\phi}^b \tag{6.58}
$$

and finally

$$
\begin{aligned}
I + II &= -i2\sqrt{2}\,\frac{\bar{\partial}_{mn}}{\partial_-^2}\phi^a\theta^m\partial^n\bar{\phi}^b\partial_-\bar{\phi}^c \\
&\qquad + \frac{1}{\partial_-}\phi^a\bar{\phi}^b\frac{d^m d^n\bar{\partial}_{mn}}{\partial_-}\bar{\phi}^c - \frac{1}{2}\phi^a\bar{\phi}^b\frac{d^m d^n\bar{\partial}_{mn}}{\partial_-^2}\bar{\phi}^c
\end{aligned} \tag{6.59}
$$

As we can see that the R.H.S. above is equal to (6.53) (from which we started) and this lead to (6.52).

The conjugate of (6.52) can be obtained by following the same procedure,

$$f^{abc}\int \frac{\partial^{mn}}{\partial_-}\bar{q}_{+n}\bar{\phi}^a\frac{1}{\partial_-^2}(\bar{q}_{+m}\phi^b\partial_-\phi^c)$$

$$= \frac{1}{3}f^{abc}\int\left(\frac{1}{\partial_-}\bar{\phi}^a\phi^b\frac{\bar{d}_m\bar{d}_n\partial^{mn}}{\partial_-}\phi^c - \frac{1}{2}\bar{\phi}^a\phi^b\frac{\bar{d}_m\bar{d}_n\partial^{mn}}{\partial_-^2}\phi^c\right) \tag{6.60}$$

Now we use the inside-out constraint on ϕ^c in the second term of (6.60) and the anti-symmetry of the structure functions allows us to write

$$f^{abc}\bar{\phi}^a\phi^b\frac{\bar{d}_m\bar{d}_n\partial^{mn}}{\partial_-^2}\phi^c = -\,f^{abc}\phi^a\bar{\phi}^b\frac{d^m d^n\bar{\partial}_{mn}}{\partial_-^2}\bar{\phi}^c \tag{6.61}$$

Therefore, the sum of (6.52) and (6.60) becomes

$$f^{abc}\int\left\{\frac{\bar{\partial}_{mn}}{\partial_-^2}q_+^n\phi^a\frac{1}{\partial_-}(q_+^m\bar{\phi}^b\partial_-\bar{\phi}^c) + \frac{\partial^{mn}}{\partial_-}\bar{q}_{+n}\bar{\phi}^a\frac{1}{\partial_-^2}(\bar{q}_{+m}\phi^b\partial_-\phi^c)\right\}$$

$$= \frac{1}{3}f^{abc}\int\left(\frac{1}{\partial_-}\phi^a\bar{\phi}^b\frac{d^m d^n\bar{\partial}_{mn}}{\partial_-}\bar{\phi}^c + \frac{1}{\partial_-}\bar{\phi}^a\phi^b\frac{\bar{d}_m\bar{d}_n\partial^{mn}}{\partial_-}\phi^c\right). \tag{6.62}$$

which exactly matches with expected Hamiltonian in (6.42). Note that we used inside-out constraint repeatedly in the above computation which suggest that maximal supersymmetry is essential for many of the simplifications presented above. Therefore the quadratic forms can be observed only in maximally supersymmetric theories, as mentioned earlier.

As superfield ϕ^a involves both A^a and $\bar{A}^a$, one might ask about the the covariance of (6.29) under the remaining gauge invariance (discussed in subsection 6.1.2). It can be answered, as the superfield $\delta_{\bar{q}_{+m}}\phi^a$ involves only $\bar{A}^a$ and (6.29) can be regarded as the covariant derivative of the superfield.

The quartic interaction vertex

The quartic interaction vertex does not contain any transverse derivatives, therefore we do not need to check it. This implies that the results in [24] for the quartic vertex holds true to our case with just two standard modifications used in this section : the fields now depend on all ten directions and the spacetime integration is over all ten directions.

Gravity

Having explained, in great detail, the concept of quadratic forms in the Yang-Mills system we now turn our attention to gravity. However, in this case, we start with the maximally supersymmetric theory of gravity. This is because the case of pure gravity remains work in progress and will be discussed after the supersymmetric case.

$\mathcal{N} = 8$ maximal supergravity

We review here the light-cone superspace formulation of $(\mathcal{N} = 8, d = 4)$ supergravity and the quadratic form structure of its Hamiltonian. It was shown in [44] that all physical degrees of freedom of $\mathcal{N} = 8$ maximally supersymmetric gravity can be captured in a scalar superfield.

