

# Model Order Reduction of Nonlinear Dynamical Systems

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by

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# Certificate

This is to certify that this dissertation entitled Model Order Reduction of Nonlinear Dynamical Systems towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Reetish Padhi at Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg under the supervision of Dr. Ion Victor Gosea, Research Associate, Computational Methods in Systems and Control Theory (CSC), during the academic year 2023-2024.

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This thesis is dedicated to my parents and friends.



# Declaration

I hereby declare that the matter embodied in the report entitled Model Order Reduction of Nonlinear Dynamical Systems are the results of the work carried out by me at the Computational Methods in Systems and Control Theory (CSC), Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, under the supervision of Dr. Ion Victor Gosea and the same has not been submitted elsewhere for any other degree.



Reetish Padhi



# Acknowledgments

This thesis, despite having just my name on it, is a collective effort of many people, each playing their own unique role. Unfortunately, the title page cannot accommodate all my unofficial ‘co-authors’, which is why I am glad this section exists!

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# Abstract

Balanced truncation (BT) is one of the oldest, most popular model reduction techniques that offers a way to construct stable, balanced, reduced order matrices that preserve the original system's dominant Hankel singular values. It also provides a priori  $\mathcal{H}_\infty$  bounds for the reduced order model. BT has been refined over the years and has been extended to mild nonlinear systems as well. Data-driven balancing or quadrature-based balanced truncation is one variant of BT that aims to approximate the Hankel singular values of the original system in a purely non-intrusive fashion using samples of the system kernels and their derivatives.

In this thesis, we first define balanced truncation for a new class of systems - bilinear systems with quadratic output (BQO). The gramians (and kernels) are first defined for these systems, which are vital for the BT algorithm. Finally, we introduce truncated Gramians and use them to implement a quick, approximate balanced truncation algorithm.

The second section of this thesis involves extending the data-driven balancing method (Quad BT) to two nonlinear classes of systems - linear systems with quadratic output (LQO) and quadratic-bilinear systems (QB). The recipe for Quad BT remains the same as that for the linear case. We construct a quadrature-based approximation of the gramians (or truncated gramians in the QB case) and replace the intrusive terms in BT with these quadrature-based approximations. Finally, we show that these approximate matrices can be expressed as data matrices with entries corresponding to samples of the system kernels or derivatives of the kernels. In addition to the theory, we propose schemes to improve the computational viability of these methods.

The theory developed in this thesis has been put to the test with numerical experiments on benchmark datasets.



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## Commonly use abbreviations

- Balanced truncation (BT)
- Quadrature-based Balanced truncation (Quad BT)
- Reduced Order Model (ROM)
- Full Order Model (FOM)
- Linear systems with Quadratic Output (LQO)
- Bilinear systems with Quadratic Output (BQO)
- Quadratic Bilinear systems (QB)
- Hankel Singular Value (HSV)
- Multiple Input Multiple Output (MIMO)
- Single Input Single Output (SISO)

# Chapter 1

## Introduction

Dynamical systems are integral tools in modelling, simulating and studying a wide range of physical processes ranging from the simulation of molecular dynamics to climate prediction. With growing scientific advancements there is an increased demand for the need to solve larger, more complex dynamical systems. Most real-world systems have an exceedingly large number of variables making them difficult to solve numerically. Thus, there is a need for reduced models which serve as smaller surrogate models while maintaining the input-output mapping and other essential features of the larger model. To be more precise, we hope that the responses of the reduced and original system (full system) to the same input  $\mathbf{u}(t)$  are uniformly close over all choices of  $\mathbf{u}$ . In addition, we hope that our reduced model is able to retain properties like stability, and passivity while also having a similar structure as that of the original system. For example, the reduced model of a spring balance system (linear) should be linear. Naturally, it is not possible to expect the reduced system to be able to preserve all these properties at once but depending on the task and quantity of interest, different approaches have been developed over the years.

The Loewner framework [1] is a data-driven structure-preserving interpolatory framework based on Lagrange rational interpolation. It is based on interpolating the transfer function (input-output map in frequency domain) at points of interest. The framework has been extended to structured nonlinear systems in Ref. [2, 3]. The AAA algorithm Ref. [4, 5] is a data-driven algorithm that approximates functions with rational functions. It involves selecting interpolation points in a greedy fashion and performing rational interpolation until

the original function is well approximated. AAA has proved to be a useful tool in obtaining ROMs for linear time-invariant systems. The RKFIT algorithm Ref. [6] also approximates a general function with a rational one, but by constructing Krylov subspaces.

In this thesis, we will be working with two projection-based methods - Balanced truncation [7] and its data-driven variant Quadrature-based Balanced Truncation [8]. We look at extending these two methods to more general classes of systems. We also present numerical experiments with our proposed algorithms in the respective chapters.

## 1.1 Problem formulation

Suppose we are given a dynamical system  $\Sigma_n$  given by the following set of differential equations -

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u})\mathbf{x} + g(\mathbf{x}, \mathbf{u})\mathbf{u} \\ \mathbf{y} &= h(\mathbf{x}, \mathbf{u})\mathbf{x}\end{aligned}$$

where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{y} \in \mathbb{R}^m$  and  $\mathbf{u} \in \mathbb{R}^p$ . We want to find a reduced order model which has the same structure as  $\Sigma_n$  with dimension  $k \ll n$ . In other words, find  $\Sigma_k$ ,

$$\begin{aligned}\dot{\mathbf{x}}_k &= \hat{f}(\mathbf{x}_k, \mathbf{u})\mathbf{x}_k + \hat{g}(\mathbf{x}_k, \mathbf{u})\mathbf{u} \\ \hat{\mathbf{y}} &= \hat{h}(\mathbf{x}_k, \mathbf{u})\mathbf{x}_k\end{aligned}$$

where  $\mathbf{x} \in \mathbb{R}^k$ ,  $\hat{\mathbf{y}} \in \mathbb{R}^m$  and  $\mathbf{u} \in \mathbb{R}^p$  subject to problem related constraints. For example, the Balanced truncation algorithm provides us with  $\Sigma_k$  which preserves stability and the Hankel singular values of the original system while also providing error bounds in the  $\mathcal{H}_\infty$  norm. More details can be found in Sec. 2.8.

In this thesis, we focus on the special case of time invariant systems i.e. systems where  $f, g, h$  are independent of  $t$ . For simplicity, we will only consider the case where  $m = p = 1$ , the single input, single output (SISO) case. The different classes of systems we will be considering are linear systems, linear systems with quadratic output (LQO), bilinear systems with quadratic output (BQO) and quadratic bilinear systems (QB). A detailed discussion of these systems will be given in the relevant sections but a rough definition of these systems

can be found in Fig. 1.1

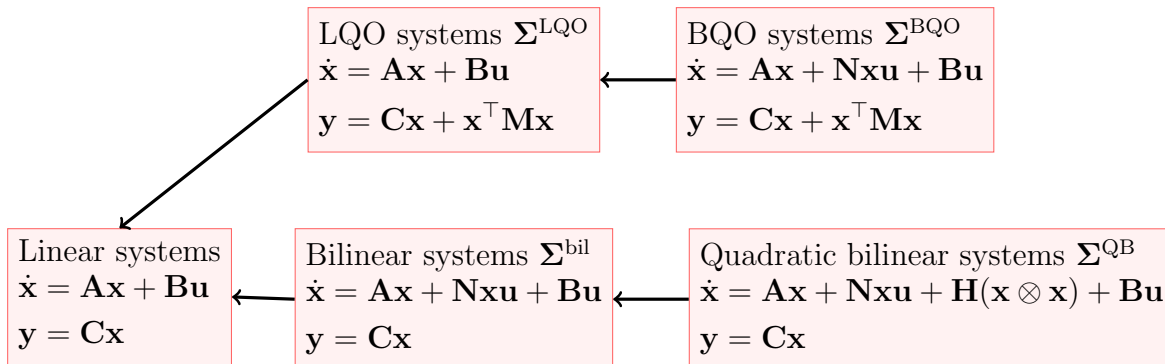


Figure 1.1: Classification of systems

## 1.2 Outline of the thesis

In Chap. 2, we introduce some of the preliminary definitions that form the basis of the work in the other sections. We begin by defining relevant norms in Sec. 2.1 and useful matrix operations in Sec. 2.2. This is followed by defining linear time-invariant systems which is used to introduce the concepts of kernels (Sec. 2.4), reachability, observability (Sec. 2.5) and gramians (Sec. 2.6). These are then used to define the energies of a state in Sec. 2.7 which motivates the balanced truncation algorithm. Sec. 2.8 describes the balanced truncation algorithm in detail. We discuss projection-based methods, the mathematical formulation of BT, the BT algorithm and some numerical examples that show the performance of BT with the ‘heat’ benchmark model. Sec. 2.9 introduces the data-driven Quad BT algorithm for linear systems. We present the Quad BT algorithm, some error bounds for the ROM obtained by Quad BT and some numerical results to demonstrate the performance of Quad BT on the iss1R benchmark model.

In Chap. 3, we motivate the need for a reduction method specifically for bilinear systems with quadratic output. Sec. 3.1 presents a method using lifting methods and highlights the problems with this naive approach. We begin working towards BT for BQO systems by deriving the expression for the kernels of the BQO systems in Sec. 3.2. and the BQO gramians in Sec. 3.3. We also derive the gramians using the adjoint system of the BQO system in Sec. 3.4 and notice that the two methods lead to the same definition. Following

this, we present the balanced truncation algorithm for BQO systems in Sec. 3.5 and some numerical results of our algorithm on the burgers bilinear benchmark model in Sec. 3.6. We propose the BT algorithm using truncated gramians in Sec. 3.7.

In Chap.4, we extend the Quadrature-based balanced truncation method to linear systems with quadratic output. The first section 4.1 defines the kernels and gramians for LQO systems. We then proceed to approximate the intrusive terms in balanced truncation with quadrature terms (data matrices) and present the Quad BT algorithm in Sec. 4.2. In Sec. 4.3 we address some of the computational issues of the method and propose ways to get around them. Sec. 4.4 shows the extension to the MIMO case and Sec. 4.5 compares the performance of Quad BT with BT on the iss1R dataset.

Chap. 5 is the main contribution of this thesis wherein we extend Quad BT to quadratic bilinear systems. First, we enumerate the kernels of QB systems (Sec. 5.1) using the Volterra series representation. This is followed by defining the gramians and truncated gramians for QB systems (Sec. 5.2). Having laid the groundwork, we define the quadrature-based gramians (Sec. 5.3) and show that the key terms appearing in QuadBT can be replaced by data-based matrices (Sec. 5.4) which ultimately gives us the Quad BT algorithm for QB systems. We offer suggestions and tricks to improve the computational feasibility of our algorithm in Sec. 5.5. Finally, we provide some numerical tests that evaluate the performance of the algorithm in Sec. 5.6

### 1.3 Original Contribution

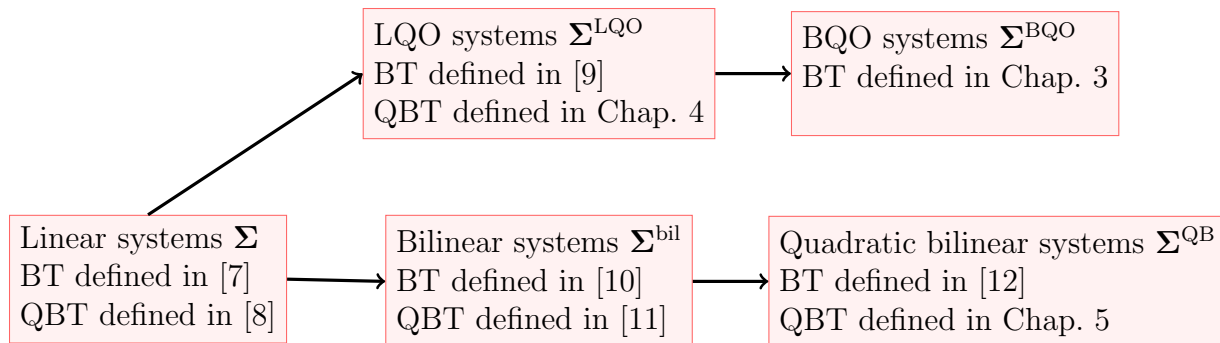


Figure 1.2: Summary of the contributions of this thesis and how they fit in the literature

Fig. 1.2 summarises the type of systems we study in this thesis. Chapters 3, 4 and 5 contain the three original contributions of this thesis. In Chap. 3 we define the kernels, and gramians of BQO systems and present the balanced truncation algorithm for the first time. We also suggest the use of truncated Gramians for faster implementation of BT to obtain ROMs. In Chap. 4 we derive the extension of Quad BT to LQO systems. We also show the efficiency of the suggested algorithm with some numerical examples.

Chap. 5 is the main contribution of the thesis where we derive the extension of Quad BT to QB systems. In Sec. 5.1, we propose a notation to enumerate the QB kernels and use these kernels in deriving the results of Sec. 5.4. As far as the author knows, such enumeration of the QB kernels has not been stated explicitly in the literature. Thm. 5.4.1, 5.4.2, 5.4.3, 5.4.4 and 5.4.5 are the main results of the thesis and form the basis of the new Quad BT algorithm presented in Algo. 6.