The superfield containing all physical degrees of freedom and in terms of Grassmann variables $\theta^m (m = 1, ..., 8)$ is defined by

$$
\begin{aligned}
\phi(y) \;&=\; \frac{1}{\partial_-^2}\, h(y) + i\,\theta^m \frac{1}{\partial_-^2}\, \bar{\psi}_m\,(y) - \frac{i}{2}\,\theta^m\,\theta^n\,\frac{1}{\partial_-}\,\bar{A}_{mn}(y), \\
&+ \frac{1}{3!}\theta^m\,\theta^n\,\theta^p \frac{1}{\partial_-}\bar{\chi}_{mnp}(y) - \frac{1}{4!}\theta^m\,\theta^n\,\theta^p\,\theta^q\,\bar{C}_{mnpq}(y) \\
&+ \frac{i}{5!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\epsilon_{mnpqrstu}\,\chi^{stu}(y), \\
&- \frac{i}{6!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\epsilon_{mnpqrstu}\,\partial_- A^{tu}(y), \\
&- \frac{1}{7!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\theta^t\,\epsilon_{mnpqrstu}\,\partial_-\psi^u(y), \\
&+ \frac{4}{8!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\theta^t\,\theta^u\,\epsilon_{mnpqrstu}\,\partial_-^2\,\bar{h}(y)
\end{aligned}
\tag{6.63}
$$

The scalar superfield contains

- $h, \bar{h} \quad \rightarrow \quad$ Two-component of graviton
- $\bar{\psi}^m \quad \rightarrow \quad$ spin$-\frac{3}{2}$ gravitinos
- $\bar{A}_{mn} \quad \rightarrow \quad$ 28 gauge fields
- $\bar{\chi}_{mnp} \quad \rightarrow \quad$ gauginos corresponding to gauge field
- $\bar{C}_{mnpq} \quad \rightarrow \quad$ 70 scalar fields

The superfield ϕ and its complex conjugate $\bar{\phi}$ satisfy chiral constraints,

$$
d^m\,\phi(y) = 0; \qquad \bar{d}_p\,\bar{\phi}(y) = 0 \tag{6.64}
$$

where chiral derivatives are

$$d^m = -\frac{\partial}{\partial\bar{\theta}_m} + \frac{i}{\sqrt{2}}\theta^m\partial_- \, ; \qquad \bar{d}_p = \frac{\partial}{\partial\theta^p} - \frac{i}{\sqrt{2}}\bar{\theta}_p\partial_- \tag{6.65}$$

Again, the superfield and its conjugate are related ("inside-out" constraint)

$$\phi = \frac{1}{4}\frac{(d)^8}{{\partial_-}^4}\bar{\phi} \tag{6.66}$$

Action to $\mathcal{O}(\kappa)$

The $\mathcal{N}=8$ supergravity action to order κ in terms of superfields reads [45]

$$-\frac{1}{64}\int d^4x \int d^8\theta \, d^8\bar{\theta}\, \mathcal{L} \,, \tag{6.67}$$

where Lagrangian is,

$$\mathcal{L} \;=\; -\bar{\phi}\frac{\Box}{\partial_-^4}\phi - 2\kappa\,(\frac{1}{\partial_-^2}\,\bar{\phi}\;\bar{\partial}\phi\,\bar{\partial}\phi + \frac{1}{\partial_-^2}\,\phi\partial\bar{\phi}\partial\bar{\phi})\;. \tag{6.68}$$

$\kappa = \sqrt{8\pi G}$ and Grassmann integration is normalized such that $\int d^8\theta\,(\theta)^8 = 1$. At this order, the dynamical supersymmetry generator is

$$\bar{Q}_m{}^{(\kappa)}\phi = \frac{1}{\partial_-}(\bar{\partial}\bar{q}_m\phi\partial_-^2\phi - \partial_-\bar{q}_m\phi\partial_-\bar{\partial}\phi). \tag{6.69}$$

Note that we have suppressed the + index on q_+ to make things easier to read. The complex conjugate yields $Q^{m\,(\kappa)}\bar{\phi}$. The "inside-out" constraints determine $Q^{m\,(\kappa)}\phi$ and $\bar{Q}_m{}^{(\kappa)}\bar{\phi}$. The anticommutator of the dynamical supersymmetry generators, yields the light-cone Hamiltonian.

Hamiltonian as a quadratic form

It was shown in [25] that the light-cone Hamiltonian is indeed a quadratic form (at order κ).

$$\mathcal{H} \;=\; \frac{1}{4\sqrt{2}}\,(\,\mathcal{W}_m\,,\,\mathcal{W}_m\,)\ , \tag{6.70}$$

with

$$\mathcal{W}_m = \bar{Q}_{-\,m}\,\phi\ , \tag{6.71}$$

where the inner product is defined as

$$(\,\phi\,,\,\xi\,)\;\equiv\;-\,2i\int d^4x\,d^8\theta\,d^8\,\bar\theta\;\bar\phi\,\frac{1}{\partial_-^3}\,\xi\ . \tag{6.72}$$

$d=4$ pure gravity

In this section we report briefly on work in progress. The pure gravity Hamiltonian corresponding to the Lagrangian (3.66) and (3.67) is

$$
\begin{aligned}
\mathcal{H} \;=\;& \partial\bar{h}\bar{\partial}h - 2\kappa\bar{\partial}h\left\{\frac{1}{\partial_-^2}\left(\bar{\partial}h\,\partial_-^2\bar{h} + h\,\bar{\partial}\partial_-^2\bar{h}\right) - \frac{1}{\partial_-}\left(\frac{\bar{\partial}}{\partial_-}h\,\partial_-^2\bar{h}\right)\right\} \\
& -2\kappa\,\partial\bar{h}\left\{\frac{1}{\partial_-^2}\left(\partial\bar{h}\,\partial_-^2 h + \bar{h}\,\partial\partial_-^2 h\right) - \frac{1}{\partial_-}\left(\frac{\partial}{\partial_-}\bar{h}\,\partial_-^2 h\right)\right\} \\
& - 4\kappa^2\left\{ -2\,\frac{1}{\partial_-^2}\left(\frac{\bar{\partial}}{\partial_-}h\,\partial_-^3\bar{h} - h\,\partial_-^2\bar{\partial}\bar{h}\right)\frac{1}{\partial_-^2}\left(\frac{\partial}{\partial_-}\bar{h}\,\partial_-^3 h - \bar{h}\,\partial_-^2\partial h\right)\right. \\
& + \frac{1}{\partial_-^2}(\bar{\partial}h\,\partial_-^2\bar{h} - \partial_- h\,\partial_-\bar{\partial}\bar{h})\,\frac{1}{\partial_-^2}(\partial\bar{h}\,\partial_-^2 h - \partial_-\bar{h}\,\partial_-\partial h) - 3\,\frac{1}{\partial_-}(\bar{\partial}h\,\partial_-\bar{h})\,\frac{1}{\partial_-}(\partial_- h\,\partial\bar{h}) \\
& + \frac{1}{\partial_-}(\bar{\partial}h\,\partial_-\bar{h} - \partial_- h\,\bar{\partial}\bar{h})\,\frac{1}{\partial_-}(\partial\bar{h}\,\partial_- h - \partial_-\bar{h}\,\partial h) + 3\,\frac{1}{\partial_-}(\partial_- h\,\partial_-\bar{h})\,\frac{1}{\partial_-}(\bar{\partial}h\,\partial\bar{h}) \\
& \left. + \left[\frac{1}{\partial_-^2}(\partial_- h\,\partial_-\bar{h}) - h\,\bar{h}\right](\bar{\partial}h\,\partial\bar{h} + \partial h\,\bar{\partial}\bar{h} - \partial_- h\,\frac{\partial\bar{\partial}}{\partial_-}\bar{h} - \partial_- h\,\frac{\partial\bar{\partial}}{\partial_-}\bar{h})\right\}, \qquad (6.73)
\end{aligned}
$$

Residual reparametrization invariance

Now we examine the effect of residual reparametrizations. To lowest order in κ, these take the form

$$x \to x \;+\; \xi(\bar{x})\ , \qquad \bar{x} \to \bar{x} \;+\; \bar{\xi}(x)\ , \qquad (6.74)$$

We know that in the covariant formulation under general coordinate transformation field h transforms as in (3.46). We find that in light-cone gauge h transforms as

$$\delta h \;=\; \frac{1}{2\kappa}\partial\xi \;+\; \xi\bar{\partial}h \;+\; \bar{\xi}\partial h \qquad (6.75)$$

where ξ satisfies

$$\partial_-\xi \;=\; 0\ , \qquad \bar{\partial}\,\xi \;=\; 0\ . \qquad (6.76)$$

Therefore to order κ^{-1} we have

$$\partial_-(\delta h) \;=\; 0 \;, \qquad \bar{\partial}(\delta h) \;=\; 0 \tag{6.77}$$

The variation of the Hamiltonian to order κ^0 gives

$$\delta\mathcal{H}^{(\kappa^0)} \;=\; \delta\;(\partial\bar{h}\,\bar{\partial}h) + \;2\kappa\;\delta^{\kappa^{-1}}\;\left\{\;\bar{h}\;\partial_-{}^2\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h\;-\;\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right)\;+\;\text{c.c.}\right\}\;, \tag{6.78}$$

The first term in (6.78) yields

$$-\;\partial\xi\;\bar{h}\;\bar{\partial}^2 h\;-\;\bar{\partial}\bar{\xi}\;h\;\partial^2\bar{h}\;. \tag{6.79}$$

We find that the variation of the second term in (6.78) with contribution from its complex conjugate exactly cancel the terms above, proving that

$$\delta\mathcal{H}^{(\kappa^0)} \;=\; 0\;. \tag{6.80}$$

The Hamiltonian upto order κ in (6.73) is therefore invariant under the following residual reparametrization transformations

$$\delta h \;=\; \frac{1}{2\kappa}\partial\xi\;+\;\xi\bar{\partial}h\;+\;\bar{\xi}\partial h\;, \tag{6.81}$$

and

$$\delta\bar{h} \;=\; \frac{1}{2\kappa}\bar{\partial}\bar{\xi}\;+\;\xi\bar{\partial}\bar{h}\;+\;\bar{\xi}\partial\bar{h}\;. \tag{6.82}$$

Also we observe that after some simplifications, the Hamiltonian (6.73) , to order κ, may be written as

$$\mathcal{H} = \int d^3x \;\; \mathcal{D}\bar{h}\, \bar{\mathcal{D}}h \;\; , \tag{6.83}$$

where

$$\mathcal{D}\bar{h} = \partial\bar{h} + 2\kappa\,\frac{1}{\partial_-^2}\,\Big(\frac{\bar{\partial}}{\partial_-}h\,\partial_-^3\,\bar{h} - h\,\partial_-^2\,\bar{\partial}\bar{h}\Big) \;, \tag{6.84}$$

and $\bar{\mathcal{D}}h$ is its complex conjugate.