# Chapter 2

## Preliminaries

In this chapter, we introduce some important definitions of system theory and model order reduction. Most of the definitions and results in this section can be found in Ref. [13, 14, 15, 16, 17]

### 2.1 Norms

In this section we introduce some of the norms that will be relevant for future discussions.

**Definition 2.1.1.** *The  $H$ -Infinity norm  $\|H\|_\infty$  of the linear system  $H$  is defined as the supremum of the largest singular number of its transfer function over the imaginary axis:*

$$\|H\|_\infty = \sup_{\omega \in \mathbf{R}} \sigma_{\max}(H(j\omega)),$$

**Definition 2.1.2.** *The Frobenius norm of a matrix  $\mathbf{A}$  (denoted by  $\|\mathbf{A}\|_F$ ) is defined as*

$$\|\mathbf{A}\|_F = \sqrt{\text{tr}(\mathbf{A}^* \mathbf{A})}$$

where  $\mathbf{A}^*$  denotes the complex conjugate of the matrix  $\mathbf{A}$ .

**Definition 2.1.3.** *The 2-norm of a matrix  $\mathbf{A}$  is defined as*

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^* \mathbf{A})}$$

## 2.2 Matrix operations

In this section we introduce some of the matrix operations that become important in the study of nonlinear dynamical systems. We begin by defining the vectorization and Kronecker product for matrices.

**Definition 2.2.1.** [15] Let  $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_m] \in \mathbb{R}^{n \times m}$  and  $\mathbf{Y} = [\mathbf{y}_1, \dots, \mathbf{y}_p] \in \mathbb{R}^{p \times q}$ .

$$\text{vec}(\mathbf{X}) := \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_m \end{bmatrix} \in \mathbb{R}^{n \cdot m \times 1}, \quad \mathbf{X} \otimes \mathbf{Y} = \begin{bmatrix} x_{11}\mathbf{Y} & \dots & x_{1m}\mathbf{Y} \\ \vdots & & \vdots \\ x_{n1}\mathbf{Y} & \dots & x_{nm}\mathbf{Y} \end{bmatrix} \in \mathbb{R}^{n \cdot p \times m \cdot q}$$

**Proposition 2.2.1.** [15] Let  $\mathbf{A} \in \mathbb{R}^{n \times m}$ ,  $\mathbf{B} \in \mathbb{R}^{p \times q}$ ,  $\mathbf{C} \in \mathbb{R}^{m \times r}$ ,  $\mathbf{D} \in \mathbb{R}^{q \times s}$  and  $\mathbf{E} \in \mathbb{R}^{r \times \ell}$ . Then the following equalities hold

- a)  $\text{vec}(\mathbf{ACE}) = (\mathbf{E}^\top \otimes \mathbf{A}) \text{vec}(\mathbf{C})$ ,
- b)  $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$ .

**Definition 2.2.2.** [16] Let  $\mathbf{X}^{\text{tens}} \in \mathbb{R}^{n \times n \times n}$  be a three-dimensional tensor which is described by the matrices  $\mathbf{X}_i^{\text{tens}} \in \mathbb{R}^{n \times n^2}$ ,  $i \in \{1, 2, \dots, n\}$  (the so-called frontal slices). Then, the mode- $\mu$  matricizations (for  $\mu \in \{1, 2, 3\}$ ) are given by

$$\begin{aligned} \mathbf{X}^{(1)} = \mathbf{X} &= \begin{bmatrix} \mathbf{X}_1^{\text{tens}} & \mathbf{X}_2^{\text{tens}} & \dots & \mathbf{X}_n^{\text{tens}} \end{bmatrix}, \quad \mathbf{X}^{(2)} = \begin{bmatrix} (\mathbf{X}_1^{\text{tens}})^\top & (\mathbf{X}_2^{\text{tens}})^\top & \dots & (\mathbf{X}_n^{\text{tens}})^\top \end{bmatrix}, \\ \mathbf{X}^{(3)} &= \begin{bmatrix} \text{vec}(\mathbf{X}_1^{\text{tens}}) & \text{vec}(\mathbf{X}_2^{\text{tens}}) & \dots & \text{vec}(\mathbf{X}_n^{\text{tens}}) \end{bmatrix}^\top. \end{aligned}$$

**Property 1.** Given vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  and consider  $\mathbf{H} \in \mathbb{R}^{n \times n^2}$  as the 1-matricization of a tensor  $\mathcal{H}$ . Then we have,

$$\mathbf{w}^\top \mathbf{H}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{u}^\top \mathcal{H}^{(2)}(\mathbf{v} \otimes \mathbf{w}) \quad (2.1)$$

where  $\mathcal{H}^{(2)}$  is the 2-matricization of  $\mathcal{H}$ .

## 2.3 Linear Time Invariant systems

Linear time invariant systems are the simplest class of systems and hence are the perfect starting place to begin our study.

**Definition 2.3.1.** An LTI system  $\Sigma$  is a dynamical system given by the following set of differential equations -

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}\tag{2.2}$$

where  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times p}$ ,  $\mathbf{C} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{u}(t) \in \mathbb{R}^p$ ,  $\mathbf{x}(t) \in \mathbb{R}^n$  and  $\mathbf{y}(t) \in \mathbb{R}^m$ . The dimension of the system is defined as the dimension of the LTI system i.e.  $\dim(\Sigma) = \dim(\mathbf{x}) = n$ .

The linear system  $\Sigma$  is stable if all the eigenvalues of  $\mathbf{A}$  are in the left half plane i.e. if the system has stable dynamics for  $\mathbf{u} = 0$ . The expression of the output can be easily calculated by solving Eq. 2.2. Let us denote the solution of Eq. 2.2 at time  $t$ , from initial point  $\mathbf{x}_0$  with input  $\mathbf{u}$ , as  $\phi(t; \mathbf{x}_0; \mathbf{u})$ . Then we have,

$$\begin{aligned}\phi(t; \mathbf{x}_0; \mathbf{u}) &= e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^t e^{\mathbf{A}(s-t)}\mathbf{B}\mathbf{u}(s)ds. \\ &= e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_0^{t-t_0} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{u}(t-\tau)d\tau, \\ &= \int_0^{t-t_0} e^{\mathbf{A}t}\mathbf{B}\mathbf{u}(t-\tau)d\tau \quad (\text{assuming } \mathbf{x}_0 = 0) \\ \mathbf{y}(t) &= \int_0^{t-t_0} \mathbf{h}(t)\mathbf{u}(t-\tau)d\tau \quad (\text{where we denote } \mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}).\end{aligned}\tag{2.3}$$

In fact, Eq. 2.3 shows that the output of the LTI system can be expressed as a convolution of  $\mathbf{h}$  and  $\mathbf{u}$ .

## 2.4 Input-Output representation of dynamical systems

Def. 2.3.1 is not the only way to define a linear time-invariant system. As seen in Eq. 2.3, the LTI system can be characterised as a function or operator that gives output  $\mathbf{y}(t)$  for input

$\mathbf{u}(t)$ . Note that the kernel  $\mathbf{h}(t)$  of the LTI system describes the impulse response of the system  $\Sigma$  i.e. the system has output  $\mathbf{h}(t)$  for  $\mathbf{u}(t) = 1$  at  $t = 0$  and  $\mathbf{u}(t) = 0$  otherwise. In the nonlinear case, we can write the output of the nonlinear system as a linear combination of multiple kernels (see Sec. 3.2 and 5.1).

**Definition 2.4.1.** *The input-output mapping of a dynamical system can be expressed using kernels  $\mathbf{h}$  which act on delayed versions of the input  $\mathbf{u}(t)$  to give the output  $\mathbf{y}(t)$ . In the linear case, we have Eq. 2.3 and more generally in the nonlinear case we can express the output as,*

$$\mathbf{y}(t) = \int_0^t \mathbf{h}(t) \mathbf{u}(t - t_1) dt_1 + \int_0^t \int_0^{t_1} \mathbf{h}(t_1, t_2) \mathbf{u}(t - t_1) \mathbf{u}(t_1 - t_2) dt_2 dt_1 + \dots$$

The kernels allow us to express the system in an input-output representation in the time domain. Similarly, we can define transfer functions which serve as the input-output transform in the frequency domain. Before defining the transfer function we must define the Laplace transform of a function  $f(t)$ .

**Definition 2.4.2.** *Laplace transform(LT) of a function  $f(t)$  is defined as*

$$\mathcal{L}(f)(s) = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

*(assuming the integral converges). The Laplace transform allows us to transform a differential equation into an algebraic one.*

**Definition 2.4.3.** *The transfer function  $\mathbf{H}(s)$  of a system is defined as the ratio of the Laplace transform of the output signal to that of the input signal.*

The transfer function is the Laplace transform of the linear kernel for LTI systems. We have the expression,

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

In this thesis, we will mainly focus on working with the time domain description - kernels but we briefly introduce the frequency domain description for completeness. It is worth noting that most of the methods introduced in subsequent chapters can be extended to the frequency domain as well.

## 2.5 Reachability and Observability

The concept of reachability provides a measure of how easy it is to reach a state from the zero state. Analogous to reachability, we also have the concept of observability which describes how easy it is to distinguish a state from a zero state. Many reduction techniques utilise the reachable and observable spaces of a system to construct reduced models. First, let us define the reachable and observable states and sub-spaces for LTI systems.

**Definition 2.5.1.** *Given a linear system  $\Sigma$ , a state  $\bar{\mathbf{x}}$  is reachable from the zero state if there exists a time  $\mathbf{T} < \infty$  and an input function  $\bar{\mathbf{u}}(t)$ , of finite norm ( $\|\bar{\mathbf{u}}\|^2 < \infty$ , where  $\|\bar{\mathbf{u}}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = \int_0^{\mathbf{T}} \mathbf{u}^\top(t) \mathbf{u}(t) dt$ ), such that  $\bar{\mathbf{x}} = \phi(\mathbf{T}; 0; \bar{\mathbf{u}})$ .*

**Definition 2.5.2.** *The reachable subspace  $\mathbb{X}^{reach} \subset \mathbb{X}$  of  $\Sigma$  is the set containing all reachable states of  $\Sigma$ .*

**Definition 2.5.3.** *A state  $\bar{\mathbf{x}} \in \mathbb{X}$  is unobservable if  $\mathbf{y}(t) = \mathbf{C}\phi(t; \bar{\mathbf{x}}; 0) = 0$  for all  $t \geq 0$ , i.e., if  $\bar{\mathbf{x}}$  is indistinguishable from the zero state for all  $t \geq 0$ .  $\Sigma$  is (completely) observable if  $\mathbb{X}^{unobs} = \{0\}$ .*

**Definition 2.5.4.** *A stable LTI system  $\Sigma$  that is both reachable and observable is called a minimal system.*

We can go one step further and define the reachable (observable) energies for a state but before that, we define the infinite time reachability and observability gramians.

## 2.6 Infinite time Gramians

Two key quantities in the study of dynamical systems are the reachability and observability infinite time gramians, simply referred to as Gramians.

**Definition 2.6.1.** *Given a stable LTI system  $\Sigma$ , the reachability gramian  $\mathbf{P}$  and the observability gramian  $\mathbf{Q}$  for  $\Sigma$  are defined as,*

$$\mathbf{P} = \int_0^{\infty} e^{\mathbf{A}t} \mathbf{B} \mathbf{B}^\top e^{\mathbf{A}^\top t} dt, \quad (2.4)$$

$$\mathbf{Q} = \int_0^{\infty} e^{\mathbf{A}^\top t} \mathbf{C}^\top \mathbf{C} e^{\mathbf{A}t} dt. \quad (2.5)$$

The definition of the gramians is realization dependent i.e. the gramians vary with change in coordinates of the original system. To see this, consider a realization  $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$  with gramians  $\mathbf{P}, \mathbf{Q}$  and a transform  $\mathbf{T}$  acting on  $\Sigma$  giving a transformed system  $\Sigma_T = (\mathbf{TAT}^{-1}, \mathbf{TB}, \mathbf{CT}^{-1})$ . The gramians of  $\Sigma_T$  -  $\mathbf{P}_T, \mathbf{Q}_T$  are given by,

$$\begin{aligned}\mathbf{P}_T &= \int_0^\infty \mathbf{T}e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top t}\mathbf{T}^\top dt, \\ &= \mathbf{TP}\mathbf{T}^\top. \\ \mathbf{Q}_T &= \int_0^\infty \mathbf{T}^{-\top}e^{\mathbf{A}^\top t}\mathbf{C}^\top\mathbf{C}e^{\mathbf{A}t}\mathbf{T}^{-1} dt, \\ &= \mathbf{T}^{-\top}\mathbf{Q}\mathbf{T}^{-1}.\end{aligned}$$

Even though the gramians are realization dependent, the eigenvalues of the product  $\mathbf{PQ}$  are preserved under such an equivalence transformation i.e.  $\lambda_{\mathbf{P}_T\mathbf{Q}_T} = \lambda_{\mathbf{TPQT}^{-1}} = \lambda_{\mathbf{PQ}}$ . These system invariants are called the Hankel singular values and are used in choosing the truncation index for balanced truncation introduced in Sec. 2.8.