The derivative introduced in (6.84) transforms “covariantly". That is

$$\delta(\bar{\mathcal{D}}h) = (\xi\bar{\partial} +\; \bar{\xi}\partial\,)\,\bar{\mathcal{D}}\,h \;. \tag{6.85}$$

This is in keeping with similar analysis of pure Yang-Mills theory in section 6.1.2.

Once quartic interaction vertices are considered, Hamiltonian in (6.73) is no longer invariant under the infinitesimal symmetry transformations. To illustrate, we start by considering contributions from the cubic and quartic vertices.

$$\delta\mathcal{H}^{(\kappa)}_{c,q} \;\;=\;\; \delta^{\kappa^0}(\text{cubic terms}) \;+\; \delta^{\kappa^{-1}}(\text{quartic terms}) \tag{6.86}$$

We start by varying the cubic terms.

$$
\begin{aligned}
\delta^{\kappa^0}(\text{cubic terms}) \;&=\; 2\kappa(\bar{\xi}\partial\bar{h} \;+\; \xi\bar{\partial}\bar{h})\partial_-{}^2\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h \;-\; \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right) \\
&+\; 2\kappa\bar{h}\,\partial_-{}^2\left((\xi\bar{\partial}h \;+\; \bar{\xi}\partial h)\frac{\bar{\partial}^2}{\partial_-{}^2}h \;+\; h\frac{\bar{\partial}^2}{\partial_-{}^2}(\xi\bar{\partial}h \;+\; \bar{\xi}\partial h) \;-\; 2\,\frac{\bar{\partial}}{\partial_-}(\xi\bar{\partial}h \;+\; \bar{\xi}\partial h)\frac{\bar{\partial}}{\partial_-}h\right)\;, \\
&=\; 2\kappa\;\bar{\xi}\;\partial\bar{h}\;\partial_-{}^2\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h \;-\; \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right) \\
&+\; 2\kappa\bar{h}\,\partial_-{}^2\left(\bar{\xi}\partial h\frac{\bar{\partial}^2}{\partial_-{}^2}h \;+\; h\frac{\bar{\partial}^2}{\partial_-{}^2}(\bar{\xi}\partial h) \;-\; 2\,\frac{\bar{\partial}}{\partial_-}(\bar{\xi}\partial h)\frac{\bar{\partial}}{\partial_-}h\right) \;+W(\xi)\;, \\
&=\; \mathcal{X} \;+\; \mathcal{Y} \;+\; W(\xi)\;,
\end{aligned}
\tag{6.87}
$$

and

$$
\begin{aligned}
W &= 2\kappa\;\xi\bar{\partial}\bar{h}\;\partial_-^2\left(h\frac{\bar{\partial}^2}{\partial_-^2}h \;-\; \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right) \\
&\quad+\; 2\kappa\;\bar{h}\;\partial_-^2\left(\xi\bar{\partial}h\frac{\bar{\partial}^2}{\partial_-^2}h \;+\; h\frac{\bar{\partial}^2}{\partial_-^2}(\xi\bar{\partial}h) \;-2\frac{\bar{\partial}}{\partial_-}(\xi\bar{\partial}h)\frac{\bar{\partial}}{\partial_-}h\right)\;, \\
&= 0\;,
\end{aligned}
\tag{6.88}
$$

by partial integrations. Similarly, from the variation of the other cubic term we do not get any $\bar{\xi}$ terms. We further simplify $\mathcal{X}$ and $\mathcal{Y}$ using partial integrations. The results are

$$
\mathcal{X} \;=\; -2\kappa\;\bar{\xi}\;\bar{h}\;\partial_-{}^2\partial\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h\right) + 2\kappa\;\bar{\xi}\;\bar{h}\;\partial_-{}^2\partial\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right)\;, \tag{6.89}
$$

and

$$
\begin{aligned}
\mathcal{Y} \;=\; & 2\kappa\bar{h}\,\bar{\xi}\partial_-{}^2\,\partial\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h\right) \;-\; 2\kappa\bar{h}\bar{\xi}\partial_-{}^2\,\partial\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right) \\
& -\; 4\kappa\;\bar{\partial}\bar{\xi}\;\frac{\partial}{\partial_-}h\;\frac{\bar{\partial}}{\partial_-}h\;\partial_-^2\,\bar{h} \;+\; 2\kappa\;\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-^2}h\;h\;\partial_-^2\,\bar{h} \;+\; 4\kappa\;\bar{\partial}\bar{\xi}\;\frac{\partial\bar{\partial}}{\partial_-^2}h\;h\;\partial_-^2\,\bar{h}\;.
\end{aligned}
\tag{6.90}
$$

The first two terms in (6.90) cancel (6.89) and we retrieve

$$
\delta^{\kappa^0}(\text{cubic terms}) \;=\; -4\kappa\;\bar{\partial}\bar{\xi}\;\frac{\partial}{\partial_-}h\;\frac{\bar{\partial}}{\partial_-}h\;\partial_-^2\,\bar{h} \;+\; 2\kappa\;\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-^2}h\;h\;\partial_-^2\,\bar{h} \;+\; 4\kappa\;\bar{\partial}\bar{\xi}\;\frac{\partial\bar{\partial}}{\partial_-^2}h\;h\;\partial_-^2\,\bar{h}
\tag{6.91}
$$

For quartic vertex, we focus on the relevant κ^{-1} variation. We consider the $\bar{\xi}$ terms as the $\xi-$dependent terms can be obtained by complex conjugation.

$$
\delta^{\kappa^{-1}}(\text{quartic terms}) \;= \mathcal{A} \;+\; \mathcal{B} \;+\; \mathcal{C} \;+\; \mathcal{D}\;,
\tag{6.92}
$$

where

$$
\begin{aligned}
\mathcal{A} \;=\; & -\;4\kappa\;\bar{\partial}\bar{\xi}\;\partial h\;\frac{1}{\partial_-{}^2}\;\left(\frac{\bar{\partial}}{\partial_-}h\;\partial_-{}^3\;\bar{h}\;-\;h\;\partial_-{}^2\bar{\partial}\bar{h}\right) \\
=\; & 4\kappa\;\bar{\partial}\bar{\xi}\;\frac{\partial}{\partial_-}h\;\frac{\bar{\partial}}{\partial_-}h\;\partial_-^2\,\bar{h} \;-\; 4\kappa\;\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-^2}h\;h\;\partial_-^2\,\bar{h} \;-\; 4\kappa\;\bar{\partial}\bar{\xi}\;\frac{\partial\bar{\partial}}{\partial_-^2}h\;h\;\partial_-^2\,\bar{h}\;,
\end{aligned}
\tag{6.93}
$$

$$
\begin{aligned}
\mathcal{D} \;=\; & -\;2\kappa^2\;\partial h\;\bar{\partial}^2\bar{\xi}\;\left(\;\frac{1}{\partial_-{}^2}\;(\partial_- h\;\partial_-\bar{h})\;-\;h\;\bar{h}\right) \\
=\; & -\;2\kappa\;\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-}h\;\partial_- h\;\bar{h} \;+\; 2\kappa\;\bar{\partial}^2\bar{\xi}\;\frac{\partial}{\partial_-^2}h\;h\;\partial_-^2\,\bar{h}\;.
\end{aligned}
\tag{6.94}
$$

Note that the terms in (6.93) along with the second term in (6.94) cancel the entire contribution from the cubic vertex. Now moving on to

$$\mathcal{B} = +\, 2\kappa\, \bar{\partial}^2 \bar{\xi}\, h\, \frac{1}{\partial_-}\, (\partial \bar{h}\, \partial_- h \,-\, \partial_- \bar{h}\, \partial h)\ , \tag{6.95}$$

and find that

$$\mathcal{B} -\, 2\kappa\, \bar{\partial}^2 \bar{\xi}\, \frac{\partial}{\partial_-} h\, \partial_- h\, \bar{h} = +\kappa \bar{\partial}^2 \bar{\xi}\, h\, h\, \partial \bar{h}\ . \tag{6.96}$$

Now we turn to the third term

$$\begin{aligned} \mathcal{C} \;&=\; +2\kappa h\, \bar{\partial}\bar{\xi} \left(\bar{\partial} h\, \partial \bar{h} \,+\, \partial h\, \bar{\partial}\bar{h} \,-\, \partial_- \bar{h}\, \frac{\partial\bar{\partial}}{\partial_-} h \,-\, \partial_- h\, \frac{\partial\bar{\partial}}{\partial_-}\, \bar{h} \right) \\ &=\; +2\kappa \bar{\partial}\bar{\xi}\, h\, \partial h\, \bar{\partial}\bar{h} - \kappa \bar{\partial}^2 \bar{\xi}\, h\, h\, \partial \bar{h} -\, 2\kappa h\, \bar{\partial}\bar{\xi}\, \partial_- \bar{h}\, \frac{\partial\bar{\partial}}{\partial_-} h\ . \end{aligned} \tag{6.97}$$

Finally we obtain the net contribution, from the cubic and quartic vertices, at κ order reads

$$\delta \mathcal{H}^{(\kappa)}_{c,q} \;=\; \left(+2\kappa\, \bar{\partial}\bar{\xi}\, h\, \partial h\, \bar{\partial}\bar{h} \,-\, 2\kappa h\, \bar{\partial}\bar{\xi}\, \partial_- \bar{h}\, \frac{\partial\bar{\partial}}{\partial_-} h \right) + \text{c.c.} \tag{6.98}$$

Therefore we deduce that the existing transformations in (6.75) do not leave the Hamiltonian invariant at order κ. In order to make the Hamiltonian invariant, we now introduce new terms of order κ to the r.h.s of (6.75). After substituting these new terms the contribution in the kinetic term of (6.73), are clearly at the same order as those in (6.98). We find

$$\delta h \;=\; \frac{1}{2\kappa} \partial \xi \,+\, \xi \bar{\partial} h \,+\, \bar{\xi} \partial h \;\,-\, \kappa\, \bar{\partial}\bar{\xi}\, h h \,+\, 2\kappa\, \partial \xi\, \frac{1}{\partial_-} (\bar{h}\, \partial_- h)\ , \tag{6.99}$$

and

$$\delta \bar{h} \;=\; \frac{1}{2\kappa} \bar{\partial} \bar{\xi} \,+\, \xi \bar{\partial} \bar{h} \,+\, \bar{\xi} \partial \bar{h} \;\,-\, \kappa\, \partial \xi\, \bar{h} \bar{h} \,+\, 2\kappa\, \bar{\partial}\bar{\xi}\, \frac{1}{\partial_-} (h\, \partial_- \bar{h})\ . \tag{6.100}$$

The variation of the kinetic term $\delta^{\kappa}(\partial \bar{h} \bar{\partial} h)$ cancels exactly against the terms in (6.98), confirming that

$$\delta \mathcal{H}^{(\kappa)} \;=\; 0 \;, \tag{6.101}$$

This proves that that the light-cone Hamiltonian to order κ^2 is invariant under the residual reparametrizations (6.99) and (6.100).