**Definition 2.6.2.** *The Hankel singular values  $(\sigma_1, \dots, \sigma_n)$  of a stable LTI system  $\Sigma$  are defined as the square roots of the eigenvalues of the product  $\mathbf{PQ}$ , where  $\mathbf{P}$  and  $\mathbf{Q}$  are the gramians of  $\Sigma$ . Assuming the eigenvalues  $(\lambda_1, \dots, \lambda_n)$  of  $\mathbf{PQ}$  are arranged in descending order such that  $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$ , we have*

$$\sigma_k = \sqrt{\lambda_k}.$$

The definition of the gramians given in Defn. 2.6.1 however is difficult to work with in practice since it involves calculating indefinite integrals. Luckily Prop. 2.6.1 below offers us a way out. It states that  $\mathbf{P}$  and  $\mathbf{Q}$  satisfy linear Lyapunov equations which can be solved efficiently to obtain the Gramians.

**Proposition 2.6.1.** *The gramians of a stable LTI system  $\Sigma$  satisfy the following Lyapunov equations,*

$$\begin{aligned}\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top &= 0, \\ \mathbf{A}^\top\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^\top\mathbf{C} &= 0.\end{aligned}$$

The solutions  $\mathbf{P}$  and  $\mathbf{Q}$  are symmetric positive-semidefinite matrices and in the case that  $\Sigma$  is minimal, the gramians are symmetric positive definite. We can also compute square factors  $\mathbf{L}, \mathbf{U} \in \mathbb{R}^{n \times n}$  (using Cholesky factorization) so that  $\mathbf{P} = \mathbf{U}\mathbf{U}^\top$  and  $\mathbf{Q} = \mathbf{L}\mathbf{L}^\top$ . We make use of this factorization extensively throughout the thesis.

**Remark 2.6.1.** *The Lyapunov equations given in Prop. 2.6.1 are linear matrix equations and hence have unique solutions when the system is stable.*

## 2.7 Energy associated with a state

Building on the concepts introduced in Sec. 2.5, we also have two energies associated with a state - reachable and observation energy.

**Definition 2.7.1.** *[13] The reachable energy of a state  $\mathbf{x}_r$  is the minimal energy input which steers the system to the desired state (not necessarily in finite time) from the zero state. Assuming stability and (complete) reachability, this minimal energy is given in terms of the reachability gramian as  $\mathbf{x}_r^\top \mathcal{P}^{-1} \mathbf{x}_r$ .*

**Definition 2.7.2.** *[13] Similarly, the observation energy produced by the state  $\mathbf{x}_o$  is equal to the largest energy output produced when the system is left at state  $\mathbf{x}_o$  and decays to the zero state. Assuming stability and (complete) observability, this maximal energy is similarly dependent on the observability gramian and is equal to  $\mathbf{x}_o^\top \mathcal{Q} \mathbf{x}_o$ .*

The energies associated with the state quantify how reachable and observable a state is. Intuitively, a state with low observation energy means that the state releases less energy when decaying to the zero state thereby making it difficult to observe. Similarly, a state with high reachability energy is one that requires high energy to reach from the zero state and hence is difficult to reach.

## 2.8 Balanced Truncation

Balanced truncation (BT) is one of the oldest, most popular model order reduction techniques that allow one to obtain ‘balanced’ lower order models of a system (Ref. [7]). It is within

the framework of projection-based methods. This method has been developed and improved over the years - see [18, 19, 20, 21].

### 2.8.1 Motivation

The core idea of balanced truncation is centred on the two energies mentioned in Sec. 2.7. We want our ROM to retain important states and disregard the less important ones. In other words, remove states that are difficult to observe (low observation energy) and states that are difficult to reach (high reachability energy). This brings us to our first dilemma - the two energies are not compatible in general! States with low observation energy might have small reachability energy and vice versa. To see this consider the case where the eigenvector (say  $\mathbf{v}$ ) of  $\mathbf{P}$  with the highest eigenvalue is the eigenvector of  $\mathbf{Q}$  with the lowest eigenvalue. Pick a point  $\mathbf{x}_0$  along  $\mathbf{v}$ . In this case, the reachability energy for  $\mathbf{x}_0$  is low making the state easy to reach but the observability energy is also low, making the state difficult to observe. The solution to this problem is to pick a coordinate transform that makes the two quantities compatible. Mathematically, this problem equates to simultaneous diagonalization of  $\mathbf{P}$  and  $\mathbf{Q}$ .

**Proposition 2.8.1.** *For a reachable, observable, stable LTI system  $\Sigma$ , there exists a linear transformation  $\mathbf{T}$  such that  $\mathbf{TPT}^\top = \mathbf{T}^{-\top}\mathbf{QT}^{-1} = \mathbf{S} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ .*

**Proof.** Since  $\mathbf{P}$  is positive semi-definite, compute the Cholesky factors  $\mathbf{P} = \mathbf{U}\mathbf{U}^\top$  and find the eigenvalue decomposition of  $\mathbf{U}^\top\mathbf{Q}\mathbf{U} = \mathbf{K}\mathbf{S}^{1/2}\mathbf{K}^\top$ . Now  $\mathbf{T} = \mathbf{S}^{1/2}\mathbf{K}^\top\mathbf{U}^{-1}$  is the desired transformation. It can be checked that  $\mathbf{TPT}^\top = \mathbf{T}^{-\top}\mathbf{QT}^{-1} = \mathbf{S}$ .

**Remark 2.8.1.** *The construction of the balancing transformation given in Prop. 2.8.1 is not very practical since the singular values of  $\mathbf{P}$  are often very small, thereby making the calculation of  $\mathbf{U}^{-1}$  ill-conditioned.*

**Definition 2.8.1.** *A reachable, observable stable LTI system  $\Sigma$  is said to be balanced if  $\mathbf{P} = \mathbf{Q}$  and principle axis balanced if  $\mathbf{P} = \mathbf{Q} = \text{diag}(\sigma_1, \dots, \sigma_n)$ . The transform  $\mathbf{T}$  that transforms  $\Sigma$  to a balanced system is called a balancing transformation.*

## 2.8.2 Relation to projection-based methods

Before mentioning the balanced truncation algorithm, we introduce the framework behind projection-based methods for LTI systems. The general idea is to construct projection matrices  $\mathcal{V} \in \mathbb{R}^{n \times r}$ ,  $\mathcal{W} \in \mathbb{R}^{n \times r}$  and use them to project the system matrices  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  to a subspace of dimension  $r$ . The projection matrices are constructed by choosing  $\mathcal{V}$  such that  $\mathbf{x} \approx \mathcal{V}\hat{\mathbf{x}}_r$  and forcing the condition that  $\mathcal{W} \perp (\mathbf{V}\dot{\mathbf{x}}_r - \mathbf{A}\mathcal{V}\mathbf{x}_r - \mathbf{B}\mathbf{u})$ . This gives us,

$$0 = \underbrace{\mathcal{W}^\top \mathcal{V}}_{\mathbf{I}_r} \dot{\mathbf{x}}_r - \underbrace{\mathcal{W}^\top \mathbf{A} \mathcal{V}}_{\mathbf{A}_r} \mathbf{x}_r + \underbrace{\mathcal{W}^\top \mathbf{B}}_{\mathbf{B}_r} \mathbf{u},$$

$$\hat{\mathbf{y}} = \underbrace{\mathbf{C} \mathcal{V}}_{\mathbf{C}_r} \mathbf{x}_r,$$

such that  $\|\mathbf{y} - \hat{\mathbf{y}}\|$  is sufficiently small. The balanced truncation algorithm fits into this framework and the projection matrices  $\mathcal{V}$  and  $\mathcal{W}$  are calculated from the square root factors of the Gramians as shown in Algorithm 1.

**Remark 2.8.2.** *The  $\mathcal{W}^\top \mathcal{V}$  term need not be orthogonal in general but is true by construction in Balanced truncation.*

## 2.8.3 Mathematical formulation

Given a balanced realization  $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$  with gramian  $\mathbf{P} = \text{diag}(\sigma_1, \dots, \sigma_n)$ , partition the system

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \\ & \mathbf{P}_2 \end{bmatrix}$$

where  $\mathbf{A}_{11} \in \mathbb{R}^{r \times r}$ ,  $\mathbf{B}_1 \in \mathbb{R}^{r \times m}$ ,  $\mathbf{C}_1 \in \mathbb{R}^{p \times r}$ ,  $\mathbf{P}_1 \in \mathbb{R}^{r \times r}$ .

**Definition 2.8.2.** *Let  $\Sigma_i = (\mathbf{A}_{ii}, \mathbf{B}_i, \mathbf{C}_i)$ ,  $i \in \{1, 2\}$  be defined as the reduced systems obtained by balanced truncation.*

**Remark 2.8.3.** *The reduced system  $\Sigma_1$  is usually a good approximation of  $\Sigma$  if  $\sigma_r \gg \sigma_{r+1}$ .*

We present the balanced truncation algorithm below. Note that we don't explicitly compute the balancing transformation like in Prop. 2.8.1 but perform the balancing and reduction in

one step. This sidesteps the numerical errors and difficulties in computing the ill-conditioned matrices in Prop. 2.8.1.

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**Algorithm 1** Balanced Truncation for LTI systems

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**Given:** LTI system given by matrices  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times m}$ , and  $\mathbf{C} \in \mathbb{R}^{p \times n}$ .

**Output:** BT reduced-system:  $\mathbf{A}_r \in \mathbb{R}^{r \times r}$ ,  $\mathbf{B}_r \in \mathbb{R}^{r \times m}$ ,  $\mathbf{C}_r \in \mathbb{R}^{p \times r}$ .

1. Compute the Lyapunov factors  $\mathbf{U}, \mathbf{L} \in \mathbb{R}^{n \times n}$  for  $\mathbf{P}, \mathbf{Q}$  and pick a truncation index,  $r$ .
2. Compute the SVD of the matrix  $\mathbf{L}^\top \mathbf{U}$ , partitioned as follows

$$\mathbf{L}^\top \mathbf{U} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1^\top \\ \mathbf{Y}_2^\top \end{bmatrix},$$

where  $\mathbf{S}_1 \in \mathbb{R}^{r \times r}$  and  $\mathbf{S}_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ .

3. Construct the model reduction bases  $\mathbf{W}_r = \mathbf{LZ}_1 \mathbf{S}_1^{-1/2}$  and  $\mathbf{V}_r = \mathbf{UY}_1 \mathbf{S}_1^{-1/2}$
4. The reduced-order system matrices are given by

$$\begin{aligned} \mathbf{A}_r &= \mathbf{W}_r^\top \mathbf{A} \mathbf{V}_r = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top (\mathbf{L}^\top \mathbf{A} \mathbf{U}) \mathbf{Y}_1 \mathbf{S}_1^{-1/2}, \\ \mathbf{B}_r &= \mathbf{W}_r^\top \mathbf{B} = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top (\mathbf{L}^\top \mathbf{B}), \mathbf{C}_r = \mathbf{C} \mathbf{V}_r = (\mathbf{C} \mathbf{U}) \mathbf{Y}_1 \mathbf{S}_1^{-1/2}. \end{aligned}$$


---

Although balanced truncation does not ensure optimality of the ROM in a norm, it offers prior error bounds in  $\mathcal{H}_\infty$  norm as shown below in Thm. 2.8.2. It is easy to see that since we are merely truncating the matrices, the ROM continues to be stable if  $\Sigma$  is stable and the HSVs of  $\Sigma_1$  are a subset of  $\Sigma$ . This is important since many other MOR methods do not preserve the stability of the original system.

**Theorem 2.8.2.** *Given the reachable, observable, and stable (poles in the open left half plane) continuous-time system  $\Sigma$ , the reduced-order systems  $\Sigma_i, i = 1, 2$ , obtained by balanced truncation have the following properties:*

- $\Sigma_i$  is balanced and has no poles in the open right half plane.
- Let the distinct singular values of  $\Sigma$  be  $\sigma_i$  with multiplicities  $m_i, i = 1, \dots, q$ . Let  $\Sigma_1$  have singular values  $\sigma_i, i = 1, \dots, k$ , with the multiplicity  $m_i, i = 1, \dots, k, k < q$ .

The  $\mathcal{H}_\infty$ -norm of the difference between the full-order system  $\Sigma$  and the reduced-order system  $\Sigma_1$  is upper bounded by twice the sum of the neglected Hankel singular values, multiplicities not included:

$$\|\Sigma - \Sigma_1\|_{\mathcal{H}_\infty} \leq 2(\sigma_{k+1} + \dots + \sigma_q).$$

### 2.8.4 Numerical example

We show the performance of balanced truncation with the [heat] model which is a benchmark single-input single-output (SISO) LTI model with dimension 200. The Hankel singular values of this model quickly decay to machine precision ( $\approx 10^{-16}$ ) as can be seen in Fig. 2.1. We pick truncation index  $r = 20$  since the singular values that are neglected are of machine precision. We follow the steps mentioned in Algorithm 2 and plot the Hankel singular values of the ROM in Fig. 2.1. As it turns out the original system with  $n = 200$  can be well approximated by a surrogate model with dimension 20. The absolute error of the FOM and ROM response to input signal  $\mathbf{u}(t) = 5 \sin(\pi t/5) e^{-t/20}$  is of the order  $10^{-15}$ .

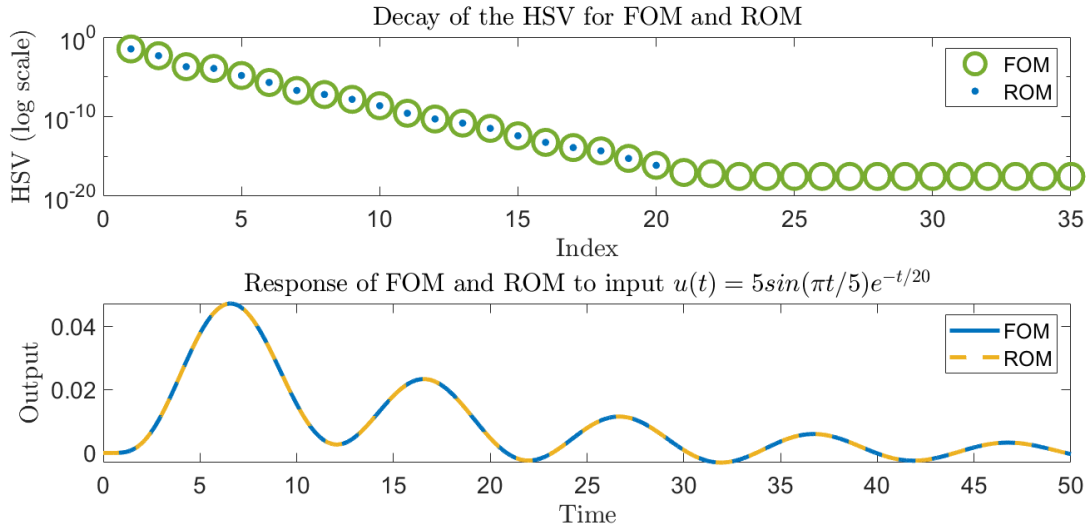


Figure 2.1: Comparison of the Hankel singular values (top) and output responses (bottom) of the FOM and ROM

## 2.9 Quadrature based Balanced truncation

In the previous sections, we mentioned balanced truncation as a technique for obtaining reduced-order models for linear systems. A drawback with this method is the fact that it requires access to the system matrices  $\mathbf{A}, \mathbf{B}, \dots$ , which are often not available for real-world systems. Quadrature-based balanced truncation is a data-driven method [8, 11] that aims at constructing reduced order models similar to those obtained by balanced truncation but constructed purely from data without knowledge about the system matrices. Data, in this case, corresponds to samples of the system kernels (in the time domain) or samples of the transfer function (in the frequency domain). In this section, we summarise the general idea and framework behind the method.

### 2.9.1 Quad BT for LTI systems

The computation of  $\mathbf{P}$  and  $\mathbf{Q}$  in classical BT is intrusive since it entails solving the Lyapunov equations (Prop. 2.6.1) involving the system matrices. The core idea of Quad BT is to replace the system gramians  $\mathbf{P}, \mathbf{Q}$  with quadrature-based approximations of the integrals. By replacing  $\mathbf{P}$  and  $\mathbf{Q}$  with quadrature based approximations -  $\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}$  we can approximate the terms  $\mathbf{L}^\top \mathbf{U}, \mathbf{L}^\top \mathbf{A} \mathbf{U}, \mathbf{L}^\top \mathbf{B}, \mathbf{C} \mathbf{U}$  appearing in Step 4 of Algo. 1 with  $\tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}, \tilde{\mathbf{L}}^\top \mathbf{A} \tilde{\mathbf{U}}, \tilde{\mathbf{L}}^\top \mathbf{B}, \mathbf{C} \tilde{\mathbf{U}}$  where  $\tilde{\mathbf{U}}, \tilde{\mathbf{L}}$  are the square root factors of the quadrature approximations of  $\tilde{\mathcal{P}}$  and  $\tilde{\mathcal{Q}}$ . Finally, it is shown that these quadrature-based matrices can be expressed exactly as samples of the kernels of the LTI system. We describe each of the steps in detail in the sections below.

### 2.9.2 Approximating $\mathbf{P}, \mathbf{Q}$

We begin by using quadrature approximation of the integral definitions of  $\mathbf{P}, \mathbf{Q}$  given in Def. 2.6.1 to construct  $\tilde{\mathcal{P}}$  and  $\tilde{\mathcal{Q}}$  respectively. The symmetric structure of  $\tilde{\mathcal{P}}$  and  $\tilde{\mathcal{Q}}$  allows us to construct their square root factors  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  as shown below.

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}\tau} \mathbf{B} \mathbf{B}^\top e^{\mathbf{A}^\top \tau} d\tau = \mathbf{U} \mathbf{U}^\top \quad (2.6)$$

$$\mathbf{Q} = \int_0^\infty e^{\mathbf{A}^\top t} \mathbf{C}^\top \mathbf{C} e^{\mathbf{A}t} dt = \mathbf{L} \mathbf{L}^\top. \quad (2.7)$$

$$\begin{aligned} \mathbf{P} &\approx \tilde{\mathcal{P}} = \sum_{i=1}^{N_p} \rho_i^2 e^{\mathbf{A}\mu_i} \mathbf{B} \mathbf{B}^\top e^{\mathbf{A}^\top \mu_i} = \tilde{\mathbf{U}} \tilde{\mathbf{U}}^\top & \mathbf{Q} &\approx \tilde{\mathcal{Q}} = \sum_{j=1}^{N_q} \phi_j^2 e^{\mathbf{A}^\top \omega_j} \mathbf{C}^\top \mathbf{C} e^{\mathbf{A} \omega_j} = \tilde{\mathbf{L}} \tilde{\mathbf{L}}^\top, \\ \mathbf{U} &\approx \tilde{\mathbf{U}} = [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{B}] & \mathbf{L} &\approx \tilde{\mathbf{L}} = [\phi_1 e^{\mathbf{A}^\top \omega_1} \mathbf{C}^\top \dots \phi_{N_q} e^{\mathbf{A}^\top \omega_{N_q}} \mathbf{C}^\top]. \end{aligned}$$

Here  $\mu_i, \omega_j$  and  $\rho_i, \phi_j$  are the quadrature nodes and associated quadrature weights for  $\mathbf{P}$  and  $\mathbf{Q}$  respectively.  $N_p$  and  $N_q$  denote the total number of quadrature nodes/points being used. In general, using higher nodes  $N_p, N_q$  results in better approximations to  $\mathbf{P}, \mathbf{Q}$ .

### 2.9.3 Replacing intrusive BT terms with data matrices

The key terms appearing in balanced truncation namely,  $\mathbf{L}^\top \mathbf{U}, \mathbf{L}^\top \mathbf{A} \mathbf{U}, \mathbf{L}^\top \mathbf{B}, \mathbf{C} \mathbf{U}$  are approximated by replacing  $\mathbf{L}$  and  $\mathbf{U}$  with  $\tilde{\mathbf{L}}$  and  $\tilde{\mathbf{U}}$ . We then show that the elements of the matrices  $\tilde{\mathbf{H}} = \tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}, \tilde{\mathbf{M}} = \tilde{\mathbf{L}}^\top \mathbf{A} \tilde{\mathbf{U}}, \tilde{\mathbf{h}} = \tilde{\mathbf{L}}^\top \mathbf{B}$  and  $\tilde{\mathbf{g}} = \mathbf{C} \tilde{\mathbf{U}}$  can be expressed as samples of the kernels of the system.

**Theorem 2.9.1.** *Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 2.6, then we have,*

$$\begin{aligned} \tilde{\mathbf{H}} &= \rho_i \phi_j \mathbf{h}_1(\mu_i + \omega_j), \text{ for } 1 \leq i \leq N_p \text{ \& } 1 \leq j \leq N_q \\ \tilde{\mathbf{M}} &= \rho_i \phi_j \mathbf{h}'_1(\mu_i + \omega_j), \text{ for } 1 \leq i \leq N_p \text{ \& } 1 \leq j \leq N_q \\ \tilde{\mathbf{h}} &= \phi_j \mathbf{h}_1(\omega_j), \text{ for } 1 \leq j \leq N_q \\ \tilde{\mathbf{g}} &= \rho_i \mathbf{h}_1(\mu_i), \text{ for } 1 \leq k \leq N_p \end{aligned}$$

The theorem above gives us a strategy to derive a reduced-order model using substitutes for the intrusive terms in balanced truncation. We summarise the Quad BT algorithm for LTI systems in Algo. 2 and note that we only require samples of the kernels and their derivatives to obtain the reduced-order model.

---

**Algorithm 2** Quadrature based Balanced Truncation for LTI systems
 

---

**Input:** Samples of  $\mathbf{h}_1$  and  $\mathbf{h}'_1$  quadrature weights  $\rho_1, \dots, \rho_{N_p}, \phi_1, \dots, \phi_{N_q}$  and nodes  $\mu_1, \dots, \mu_{N_p}, \omega_1, \dots, \omega_{N_q}$

**Output:** The reduced order matrices  $\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r$

1. Construct the data matrices  $\tilde{\mathbf{H}}, \tilde{\mathbf{M}}, \tilde{\mathbf{h}}, \tilde{\mathbf{g}}$  using the appropriate quadrature nodes  $\rho_i, \phi_j$  as per Thm. 4.2.1, 4.2.2 and 4.2.3. Choose an appropriate truncation index  $r$  while ensuring that  $r \leq \min(N_p, N_q)$ .
2. Compute SVD of  $\tilde{\mathbf{L}}$  and partition the matrices as follows,

$$\tilde{\mathbf{L}} = \begin{bmatrix} \tilde{\mathbf{Z}}_1 & \tilde{\mathbf{Z}}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{S}}_1 & \\ & \tilde{\mathbf{S}}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{Y}}_1^\top \\ \tilde{\mathbf{Y}}_2^\top \end{bmatrix}$$

where  $\tilde{\mathbf{Z}}_1 \in \mathbb{R}^{N_q \times r}$ ,  $\tilde{\mathbf{Z}}_2 \in \mathbb{R}^{N_q \times (N_q - r)}$ ,  $\tilde{\mathbf{S}}_1 \in \mathbb{R}^{r \times r}$ ,  
 $\tilde{\mathbf{S}}_2 \in \mathbb{R}^{(N_q - r) \times (N_q - r)}$ ,  $\tilde{\mathbf{Y}}_1^\top \in \mathbb{R}^{r \times N_p}$  and  $\tilde{\mathbf{Y}}_2^\top \in \mathbb{R}^{(N_q - r) \times N_p}$ .

3. Compute the reduced matrices as follows

$$\begin{aligned} \mathbf{A}_r &= \mathbf{S}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{M}} \tilde{\mathbf{Y}}_1 \mathbf{S}_1^{-1/2} & \mathbf{B}_r &= \mathbf{S}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{h}} \\ \mathbf{C}_r &= \tilde{\mathbf{g}}^\top \tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{S}}_1^{-1/2} \end{aligned}$$


---

### 2.9.4 Error bounds for Quad BT

In Ref. [8], the authors provide some error bounds on the singular values of the ROM. They show that for sufficiently large and well-chosen quadrature nodes, the singular values of the FOM are well approximated by those of the ROM.

**Proposition 2.9.2.** [8] *Suppose the quadratures in (3.1) and (3.3) produce approximations  $\tilde{\mathbf{P}}$  and  $\tilde{\mathbf{Q}}$  to  $\mathbf{P}$  and  $\mathbf{Q}$ , respectively, that satisfy  $\|\mathbf{Q} - \tilde{\mathbf{Q}}\|_F \leq \frac{\delta}{1+\delta} \sigma_{\min}(\mathbf{Q})$  and  $\|\mathbf{P} - \tilde{\mathbf{P}}\|_F \leq \frac{\delta}{1+\delta} \sigma_{\min}(\mathbf{P})$  for some  $\delta \in (0, 1)$ , where  $\sigma_{\min}(\cdot)$  denotes the smallest singular value. Then there exist isometries  $\Psi_p \in \mathbb{C}^{N_p \times n}$  and  $\Psi_q \in \mathbb{C}^{N_q \times n}$  such that*

$$\left\| \tilde{\mathbf{L}}^* \mathbf{E} \tilde{\mathbf{U}} - \Psi_q (\mathbf{L}^T \mathbf{E} \mathbf{U}) \Psi_p^* \right\|_F \leq 2\delta \|\mathbf{E}\|_2 \|\mathbf{L}\|_2 \|\mathbf{U}\|_2$$

Let  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$  denote singular values of  $\mathbf{L}^T \mathbf{E} \mathbf{U}$  (i.e., Hankel singular values of (2.1)), and let  $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \dots \geq \tilde{\sigma}_n$  denote the singular values of  $\tilde{\mathbf{L}}^* \mathbf{E} \tilde{\mathbf{U}}$ . Then,

$$\left( \sum_{k=1}^n (\sigma_k - \tilde{\sigma}_k)^2 \right)^{\frac{1}{2}} \leq 2\delta \|\mathbf{E}\|_2 \|\mathbf{L}\|_2 \|\mathbf{U}\|_2$$

The proposition above holds not only for linear systems but also for other systems like the ones mentioned in Chap. 4 and Chap. 5.