Quadratic form structure

We show that the Hamiltonian in (6.73), to order κ^2, can also be written as a quadratic form. We already know $\mathcal{D}\bar{h}$ to order κ. The product $\mathcal{D}\bar{h}\,(\kappa)\,\bar{\mathcal{D}}h\,(\kappa)$ gives one-half of the third line in (6.73). We now show that (the remaining) half of the third line and all other terms at order κ^2, in (6.73) can be written in the form

$$\mathcal{D}\bar{h}\,(\kappa^2)\,\bar{\partial}h + \partial\bar{h}\,\bar{\mathcal{D}}h\,(\kappa^2)\ . \tag{6.102}$$

Where at order κ^2, $\mathcal{D}\bar{h}$ reads

$$\begin{aligned}
&+2\,\kappa^2\,\frac{1}{\partial_-}\{\partial_-^2\,\bar{h}\frac{1}{\partial_-^3}(\partial_-^3\,h\frac{\partial}{\partial_-}\bar{h}-\partial_-^2\,\partial h\bar{h}\,)\,\}+2\,\kappa^2\,\frac{1}{\partial_-}\{\,\frac{\partial}{\partial_-^4}(\bar{h}\partial_-^2\,h)\partial_-^3\,\bar{h}\}\\
&-2\,\kappa^2\,\partial_-^2\,\bar{h}\,\frac{1}{\partial_-^4}(\partial_-^2\,h\partial\bar{h}-2\partial_-\partial h\partial_-\bar{h}\,)+2\,\kappa^2\,\partial_-\bar{h}\frac{1}{\partial_-^2}(\partial_-h\partial\bar{h}-2\partial h\partial_-\bar{h})\\
&+6\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\,\partial\bar{h}-6\,\kappa^2\,\partial_-\bar{h}\,\frac{1}{\partial_-^2}(\partial_-h\partial\bar{h})-2\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})\partial\bar{h}\\
&+4\kappa^2\,h\bar{h}\,\partial\bar{h}+4\kappa^2\,\frac{\partial}{\partial_-}\{\partial_-\bar{h}\,(\frac{1}{\partial_-^2}(\partial_-h\partial_-\bar{h})-h\bar{h})\,\}+2\kappa^2\,\partial_-^2\,\bar{h}\frac{1}{\partial_-^4}(\partial_-^2\,\partial h\bar{h})\\
&-2\kappa^2\,\partial_-\{\partial_-\bar{h}\frac{1}{\partial_-^2}(\bar{h}\partial h)\}-2\kappa^2\,\partial\{\bar{h}\frac{1}{\partial_-^2}(\partial_-\bar{h}\partial_-h)\}-2\kappa^2\,\partial_-^2\,\bar{h}\frac{1}{\partial_-^3}(\partial_-\partial h\bar{h})\\
&+2\kappa^2\partial_-\partial\{\bar{h}\frac{1}{\partial_-^3}(h\partial_-^2\,\bar{h})\}+2\kappa^2\partial\{\partial_-\bar{h}\frac{1}{\partial_-^3}(\bar{h}\partial_-^2\,h)\}+2\kappa^2\partial_-^2\,\{\bar{h}\frac{1}{\partial_-^3}(\partial_-\bar{h}\partial h)\}
\end{aligned} \tag{6.103}$$

Therefore the Hamiltonian for pure gravity in $d=4$ flat spacetime in light-cone gauge can be expressed as a quadratic form up to order κ^2. It can be shown that any other form derived for $\mathcal{D}\bar{h}$ is equivalent to the one above (this involves substituting $\mathcal{D}\bar{h}$ back in the Hamiltonian and using multiple partial integration). We do not offer proof of uniqueness here. However it is straightforward to verify that the $\mathcal{D}\bar{h}$ defined above is unique. This is similar to what we observed in $d=4$ pure Yang-Mills theory, where $\bar{\mathcal{D}}A^a$ defined in (6.3) is unique.