### 2.9.5 Numerical examples

In this section, we show the performance of the Quad BT algorithm on a benchmark dataset. The first system we use for testing is the International Space Station benchmark (referred to as [iss1r]) [22] modelling the 1r component of the International Space Station. It has a system dimension of 270, an input dimension of 3 and an output dimension of 3. We only consider the first component of the input and output to obtain a SISO linear system. In Fig. 2.2, we show the output response of the FOM, ROM obtained by BT ( $r = 30$ ) and ROM ( $r = 30$ ) obtained by Quad BT to input  $\mathbf{u}(t) = 5(\cos(5\pi t) + \sin(12\pi t) \exp(-0.4t))$ . The absolute errors of the outputs are comparable and of the order  $10^{-6}$ . Fig. 2.3 on the other hand, shows the Hankel singular values of the three systems. We notice that the QBT ROM closely approximates the true HSVs nearly as well as balanced truncation.

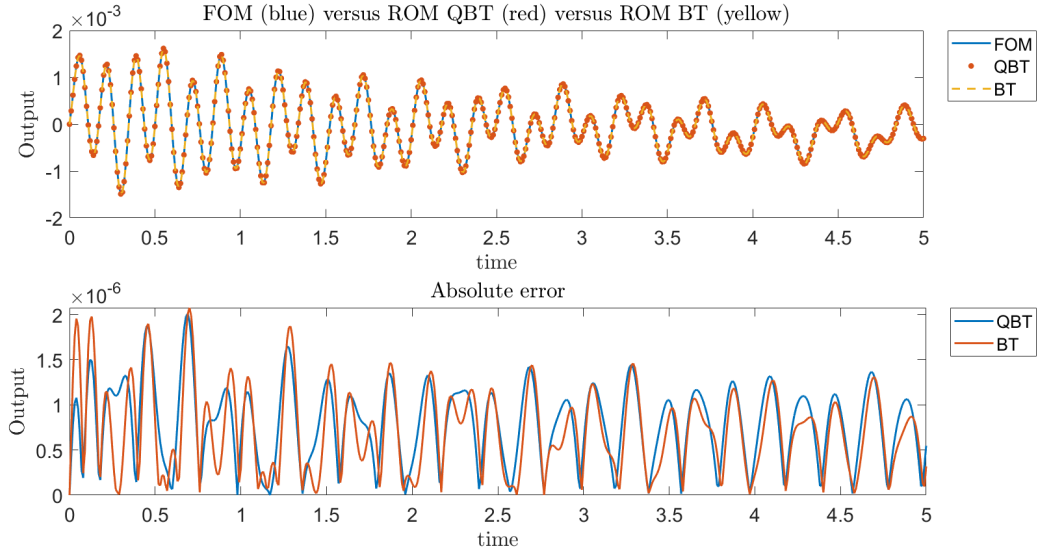


Figure 2.2: Comparison of the output responses of the FOM and ROM for iss1R system(top) and comparison of the absolute error between the ROM and FOM for balanced truncation and QBT

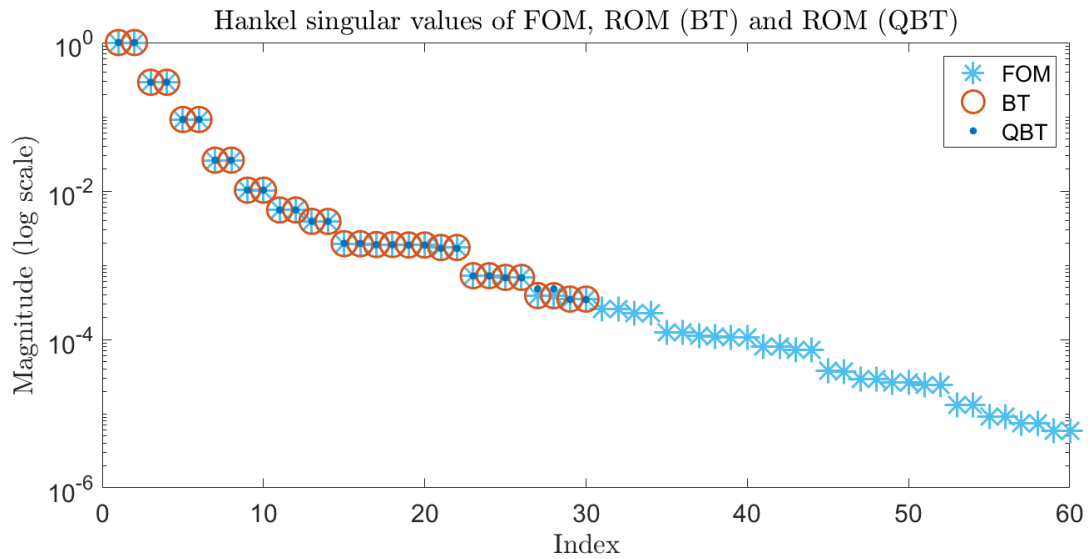


Figure 2.3: Comparison of the Hankel singular values (top) and output responses (bottom) of the FOM and ROM for iss1R system

# Chapter 3

## BT for bilinear systems with quadratic output (BQO)

Bilinear control systems are an important class of systems with a wide range of applications in many fields of applied sciences. For example, bilinear systems represent surrogates for virtually any class of nonlinear systems with analytic nonlinearities. In this chapter, we focus on bilinear systems with quadratic terms in their output - bilinear systems with quadratic output (BQO). A BQO system  $\Sigma^{\text{BQO}}$ , is given by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{x}(t)^\top \mathbf{M}\mathbf{x}(t)\end{aligned}\tag{3.1}$$

where  $\mathbf{M} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times p}$ ,  $\mathbf{C} \in \mathbb{R}^{1 \times n}$ ,  $\mathbf{N} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{u}(t) \in \mathbb{R}^p$ ,  $\mathbf{x}(t) \in \mathbb{R}^n$  and  $\mathbf{y}(t) \in \mathbb{R}$ . In the literature, the term ‘bilinear systems’ refers to BQO systems with  $\mathbf{M} = \mathbf{0}$ . BQO systems (with  $\mathbf{M} \neq \mathbf{0}$ ) are important when we are interested in studying some form of energies of a bilinear system. Note that, without loss of generality, we can assume  $\mathbf{M}$  to be a symmetric matrix. This follows by observing that replacing  $\mathbf{x}^\top \mathbf{M}\mathbf{x}$  with  $\mathbf{x}^\top \frac{(\mathbf{M} + \mathbf{M}^\top)}{2} \mathbf{x}$  does not change the output of the system. Even though there have been many studies on bilinear systems (with linear output), the theory for BQO systems hasn’t been explored enough (see Ref. [23]). The naive approach would be to transform these BQO systems into bilinear systems with linear output (as shown in Sec. 3.1) using lifting transforms. This would allow us to study BQO systems using classical methods for bilinear systems. However, enforcing

the bilinear structure comes at the expense of having a higher state dimension of the lifted bilinear system. As an alternative, we derive an extension of the classic balanced truncation algorithm for BQO systems. We begin by using the Volterra series representation for BQO systems (Sec. 3.2) to define the Gramians of BQO systems (Sec. 3.3). Moreover, we use the newly defined Gramians to present the balanced truncation algorithm for BQO systems in Sec. 3.5.

### 3.1 Lifting BQO systems to bilinear system with linear output

The system  $\Sigma^{\text{BQO}}$  can be lifted to a bilinear system  $\Sigma^{\text{B}}$  by lifting transforms. The  $\mathbf{x}^\top \mathbf{M} \mathbf{x}$  term appearing in the expression for  $\mathbf{y}$  can be written as  $\mathbf{K}(\mathbf{x} \otimes \mathbf{x})$  where  $\mathbf{K} = \text{vec}(\mathbf{M})^\top$ . Now introduce the variable  $\mathbf{z} = (\mathbf{x} \otimes \mathbf{x})$ . Then we have,

$$\begin{aligned}
\dot{\mathbf{z}} &= \dot{\mathbf{x}} \otimes \mathbf{x} + \mathbf{x} \otimes \dot{\mathbf{x}} \\
&= (\mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}) \otimes \mathbf{x} + \mathbf{x} \otimes (\mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}) \\
&= (\mathbf{A} \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{A})(\mathbf{x} \otimes \mathbf{x}) + (\mathbf{N} \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{N})(\mathbf{x} \otimes \mathbf{x})\mathbf{u} + (\mathbf{B} \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{B})\mathbf{x}\mathbf{u} \\
&= (\mathbf{A} \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{A})\mathbf{z} + (\mathbf{N} \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{N})\mathbf{z}\mathbf{u} + (\mathbf{B} \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{B})\mathbf{x}\mathbf{u} \\
&= \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{N}}\mathbf{z}\mathbf{u} + \bar{\mathbf{B}}\mathbf{x}\mathbf{u}.
\end{aligned}$$

Define the new system  $\Sigma^{\text{B}}$  as,

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}, \tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \bar{\mathbf{A}} \end{bmatrix}, \tilde{\mathbf{N}} = \begin{bmatrix} \mathbf{N} & 0 \\ \bar{\mathbf{B}} & \bar{\mathbf{N}} \end{bmatrix}, \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & \mathbf{K} \end{bmatrix}$$

We have the state equations of  $\Sigma^{\text{B}}$  given below,

$$\begin{aligned}
\dot{\tilde{\mathbf{x}}} &= \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \bar{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{N} & 0 \\ \bar{\mathbf{B}} & \bar{\mathbf{N}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= \begin{bmatrix} \mathbf{C} & \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}
\end{aligned}$$

Note here that  $\mathbf{u}$  and  $\mathbf{y}$  are the same as in  $\Sigma^{\text{BQO}}$ . The lifted system  $\Sigma^{\text{B}}$  has the same input-output response but different state dynamics and dimensions. The gramians  $\mathbf{P}^{\text{B}}$  and  $\mathbf{Q}^{\text{B}}$  of  $\Sigma^{\text{B}}$  satisfy the linear matrix equations,

$$\bar{\mathbf{A}}\mathbf{P}^{\text{B}} + \mathbf{P}^{\text{B}}\bar{\mathbf{A}}^{\top} + \bar{\mathbf{N}}\mathbf{P}^{\text{B}}\bar{\mathbf{N}}^{\top} + \bar{\mathbf{B}}\bar{\mathbf{B}}^{\top} = 0 \quad \bar{\mathbf{A}}^{\top}\mathbf{Q}^{\text{B}} + \mathbf{Q}^{\text{B}}\bar{\mathbf{A}} + \bar{\mathbf{N}}^{\top}\mathbf{Q}^{\text{B}}\bar{\mathbf{N}} + \bar{\mathbf{C}}^{\top}\bar{\mathbf{C}} = 0$$

The lifted bilinear system has dimension  $n+n^2$  if the dimension of the original system was  $n$ . This is bad news especially when  $n$  becomes large. This highlights the need for an algorithm to obtain reduced-order models specifically for BQO systems. With this motivation in mind, we proceed by defining the Gramians for BQO systems and build towards formulating the balanced truncation algorithm for BQO systems.

## 3.2 Kernels of BQO systems

Consider a bilinear system  $\Sigma^{\text{B}}$ . The idea is to rewrite the bilinear system as a sum of infinite subsystems. We consider inputs of the form  $\alpha\mathbf{u}(t)$  where  $\alpha$  is an arbitrary scalar. Since the system  $\Sigma^{\text{B}}$  is analytic, we can write the solution as a power series in  $\alpha$ ,  $\mathbf{x}(t) = \sum_{k=1}^{\infty} \alpha^k \mathbf{x}_k(t)$ , where  $\mathbf{x}_k(t)$  denotes the  $k$ -th sublevel of the system. The solution of the system with input  $\alpha\mathbf{u}(t)$  is now given by

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{\infty} \alpha^k \dot{\mathbf{x}}_k(t) \tag{3.2}$$

$$= \mathbf{A}\mathbf{x}(t) + \alpha\mathbf{B}\mathbf{u}(t) + \alpha\mathbf{N}\mathbf{x}(t)\mathbf{u}(t)$$

$$= \mathbf{A} \left( \sum_{k=1}^{\infty} \alpha^k \mathbf{x}_k(t) \right) + \alpha\mathbf{B}\mathbf{u}(t) + \alpha\mathbf{N} \left( \sum_{k=1}^{\infty} \alpha^k \mathbf{x}_k(t) \right) \mathbf{u}(t) \tag{3.3}$$

Since the choice of  $\alpha$  is arbitrary, we must have that the terms corresponding to  $\alpha^k$  in Eq. 3.2 and Eq. 3.3 are equal for all values of  $k$ . Using this, we obtain  $\mathbf{x}(t) = \sum_{k=1}^{\infty} \mathbf{x}_k(t)$  with the

equations for  $\mathbf{x}_k$  given by -

$$\begin{aligned}\dot{\mathbf{x}}_1(t) &= \mathbf{A}\mathbf{x}_1(t) + \mathbf{B}\mathbf{u}(t) \\ &\vdots \\ \dot{\mathbf{x}}_k(t) &= \mathbf{A}\mathbf{x}_k(t) + \mathbf{N}\mathbf{x}_{k-1}(t)\mathbf{u}(t) \quad \text{for } k > 1\end{aligned}$$

Let  $z(t) = e^{-\mathbf{A}t}\mathbf{x}(t)$ . Then computing  $\dot{z}(t)$  we get,

$$\dot{z}(t) = -\mathbf{A}e^{-\mathbf{A}t}\mathbf{x}(t) + e^{-\mathbf{A}t}\dot{\mathbf{x}}(t)$$

For  $k = 1$ , we have  $\mathbf{x}_1(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$ . For  $k > 1$ ,

$$\begin{aligned}\dot{\mathbf{z}}_k(\tau) &= -e^{-\mathbf{A}\tau}\mathbf{A}\mathbf{x}_k(\tau) + e^{-\mathbf{A}\tau}[\mathbf{A}\mathbf{x}_k(\tau) + \mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)] \\ \mathbf{z}_k(t) &= \int_0^t e^{-\mathbf{A}\tau}\mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)d\tau \\ \mathbf{x}_k(t) &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)d\tau\end{aligned}$$

Summing up the  $\mathbf{x}_k$  terms we get,

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \sum_{k=2}^{\infty} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)d\tau$$

Using this expression to solve for  $\mathbf{y}$  gives us,

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{C} \sum_{i=1}^{\infty} \mathbf{x}_i(t) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{x}_i(t)^\top \mathbf{M} \mathbf{x}_j(t) \\
&= \boxed{\int_{t=0}^{\infty} \mathbf{C} e^{\mathbf{A}t} \mathbf{B} \mathbf{u} dt + \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \mathbf{C} e^{\mathbf{A}t_1} \mathbf{N} e^{\mathbf{A}t_2} \mathbf{B} \mathbf{u} dt + \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \int_{t_3=0}^{\infty} \mathbf{C} e^{\mathbf{A}t_1} \mathbf{N} e^{\mathbf{A}t_2} \mathbf{N} e^{\mathbf{A}t_3} \mathbf{B} \mathbf{u} dt + \dots} \\
&+ \boxed{\int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \mathbf{B}^\top e^{\mathbf{A}^\top t_1} \mathbf{M} e^{\mathbf{A}t_2} \mathbf{B} \mathbf{u} dt} + \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \int_{t_3=0}^{\infty} \mathbf{B}^\top e^{\mathbf{A}^\top t_1} \mathbf{M} e^{\mathbf{A}t_2} \mathbf{N} e^{\mathbf{A}t_3} \mathbf{B} \mathbf{u} dt + \dots \\
&+ \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \int_{t_3=0}^{\infty} \mathbf{B}^\top e^{\mathbf{A}^\top t_1} \mathbf{N}^\top e^{\mathbf{A}^\top t_2} \mathbf{M} e^{\mathbf{A}t_3} \mathbf{B} \mathbf{u} dt + \dots \\
&\vdots \\
&+ \dots + \int_{t_1=0}^{\infty} \dots \int_{t_i=0}^{\infty} \int_{t_{i+1}=0}^{\infty} \dots \int_{t_{i+j}=0}^{\infty} e^{\mathbf{A}^\top t_1} \mathbf{N}^\top \dots \mathbf{N}^\top e^{\mathbf{A}^\top t_i} \mathbf{M} e^{\mathbf{A}t_{i+1}} \mathbf{N} \dots \mathbf{N} e^{\mathbf{A}t_{i+j}} \mathbf{B} \mathbf{u} dt + \dots
\end{aligned} \tag{3.4}$$

**Notation 3.2.1.** For ease of notation, instead of denoting the  $dt_i$  in the integrals, we just write  $dt$ . We represent the input factors in the kernels with  $\mathbf{u}$ .

In Eq. 3.4, we can see that we have an infinite number of kernels as is the case for nonlinear systems. Even though we represent the input terms by  $\mathbf{u}$ , there are actually multiple  $\mathbf{u}(t - t_1), \mathbf{u}(t - t_2), \dots$  terms appearing in the expression. The key takeaway from the equation is the kernels since we will define the gramians using the structure of these kernels.

**Remark 3.2.1.** We notice that setting  $\mathbf{M} = 0$  recovers the kernels of the bilinear system (terms in the black box) and similarly, setting  $\mathbf{N} = \mathbf{C} = 0$  recovers the kernels of the LQO system (term in the brown box).

### 3.3 Gramians of BQO system

Let  $\mathbf{P}^B$  and  $\mathbf{Q}^B$  represent the Gramians of the bilinear system  $\Sigma^B = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{N})$  which satisfy the following equations,

$$\begin{aligned}\mathbf{A}\mathbf{P}^B + \mathbf{P}^B\mathbf{A}^\top + \mathbf{N}\mathbf{P}^B\mathbf{N}^\top + \mathbf{B}\mathbf{B}^\top &= 0 \\ \mathbf{A}^\top\mathbf{Q}^B + \mathbf{Q}^B\mathbf{A} + \mathbf{N}^\top\mathbf{Q}^B\mathbf{N} + \mathbf{C}^\top\mathbf{C} &= 0\end{aligned}$$

The expression in Eq. 3.4 can provide us with some insight into what the definition for the BQO Gramians should be. Since the state equation is the same as in the bilinear case, the reachability Gramian  $\mathbf{P}$  for the BQO system and the bilinear system  $\mathbf{P}^B$  match i.e.  $\mathbf{P} = \mathbf{P}^B$ .

**Definition 3.3.1.** [10] *The reachability Gramian  $\mathbf{P}$  for the BQO system (and bilinear system) is defined as*

$$\begin{aligned}\mathbf{P}_k^B = \mathbf{P}_k &= \int_0^\infty \hat{\mathbf{P}}_k(t_1, \dots, t_k) \hat{\mathbf{P}}_k(t_1, \dots, t_k)^\top dt_1 \dots dt_k \\ \mathbf{P}^B = \mathbf{P} &= \sum_{k=1}^\infty \mathbf{P}_k\end{aligned}$$

where

$$\begin{aligned}\hat{\mathbf{P}}_1(t) &= e^{\mathbf{A}t_1}\mathbf{B}, \\ \hat{\mathbf{P}}_k(t_1, \dots, t_k) &= e^{\mathbf{A}t_k}\mathbf{N}\hat{\mathbf{P}}_{k-1}.\end{aligned}$$

The Gramian satisfies the matrix equation

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{N}\mathbf{P}\mathbf{N}^\top + \mathbf{B}\mathbf{B}^\top = 0$$

We define the observability Gramian of the BQO system  $\mathbf{Q}$  as follows.

**Definition 3.3.2.** *Let  $\mathbf{Q}^B$  be the bilinear observability Gramian which satisfies*

$$\mathbf{A}^\top\mathbf{Q}^B + \mathbf{Q}^B\mathbf{A} + \mathbf{N}^\top\mathbf{Q}^B\mathbf{N} + \mathbf{C}^\top\mathbf{C} = 0$$

and  $\mathbf{Q}_{ij}$  be defined as

$$\begin{aligned}\hat{\mathbf{Q}}_{1,j} &= e^{\mathbf{A}^\top t} \mathbf{M} \hat{\mathbf{P}}_j && (\text{for } i = 1) \\ \hat{\mathbf{Q}}_{i,j} &= e^{\mathbf{A}^\top t} \mathbf{N}^\top \hat{\mathbf{Q}}_{i-1,j} && (\text{for } i > 1) \\ \mathbf{Q}_{i,j} &= \int_0^\infty \cdots \int_0^\infty \hat{\mathbf{Q}}_{i,j} \hat{\mathbf{Q}}_{i,j}^\top dt_1 \dots dt_{i+j}\end{aligned}$$

Then observability Gramian  $\mathbf{Q}$  is given by  $\mathbf{Q} = \mathbf{Q}^B + \sum_{i=1}^\infty \sum_{j=1}^\infty \mathbf{Q}_{ij}$ .

**Remark 3.3.1.** The  $\mathbf{Q}$  term can be expressed as a sum of the  $\mathbf{Q}^B$  term, which corresponds to the bilinear Gramian and the newly defined  $\mathbf{Q}_{ij}$  terms. Each  $\mathbf{Q}_{ij}$  term corresponds to the contribution arising from  $\mathbf{x}_i^\top \mathbf{M} \mathbf{x}_j$ .

The definitions of these Gramian components are inspired by the kernels appearing in the expression for  $\mathbf{y}$  in Eq. 3.4. Now just like in the linear case, we want to express these Gramians as the solution of matrix equations. But first, we notice that for  $i = j = 1$  the  $\mathbf{Q}_{11}$  component corresponds to the LQO Gramian and satisfies the following Lyapunov equation,

$$\mathbf{A}^\top \mathbf{Q}_{11} + \mathbf{Q}_{11} \mathbf{A} + \mathbf{M} \mathbf{P}_1 \mathbf{M} = 0$$

We divide the rest of the components into two cases,  $(i = 1, j > 1)$  and  $(i > 1, j \geq 1)$ .

**Proposition 3.3.1.** Suppose we have a stable BQO system  $\Sigma$ , the Gramian components  $\mathbf{Q}_{1j}$  for  $j \geq 2$  in Defn. 3.3.2 satisfy the following Lyapunov equations,

$$\mathbf{A}^\top \mathbf{Q}_{1j} + \mathbf{Q}_{1j} \mathbf{A} + \mathbf{M} \mathbf{P}_j \mathbf{M} = 0$$

**Proof.** We begin with the case  $(i = 1)$ ,  $\mathbf{Q}_{1j}$  for  $j \geq 2$ . We make use of the fact that  $\mathbf{A}$  has

negative eigenvalues ( $\Sigma$  is stable) and hence  $e^{\mathbf{A}t}$  approaches 0 as  $t \rightarrow \infty$ .

$$\begin{aligned}
\mathbf{A}^\top \mathbf{Q}_{1j} + \mathbf{Q}_{1j} \mathbf{A} &= \mathbf{A}^\top \int_{t_1=0}^{\infty} e^{\mathbf{A}^\top t_1} \mathbf{M} \mathbf{P}_j \mathbf{M} e^{\mathbf{A} t_1} dt_1 \\
&+ \int_{t=0}^{\infty} e^{\mathbf{A}^\top t} \mathbf{M} \mathbf{P}_j \mathbf{M} e^{\mathbf{A} t} \mathbf{A} dt \\
&= \int_{t=0}^{\infty} d(e^{\mathbf{A}^\top t} \mathbf{M} \mathbf{P}_j \mathbf{M} e^{\mathbf{A} t}) \\
&= \left[ e^{\mathbf{A}^\top t} \mathbf{M} \mathbf{P}_j \mathbf{M} e^{\mathbf{A} t} \right]_{t=0}^{\infty} \\
&= -\mathbf{M} \mathbf{P}_j \mathbf{M} \quad \square
\end{aligned}$$

Now let's consider the  $i > 1$  case,

**Proposition 3.3.2.** *Suppose we have a stable BQO system  $\Sigma$ , the Gramian components  $\mathbf{Q}_{ij}$  for  $i \geq 2$  in Defn. 3.3.2 satisfy the following Lyapunov equations*

$$\mathbf{A}^\top \mathbf{Q}_{ij} + \mathbf{Q}_{ij} \mathbf{A} + \mathbf{N}^\top \mathbf{Q}_{i-1,j} \mathbf{N} = 0$$

**Proof.** The expression for  $\mathbf{Q}_{ij}$  is given by,

$$\mathbf{Q}_{ij} = \int_{t=0}^{\infty} e^{\mathbf{A}^\top t} \mathbf{N}^\top \mathbf{Q}_{i-1,j} \mathbf{N} e^{\mathbf{A} t} dt$$

Now we follow the same procedure as in Prop. 3.3.1 and make use of the stability of  $\Sigma$ .

$$\begin{aligned}
\mathbf{A}^\top \mathbf{Q}_{ij} + \mathbf{Q}_{ij} \mathbf{A} &= \mathbf{A}^\top \int_{t_1=0}^{\infty} e^{\mathbf{A}^\top t_1} \mathbf{N}^\top \mathbf{Q}_{i-1,j} \mathbf{N} e^{\mathbf{A} t_1} dt_1 \\
&+ \int_{t_1=0}^{\infty} e^{\mathbf{A}^\top t_1} \mathbf{N}^\top \mathbf{Q}_{i-1,j} \mathbf{N} e^{\mathbf{A} t_1} \mathbf{A} dt_1 \\
&= \int_{t_1=0}^{\infty} d(e^{\mathbf{A}^\top t_1} \mathbf{N}^\top \mathbf{Q}_{i-1,j} \mathbf{N} e^{\mathbf{A} t_1}) \\
&= \left[ e^{\mathbf{A}^\top t} \mathbf{N}^\top \mathbf{Q}_{i-1,j} \mathbf{N} e^{\mathbf{A} t} \right]_{t=0}^{\infty} \\
&= -\mathbf{N}^\top \mathbf{Q}_{i-1,j} \mathbf{N} \quad \square
\end{aligned}$$

**Corollary 3.3.3.** *Suppose we have a stable BQO system  $\Sigma$ , the observability and reachability*

Gramians  $\mathbf{P}, \mathbf{Q}$  satisfy the following Lyapunov equations

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{N}\mathbf{P}\mathbf{N}^\top + \mathbf{B}\mathbf{B}^\top = 0 \quad (3.5)$$

$$\mathbf{A}^\top\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{M}\mathbf{P}\mathbf{M} + \mathbf{N}^\top\mathbf{Q}\mathbf{N} + \mathbf{C}^\top\mathbf{C} = 0 \quad (3.6)$$

**Remark 3.3.2.** We notice that setting  $\mathbf{N} = 0$  recovers the Lyapunov equation for the LQO Gramian. Similarly, setting  $\mathbf{M} = 0$  recovers the matrix equations for the bilinear Gramians.