As at order κ, one would expect $\mathcal{D}\bar{h}$ in (6.103) to transform “covariantly". Unfortunately, at this order, the $\mathcal{D}\bar{h}$ does not transform like the field. The variation of (6.103) yields

$$\begin{aligned}\delta\,(\mathcal{D}\bar{h})^\kappa = \quad & +\kappa\,\partial\xi\,\partial\{\partial_-\bar{h}\frac{1}{\partial_-}\bar{h}\} \\ & +2\kappa\,\bar{\partial}\bar{\xi}\,h\,\partial\,\bar{h} + \kappa\,\bar{\partial}\bar{\xi}\,\partial\,\partial_-\,\bar{h}\,\frac{1}{\partial_-}\,h - \kappa\,\bar{\partial}\bar{\xi}\,\frac{1}{\partial_-}\,\{\partial_-\,\partial\,\bar{h}\,h\}\ ,\end{aligned} \tag{6.104}$$

and it can be simply verified verified that

$$\delta\mathcal{H}^\kappa = \int d^3x\ [\delta(\mathcal{D}\bar{h})\,\bar{\mathcal{D}}h\ +\ \mathcal{D}\bar{h}\,\delta(\bar{\mathcal{D}}h)]^\kappa = 0\ . \tag{6.105}$$

In the next section we explain why this transformation property in (6.105) is not unexpected.

Transformation properties of $\mathcal{D}\bar{h}$

We consider the following ansatz for $\delta(\bar{\mathcal{D}}h)$ and $\delta(\mathcal{D}\bar{h})$ from (6.99) and (6.100),ignoring all the ξ-dependent terms

$$\delta(\bar{\mathcal{D}}h)\ =\ 0\ +(\xi\bar{\partial}+\bar{\xi}\partial)\,\bar{\mathcal{D}}h - \kappa\,\bar{\partial}\bar{\xi}\,\sum_i \alpha_i\,\hat{A}_i\ (\hat{B}_i h\ \hat{C}_i h) + 2\kappa\,\partial\xi\,\sum_j \beta_j\,\hat{P}_j(\hat{Q}_j\bar{h}\,\hat{R}_j h) \tag{6.106}$$

and

$$\delta(\mathcal{D}\bar{h}) \;=0+(\xi\bar{\partial}+\bar{\xi}\partial)\mathcal{D}\bar{h}-\kappa\,\partial\xi\sum_i \alpha_i\,\bar{\hat{A}}_i\,(\bar{\hat{B}}_i\bar{h}\,\bar{\hat{C}}_i\bar{h})+\;2\kappa\;\bar{\partial}\bar{\xi}\;\sum_j \beta_j\,\bar{\hat{P}}_j(\bar{\hat{Q}}_j h\bar{\hat{R}}_j\bar{h})\;. \tag{6.107}$$

Where, the $\hat{A}_i,\ldots$ are operators that need to be determined later while α and β are constants. Note that the above ansatz transforms "covariantly" (like the field) if we choose

$$\begin{aligned}
&\alpha=1,\;\hat{A}\;=\;\bar{\partial}\;,\;\hat{B}\;=\;\hat{C}\;=\;1\;,\\
&\beta=1,\;\hat{P}\;=\;\frac{1}{\partial_-}\;,\;\hat{Q}\;=\;1\;,\;\hat{R}\;=\;\partial_-\bar{\partial}\;.
\end{aligned} \tag{6.108}$$

As the Hamiltonian is invariant under (6.99) and (6.100), therefore we have

$$\delta\mathcal{H}=0\;\;\Longrightarrow\int d^3x\;[\delta(\mathcal{D}\bar{h})\bar{\mathcal{D}}h\;+\;\mathcal{D}\bar{h}\,\delta(\bar{\mathcal{D}}h)]\;. \tag{6.109}$$

Now we check (6.109) at order κ^0

$$\delta\mathcal{H}=\;\int d^3x\;[(\delta(\mathcal{D}\bar{h}))^{\kappa^0}\bar{\partial}h\;+\partial\bar{h}(\delta(\bar{\mathcal{D}}h))^{\kappa^0}]\;, \tag{6.110}$$

$$=\;\int d^3x\;[\bar{\xi}\partial^2\bar{h}\bar{\partial}h\;+\;\partial\bar{h}\xi\partial\bar{\partial}h]\;. \tag{6.111}$$

After integrating a ∂ from the $\bar{h}$ in the first term, we get

$$\delta\mathcal{H})^{\kappa^0}\;=\;0 \tag{6.112}$$

At order $\mathcal{O}(\kappa)$, we have

$$
\begin{aligned}
(\delta\mathcal{H})^\kappa &= \int d^3x\,[(\delta(\mathcal{D}\bar{h}))^\kappa\bar{\partial}h \,+\, (\delta(\mathcal{D}\bar{h}))^{\kappa^0}(\bar{\mathcal{D}}h)^\kappa \,+\, (\mathcal{D}\bar{h})^\kappa(\delta(\bar{\mathcal{D}}h))^{\kappa^0} \,+\partial\bar{h}(\delta(\bar{\mathcal{D}}h))^\kappa] \\
&= \kappa\int d^3x\,\{[\bar{\xi}\partial(\mathcal{D}\bar{h})^\kappa + \,2\bar{\partial}\bar{\xi}\,\sum_j\beta_j\,\bar{\hat{P}}_j(\bar{\hat{Q}}_j h\bar{\hat{R}}_j\bar{h})]\bar{\partial}h \,+\, \bar{\xi}\partial^2\bar{h}(\bar{\mathcal{D}}h)^\kappa \qquad (6.113) \\
&\qquad + [\bar{\xi}\partial(\bar{\mathcal{D}}h)^\kappa - \bar{\partial}\bar{\xi}\,\sum_i\alpha_i\,\hat{A}_i\,(\hat{B}_i h\,\hat{C}_i h)]\,\partial\bar{h} \,+\, (\mathcal{D}\bar{h})^\kappa\bar{\xi}\partial\bar{\partial}h\} \qquad (6.114)
\end{aligned}
$$