**Theorem 3.3.4.** Let reachability Gramian  $\mathbf{P}$  and observability Gramian  $\mathbf{Q}$  solve Eq. 3.5 and 3.6. Then, we have the following results

(a) If  $\mathbf{x} \in \ker(\mathbf{P})$ , then  $\mathbf{x}$  is unreachable.

(b) If  $\mathbf{P} > 0$  and the state  $\mathbf{x} \in \ker(\mathbf{Q})$ , then  $\mathbf{x}$  is unobservable.

**Proof.** (a) The proof for this part is the same as in the bilinear case given in Ref. [10]. (b) The observability Gramian  $\mathbf{Q}$  satisfies the equation,

$$\mathbf{A}^\top\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{M}\mathbf{P}\mathbf{M} + \mathbf{N}^\top\mathbf{Q}\mathbf{N} + \mathbf{C}^\top\mathbf{C} = 0. \quad (3.7)$$

Let  $\mathbf{v} \in \ker(\mathbf{Q})$ , multiply the above equation by  $\mathbf{v}$  from the right and  $\mathbf{v}^\top$  from the left.

$$\begin{aligned} \mathbf{v}^\top (\mathbf{A}^\top\mathbf{Q} + \mathbf{Q}\mathbf{A}\mathbf{v} + \mathbf{M}\mathbf{P}\mathbf{M} + \mathbf{N}^\top\mathbf{Q}\mathbf{N} + \mathbf{C}^\top\mathbf{C}) \mathbf{v} &= 0, \\ \Rightarrow \mathbf{v}^\top\mathbf{M}\mathbf{P}\mathbf{M}\mathbf{v} + \mathbf{v}^\top\mathbf{N}^\top\mathbf{Q}\mathbf{N}\mathbf{v} + \mathbf{v}^\top\mathbf{C}^\top\mathbf{C}\mathbf{v} &= 0, \end{aligned}$$

Since the terms  $\mathbf{M}\mathbf{P}\mathbf{M}$ ,  $\mathbf{N}^\top\mathbf{Q}\mathbf{N}$  and  $\mathbf{C}^\top\mathbf{C}$  are positive semi-definite matrices, we have  $\mathbf{P}\mathbf{M}\mathbf{v} = 0$ ,  $\mathbf{Q}\mathbf{N}\mathbf{v} = 0$  and  $\mathbf{C}\mathbf{v} = 0$ . Multiply Eq. 3.7 by  $\mathbf{v}$  from the right,

$$\mathbf{A}^\top\mathbf{Q}\mathbf{v} + \mathbf{Q}\mathbf{A}\mathbf{v} + \mathbf{M}\mathbf{P}\mathbf{M}\mathbf{v} + \mathbf{N}^\top\mathbf{Q}\mathbf{N}\mathbf{v} + \mathbf{C}^\top\mathbf{C}\mathbf{v} = 0$$

This gives us  $\mathbf{Q}\mathbf{A}\mathbf{v} = 0$ . Let  $\mathbf{q}^\top \in \text{Im}(\mathbf{Q})$  and  $\mathbf{x} \in \ker(\mathbf{Q})$ ,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) \\ \mathbf{q}^\top\dot{\mathbf{x}}(t) &= \mathbf{q}^\top\mathbf{A}\mathbf{x}(t) + \mathbf{q}^\top\mathbf{N}\mathbf{x}(t)\mathbf{u}(t) \\ &= 0 \quad (\text{Since } \mathbf{Q}\mathbf{A}\mathbf{x} = 0 \text{ and } \mathbf{Q}\mathbf{N}\mathbf{x} = 0) \end{aligned}$$

This implies that if  $\mathbf{x} \in \ker(\mathbf{Q})$ , then  $\dot{\mathbf{x}} \in \ker(\mathbf{Q})$ . Thus a state in the kernel of  $\mathbf{Q}$  remains in the kernel. We assumed  $\mathbf{P} > 0$  which means the system is controllable and hence we can

write  $\mathbf{x} = \mathbf{P}\mathbf{x}_0$ .

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{C}\mathbf{x} + \mathbf{x}^\top \mathbf{M}\mathbf{x} \\ &= \mathbf{C}\mathbf{x} + \mathbf{x}_0^\top \mathbf{P}\mathbf{M}\mathbf{x} \\ &= 0 \quad (\text{Since } \mathbf{P}\mathbf{M}\mathbf{x} = 0, \mathbf{C}\mathbf{x} = 0 \text{ for } \mathbf{x} \in \ker(\mathbf{Q})) \end{aligned}$$

The output  $\mathbf{y}(t)$  is 0 whenever  $\mathbf{x} \in \ker(\mathbf{Q})$  and a state in the kernel of  $\mathbf{Q}$  continues to be in the kernel. Thus, the state  $\mathbf{x}$  is indistinguishable from the zero state making it unobservable.  $\square$

### 3.4 Observability from the dual system

We defined the observability Gramian for the BQO system in the previous section. In this section, we show that the Gramians can also be derived as the reachability Gramian of the dual system. With this motivation in mind, consider a nonlinear system of the form

$$\Sigma := \begin{cases} \dot{\mathbf{x}}(t) = \mathcal{A}(\mathbf{x}, \mathbf{u}, t)\mathbf{x}(t) + \mathcal{B}(\mathbf{x}, \mathbf{u}, t)\mathbf{u}(t), \\ \mathbf{y}(t) = \mathcal{C}(\mathbf{x}, \mathbf{u}, t)\mathbf{x}(t) + \mathcal{D}(\mathbf{x}, \mathbf{u}, t)\mathbf{u}(t), \quad \mathbf{x}(0) = 0 \end{cases} \quad (3.8)$$

We make use of the result mentioned in Ref. [24] to construct the state space realisation of the dual system. In the BQO case we have,  $\mathcal{A}(\mathbf{x}, \mathbf{u}, t) = \mathbf{A} + \mathbf{N}\mathbf{u}$ ,  $\mathcal{B}(\mathbf{x}, \mathbf{u}, t) = \mathbf{B}$ ,  $\mathcal{C}(\mathbf{x}, \mathbf{u}, t) = \mathbf{C} + \mathbf{x}^\top \mathbf{M}$  and  $\mathcal{D}(\mathbf{x}, \mathbf{u}, t) = 0$ .

**Lemma 3.4.1.** [24] *Consider the system given in Eq. 3.8 with the initial condition  $\mathbf{x}(0) = 0$ . Assume that the input-output mapping is given by operator  $\Sigma: L_2^m(\Omega) \rightarrow L_2^p(\Omega)$ . Then, the state-space realization of the nonlinear Hilbert adjoint operator  $\Sigma^*: L_2^{p+m}(\Omega) \rightarrow L_2^m(\Omega)$  is given by*

$$\Sigma^* := \begin{cases} \dot{\mathbf{x}}(t) &= \mathcal{A}(\mathbf{x}, \mathbf{u}, t)\mathbf{x}(t) + \mathcal{B}(\mathbf{x}, \mathbf{u}, t)\mathbf{u}(t), \\ \dot{\mathbf{x}}_d(t) &= -\mathcal{A}^\top(\mathbf{x}, \mathbf{u}, t)\mathbf{x}_d(t) - \mathcal{C}^\top(\mathbf{x}, \mathbf{u}, t)\mathbf{u}_d(t), \\ \mathbf{y}_d(t) &= \mathcal{B}^\top(\mathbf{x}, \mathbf{u}, t)\mathbf{x}_d(t) + \mathcal{D}^\top(\mathbf{x}, \mathbf{u}, t)\mathbf{u}_d(t), \end{cases}$$

where  $\mathbf{x}_d(t) \in \mathbb{R}^n$ ,  $\mathbf{u}_d(t) \in \mathbb{R}^p$ ,  $\mathbf{x}(0) = 0$ ,  $\mathbf{x}_d(\infty) = 0$  and  $\mathbf{y}_d(t) \in \mathbb{R}^m$  can be interpreted as the dual state, dual input and dual output vectors of the system, respectively.

The dual system is given by,

$$\Sigma^* := \begin{cases} \dot{\mathbf{x}}(t) &= (\mathbf{A} + \mathbf{N}\mathbf{u}(t))^\top \mathbf{x}_d(t) - (\mathbf{C}^\top + \mathbf{M}\mathbf{x}(t))\mathbf{u}_d(t) \\ \dot{\mathbf{x}}_d(t) &= -(\mathbf{A} + \mathbf{N}\mathbf{u}(t))^\top \mathbf{x}_d(t) - \mathbf{C}^\top \mathbf{u}_d(t) - \mathbf{M}\mathbf{x}\mathbf{u}_d(t), \\ &= -\mathbf{A}^\top \mathbf{x}_d(t) - \mathbf{N}^\top \mathbf{u}(t)\mathbf{x}_d(t) - \mathbf{C}^\top \mathbf{u}_d(t) - \mathbf{M}\mathbf{x}(t)\mathbf{u}_d(t), \\ \mathbf{y}_d(t) &= \mathbf{B}^\top \mathbf{x}_d(t) \end{cases}$$

Now, we solve for the  $\mathbf{x}_d$  and calculate the Volterra series representation. Using the Volterra representation, we derive the reachability Gramians for the dual system which we realise gives us the observability Gramians for the BQO systems.

Define the term  $\mathbf{z}(t)$  as  $\mathbf{z}(t) = e^{\mathbf{A}^\top t} \mathbf{x}_d(t)$ . Then we have,

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}^\top e^{\mathbf{A}^\top t} \mathbf{x}_d + e^{\mathbf{A}^\top t} \dot{\mathbf{x}}_d \\ &= -e^{\mathbf{A}^\top t} \mathbf{N}^\top \mathbf{u} \mathbf{x}_d - e^{\mathbf{A}^\top t} \mathbf{C}^\top \mathbf{u}_d - e^{\mathbf{A}^\top t} \mathbf{M} \mathbf{x} \mathbf{u}_d \end{aligned} \quad (3.9)$$

Integrating  $\dot{\mathbf{z}}(t)$  backwards in time we get,

$$\mathbf{z}(t) = \int_{\infty}^t e^{\mathbf{A}^\top \sigma} \mathbf{N}^\top \mathbf{u}(\sigma) \mathbf{x}_d(\sigma) d\sigma + \int_{\infty}^t e^{\mathbf{A}^\top \sigma} \mathbf{C}^\top \mathbf{u}_d(\sigma) d\sigma + \int_{\infty}^t e^{\mathbf{A}^\top \sigma} \mathbf{M} \mathbf{x}(\sigma) \mathbf{u}_d(\sigma) d\sigma \quad (3.10)$$

We obtain the following expression for  $\mathbf{x}_d(t)$  using  $\mathbf{z}(t) = e^{\mathbf{A}^\top t} \mathbf{x}_d(t)$  and performing a suitable variable transform.

$$\begin{aligned} \mathbf{x}_d(t) &= \int_{\infty}^0 \underbrace{e^{\mathbf{A}^\top \tau} \mathbf{N}^\top \mathbf{u}(\tau + t) \mathbf{x}_d(\tau + t)}_{\Gamma_1} d\tau + \int_{\infty}^0 \underbrace{e^{\mathbf{A}^\top \tau} \mathbf{C}^\top \mathbf{u}_d(\tau + t)}_{\Gamma_2} d\tau \\ &\quad + \int_{\infty}^0 \underbrace{e^{\mathbf{A}^\top \tau} \mathbf{M} \mathbf{x}(\tau + t) \mathbf{u}_d(\tau + t)}_{\Gamma_3} d\tau \end{aligned} \quad (3.11)$$

There are three terms in the expression for  $\mathbf{x}_d$  and we examine them one by one. The term  $\Gamma_2$  corresponds to the linear observability kernel, using this we obtain the Gramian component -  $\mathbf{Q}_1^B = \int_0^{\infty} e^{\mathbf{A}^\top t} \mathbf{C}^\top \mathbf{C} e^{\mathbf{A}t} dt$ . Let's move on to the  $\Gamma_3$  term. Substitute the expression for

$\mathbf{x}(t + \tau)$  to get,

$$\begin{aligned}\Gamma_3 &= \int_{-\infty}^0 e^{\mathbf{A}^\top \tau} \mathbf{M} \left( \int_0^{t+\tau} e^{\mathbf{A} \sigma_1} \mathbf{B} \mathbf{u}(t + \tau - \sigma_1) \mathbf{u}_d(t + \tau) d\sigma \right) d\tau \\ &+ \int_{-\infty}^0 e^{\mathbf{A}^\top \tau} \mathbf{M} \left( \int_0^{t+\tau} \int_0^{\sigma_1} e^{\mathbf{A} \sigma_1} \mathbf{N} e^{\mathbf{A} \sigma_2} \mathbf{B} \mathbf{u}_d(t + \tau) \dots d\sigma_1 \right) d\tau \\ &\vdots\end{aligned}$$