Now we integrate a ∂ from $\bar{h}$ in the last term of (6.113) and cancel it against the first term in (6.114). Later we cancel last term of (6.114) with the first term of (6.113) by integrating a ∂. Finally we get

$$
(\delta\mathcal{H})^\kappa \;=\; \kappa\int d^3x\,[2\bar{\partial}\bar{\xi}\,\sum_j\beta_j\,\bar{\hat{P}}_j(\bar{\hat{Q}}_j h\bar{\hat{R}}_j\bar{h})\bar{\partial}h \,-\, \bar{\partial}\bar{\xi}\,\sum_i\alpha_i\,\hat{A}_i\,(\hat{B}_i h\,\hat{C}_i h)\,\partial\bar{h}] \qquad (6.115)
$$

Substituting (6.108) into (6.115) , gives us

$$
(\delta\mathcal{H})^\kappa = +2\,\kappa\int d^3x\;\bar{\partial}\bar{\xi}\,\frac{1}{\partial_-}(\partial_- h\,\partial\bar{h})\,\bar{\partial}h + \text{c.c.} \neq 0 \qquad (6.116)
$$

Therefore we deduce that the Hamiltonian for gravity is not the “square” of a “covariant derivative”. Instead, if we substitute (6.104) into (6.115), we find

$$
(\delta\mathcal{H})^\kappa = 0\;. \qquad (6.117)
$$

This is in contrast to Yang-Mills theory where both, the pure and maximally supersymmetric theories may be described by covariant derivatives. Also from the MHV literature [16, 31–33], we know that all tree-level amplitudes in Yang-Mills theory may be expressed entirely in terms of the “square" or “angular" brackets as shown in (3.33) . However in the case of gravity, the cubic amplitude does indeed have the same property but the quartic and higher vertices involve a mixture of both brackets. Therefore, the lesson from the amplitude structures may be the another way of looking at the fact that the $(\mathcal{D}\bar{h})$ introduced in gravity does not transform as expected (like the field) beyond order κ^2.

Chapter 7

Discussion and outlook

The kinetic and cubic vertices of the Einstein-Hilbert action on AdS_4 are more complicated than their flat spacetime counterparts. This deviation is observed even more significantly in the case of the four dimensional de Sitter background. However, as expected, if we strip-off coordinate-dependent conformal factors and ignore cosmological contributions, the closed form expression for the gravity Lagrangian in light-cone gauge on these curved backgrounds is structurally similar to that on flat spacetime. Thus the closed form expression is an ideal starting point for any further studies involving light-cone gravity on any background.

We wish to extend our analysis and obtain quartic interaction vertices but this seems to be trickier because time derivatives begin to appear. On flat backgrounds, the time derivatives are eliminated by field redefinitions which appears to be quite complicated to perform in the case of AdS_4. In the case of dS_4, this may become even more complicated or simply impractical to do due to the presence of the time-dependent conformal factors.

We are also, at present, extending the results presented in this thesis to more general conformal background spacetimes. Once this work is completed, we hope to show that the AdS_4 and dS_4 results are special cases of this more general result [46].

We hope to return to many of these issues in the future.

Amplitudes on curved backgrounds

The KLT relations are observed at tree-level in flat spacetime. One important and interesting future direction is ask whether such relations (or some similar version of them) are valid on curved backgrounds. The tree-level amplitudes or interaction vertices are computed by Fourier transforming the vertices in the light-cone Lagrangian. Unfortunately, the usual technique of Fourier transform does not work on curved backgrounds (such as AdS_4) due to the presence of coordinate-dependant conformal factors. This could prove to be a serious road-block to further progress. One hope we have is that integral transforms, like the Mellin transform [47] for example, could prove to be extremely useful in this context.

Quadratic forms

In the below table I summarize the status of theories which can be expressed as quadratic forms.

Theories	Quadratic form
$\mathcal{N} = 4, d = 4$ SYM	Yes
$d = 4$ Pure YM	Yes
$\mathcal{N} = 8, d = 4$ SUGRA	Yes (upto order κ)
$d = 4$ Pure gravity	Yes (upto order κ^2)
$d = 4$ Pure gravity in curved backgrounds	Don't Know, in progress.
Non-maximally supersymmetric theories	No

There are a number of open questions as far as these quadratic forms are concerned. Firstly, what is the physical significance of these forms? In particular, can we connect them to some physical symmetry in the theory (infinitesimal or otherwise). Secondly, why do these quadratic form structures appear only in the pure and maximally supersymmetric cases (as shown in the table)? Thirdly, in the context of $\mathcal{N}=8$ supergravity in $d=4$, can we relate this quadratic form strcutre to the improved ultra-violet behavior observed in the theory? And finally, do these quadratic form structures persist/appear in curved spacetime backgrounds such as AdS_4 and dS_4? It would be wonderful if these elegant and interesting algebraic structures in the Hamiltonians describing gravity theories could somehow lead us to some new and hidden symmetries in the theory.

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