Observing the trend we notice that the terms appearing in  $\Gamma_3$  correspond to the Gramian components  $\mathbf{Q}_{1j}$  for  $j \geq 1$ . Finally, we have the  $\Gamma_1$  term which can be simplified by replacing the  $\mathbf{x}_d(t + \tau)$  with the expression in Eq. 3.11. This gives us,

$$\begin{aligned}\Gamma_1 &= \int_{-\infty}^0 \int_{-\infty}^0 \underbrace{e^{\mathbf{A}^\top \tau} \mathbf{N}^\top e^{\mathbf{A}^\top \sigma_1} \mathbf{N}^\top \mathbf{x}_d(\tau + t + \sigma_1)}_{\Gamma_{1,1}} \mathbf{u}(t + \tau) \dots d\sigma_1 d\tau \\ &+ \int_{-\infty}^0 \int_{-\infty}^0 \underbrace{e^{\mathbf{A}^\top \tau} \mathbf{N}^\top e^{\mathbf{A}^\top \sigma_1} \mathbf{C}^\top}_{\Gamma_{1,2}} \mathbf{u}_d(\tau + t + \sigma_1) \mathbf{u}(t + \tau) d\sigma_1 d\tau \\ &+ \int_{-\infty}^0 \int_{-\infty}^0 \underbrace{e^{\mathbf{A}^\top \tau} \mathbf{N}^\top e^{\mathbf{A}^\top \sigma_1} \mathbf{M} \mathbf{x}(\tau + t)}_{\Gamma_{1,3}} \mathbf{u}_d(\tau + t) \mathbf{u}(t + \tau) d\sigma_1 d\tau\end{aligned}$$

We notice a similar pattern as in Eq. 3.11, the  $\Gamma_{1,2}$  term corresponds to  $\mathbf{Q}_2^B$  Gramian component and  $\Gamma_{1,3}$  component gives us the  $\mathbf{Q}_{2,j}$  components for  $j \geq 1$ . Continuing this method of substituting Eq. 3.11 in  $\Gamma_{1,\dots,k}$  terms gives us the components  $\mathbf{Q}_k^B$ ,  $\mathbf{Q}_{k,j}$  for  $j \geq 1$ . By summing all the Gramian components we obtain the same expression as in Def. 3.3.2.,

$$\mathbf{Q} = \sum_{k=1}^{\infty} \mathbf{Q}_k^B + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{Q}_{kj}.$$

### 3.5 Balanced truncation for BQO systems

Now we follow the standard BT procedure. Calculate the low rank factors of  $\mathbf{P}$  and  $\mathbf{Q}$  as  $\mathbf{P} = \mathbf{U}\mathbf{U}^\top$  and  $\mathbf{Q} = \mathbf{L}\mathbf{L}^\top$ . Just like in the linear case, we construct projection matrices  $\mathcal{V}$ ,  $\mathcal{W}$  using the SVD of the term  $\mathbf{L}^\top \mathbf{U}$ . The form for the reduced terms for  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{N}$  remain the same as in Ref. [10] -  $\mathcal{W}^\top \mathbf{A} \mathcal{V}$ ,  $\mathcal{W}^\top \mathbf{B}$ ,  $\mathbf{C} \mathcal{V}$ ,  $\mathcal{W}^\top \mathbf{N} \mathcal{V}$ , the final term  $\mathbf{M}$  becomes  $\mathcal{V}^\top \mathbf{M} \mathcal{V}$ . The balanced truncation algorithm for BQO systems is stated below.

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**Algorithm 3** Balanced Truncation for BQO systems
 

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**Input:** LTI system given by matrices  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times m}$ , and  $\mathbf{C} \in \mathbb{R}^{p \times n}$ ,  $\mathbf{M} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{N} \in \mathbb{R}^{n \times n}$ .

**Output:** BT reduced-system:  $\mathbf{A}_r \in \mathbb{R}^{r \times r}$ ,  $\mathbf{B}_r \in \mathbb{R}^{r \times m}$ ,  $\mathbf{C}_r \in \mathbb{R}^{p \times r}$ ,  $\mathbf{M}_r \in \mathbb{R}^{r \times r}$ ,  $\mathbf{N}_r \in \mathbb{R}^{r \times r}$ .

1. Compute the Lyapunov factors  $\mathbf{U}, \mathbf{L} \in \mathbb{R}^{n \times n}$  for  $\mathbf{P}, \mathbf{Q}$  and pick a truncation index,  $r$ .
2. Compute the SVD of the matrix  $\mathbf{L}^\top \mathbf{U}$ , partitioned as follows

$$\mathbf{L}^\top \mathbf{U} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1^\top \\ \mathbf{Y}_2^\top \end{bmatrix},$$

where  $\mathbf{S}_1 \in \mathbb{R}^{r \times r}$  and  $\mathbf{S}_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ .

3. Construct the model reduction bases  $\mathbf{W}_r = \mathbf{LZ}_1 \mathbf{S}_1^{-1/2}$  and  $\mathbf{V}_r = \mathbf{UY}_1 \mathbf{S}_1^{-1/2}$
4. The reduced-order system matrices are given by

$$\begin{aligned} \mathbf{A}_r &= \mathbf{W}_r^\top \mathbf{A} \mathbf{V}_r = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top (\mathbf{L}^\top \mathbf{A} \mathbf{U}) \mathbf{Y}_1 \mathbf{S}_1^{-1/2} & \mathbf{I}_r &= \mathbf{W}_r^\top \mathbf{V}_r = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top (\mathbf{L}^\top \mathbf{U}) \mathbf{Y}_1 \mathbf{S}_1^{-1/2} \\ \mathbf{M}_r &= \mathbf{V}_r^\top \mathbf{M} \mathbf{V}_r = \mathbf{S}_1^{-1/2} \mathbf{Y}_1^\top (\mathbf{U}^\top \mathbf{M} \mathbf{U}) \mathbf{Y}_1 \mathbf{S}_1^{-1/2} & \mathbf{N}_r &= \mathbf{W}_r^\top \mathbf{N} \mathbf{V}_r = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top (\mathbf{L}^\top \mathbf{N} \mathbf{U}) \mathbf{Y}_1 \mathbf{S}_1^{-1/2} \\ \mathbf{B}_r &= \mathbf{W}_r^\top \mathbf{B} = \mathbf{S}_1^{-1/2} \mathbf{Z}_1^\top (\mathbf{L}^\top \mathbf{B}), & \mathbf{C}_r &= \mathbf{C} \mathbf{V}_r = (\mathbf{C} \mathbf{U}) \mathbf{Y}_1 \mathbf{S}_1^{-1/2}. \end{aligned}$$


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**Remark 3.5.1.** *In BT for BQO systems, we only have to solve for matrix equations of dimension  $n$  instead of  $n + n^2$  like in the case we lift to a bilinear system. Moreover, since the algorithm doesn't require  $\mathbf{P}$  and  $\mathbf{Q}$  explicitly we can use low-rank approximations of the square root factors for faster implementation.*

Given a system  $\Sigma^{\text{BQO}}$  with positive definite Gramians  $\mathbf{P}, \mathbf{Q} > 0$ , we can always find a  $\mathbf{T}$  such that the Gramians of the transformed system is balanced (see Sec. 2.8). Now consider the system  $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{N}, \mathbf{M})$  is balanced and partition the system as follow:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}, \\ \mathbf{N} &= \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix}, \mathbf{P} = \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \end{aligned} \quad (3.12)$$

In balanced truncation, the subsystem  $\Sigma_r^{\text{BQO}} = (\mathbf{A}_{11}, \mathbf{B}_1, \mathbf{C}_1, \mathbf{N}_{11}, \mathbf{M}_{11})$  is the reduced order model of  $\Sigma^{\text{BQO}}$ . Now assuming we start with a balanced system, we have that the system  $\Sigma^{\text{BQO}}$  satisfies the observability Lyapunov equation for BQO systems in Eq. 3.6.

$$\begin{aligned} & \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^\top \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}^\top \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \\ & + \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix}^\top \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix} = 0 \end{aligned}$$

The block matrix representation gives us four matrix equations, but we are mostly interested in the first equation, which involves terms from the upper left subsystem.

$$\begin{aligned} & \mathbf{A}_{11}^\top \mathbf{S}_1 + \mathbf{S}_1 \mathbf{A}_{11} + \mathbf{M}_{11} \mathbf{S}_1 \mathbf{M}_{11} + \mathbf{M}_{12} \mathbf{S}_2 \mathbf{M}_{21} + \mathbf{C}_1^\top \mathbf{C}_1 + \mathbf{N}_{11}^\top \mathbf{S}_1 \mathbf{N}_{11} + \mathbf{N}_{21}^\top \mathbf{S}_2 \mathbf{N}_{12} = 0, \\ & \Rightarrow \mathbf{A}_{11}^\top \mathbf{S}_1 + \mathbf{S}_1 \mathbf{A}_{11} + \mathbf{M}_{11} \mathbf{S}_1 \mathbf{M}_{11} + \mathbf{C}_1^\top \mathbf{C}_1 + \mathbf{N}_{11}^\top \mathbf{S}_1 \mathbf{N}_{11} = -(\mathbf{N}_{12}^\top \mathbf{S}_2 \mathbf{N}_{12} + \mathbf{M}_{12} \mathbf{S}_2 \mathbf{M}_{21}). \end{aligned} \quad (3.13)$$

Similarly, for the reachability Gramian, we have that the reachability gramian satisfies the Eq. 3.5,

$$\begin{aligned} & \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^\top + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}^\top \\ & + \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix}^\top = 0 \end{aligned}$$

Using the block matrix representation in Eq. 3.12 gives us four matrix equations, but we are mostly interested in the first equation, which involves terms from the upper left subsystem.

$$\begin{aligned} & \mathbf{A}_{11}^\top \mathbf{S}_1 + \mathbf{S}_1 \mathbf{A}_{11} + \mathbf{C}_1^\top \mathbf{C}_1 + \mathbf{N}_{11}^\top \mathbf{S}_1 \mathbf{N}_{11} + \mathbf{N}_{21}^\top \mathbf{S}_2 \mathbf{N}_{12} = 0, \\ & \Rightarrow \mathbf{A}_{11}^\top \mathbf{S}_1 + \mathbf{S}_1 \mathbf{A}_{11} + \mathbf{C}_1^\top \mathbf{C}_1 + \mathbf{N}_{11}^\top \mathbf{S}_1 \mathbf{N}_{11} = -(\mathbf{N}_{12}^\top \mathbf{S}_2 \mathbf{N}_{12}). \end{aligned} \quad (3.14)$$

We now try to find under what conditions our reduced order model is balanced i.e.  $\mathbf{P}_r = \mathbf{Q}_r = \mathbf{S}_1$ . The reduced system is also a bilinear with quadratic output and hence its Gramians must satisfy the BQO Gramian equations given in Eq. 3.6 and 3.5.

$$\mathbf{A}_{11}^\top \mathbf{Q}_r + \mathbf{Q}_r \mathbf{A}_{11} + \mathbf{M}_{11} \mathbf{P}_r \mathbf{M}_{11} + \mathbf{N}_{11}^\top \mathbf{Q}_r \mathbf{N}_{11} + \mathbf{C}_1^\top \mathbf{C}_1 = 0 \quad (3.15)$$

$$\mathbf{A}_{11} \mathbf{P}_r + \mathbf{P}_r \mathbf{A}_{11}^\top + \mathbf{N}_{11} \mathbf{P}_r \mathbf{N}_{11}^\top + \mathbf{B}_1 \mathbf{B}_1^\top = 0 \quad (3.16)$$

Comparing Eq. 3.13, 3.14 with Eq. 3.15 and 3.16 we notice that the solutions  $\mathbf{Q}_r$  and  $\mathbf{P}_r$  are equal to  $\mathbf{S}_1$  only when  $\mathbf{N}_{12}, \mathbf{N}_{21}$  and  $\mathbf{M}_{12}$  are all zero. This is not true in general, so the reduced model is not balanced. However, we notice that the matrices on the right-hand side in Eq. 3.13 and 3.14 are symmetric positive semi-definite. This means that  $\mathbf{S}_1$  is a generalised Gramian of the reduced system  $\Sigma_r^{\text{BQO}}$  and the reduced system is balanced in the generalised sense as mentioned in Ref. [25].

## 3.6 Numerical results

In this section, we test the performance of the proposed BT algorithm for BQO systems given in Algorithm 3. The benchmark model we consider is the Burgers bilinear model [26]. The model has a state dimension of 240 with a single input and a single output. As shown below, we construct a reduced-order model with truncation index  $r = 18$ . The following figure shows the output response of the FOM, ROM and FOM with  $\mathbf{M} = 0$  to the input signal  $\mathbf{u}(t) = 5(\cos(2\pi t) + \sin(12\pi t)e^{-0.4t})$  and the Hankel singular values of the FOM and the ROM. We notice that the HSVs of the ROM match with the first 18 HSVs of FOM and the output response of the ROM is similar to that of the FOM. The error between the FOM and ROM is of the order 1. The absolute error can be made smaller by increasing the truncation index but that comes at the expense of having to work with a larger reduced model.

## 3.7 Truncated Gramians for BQO system

The solution of the matrix equations given in Eq. 3.5 and Eq. 3.6 involve using iterative methods like ADI solvers which can be difficult to solve efficiently. We propose using truncated

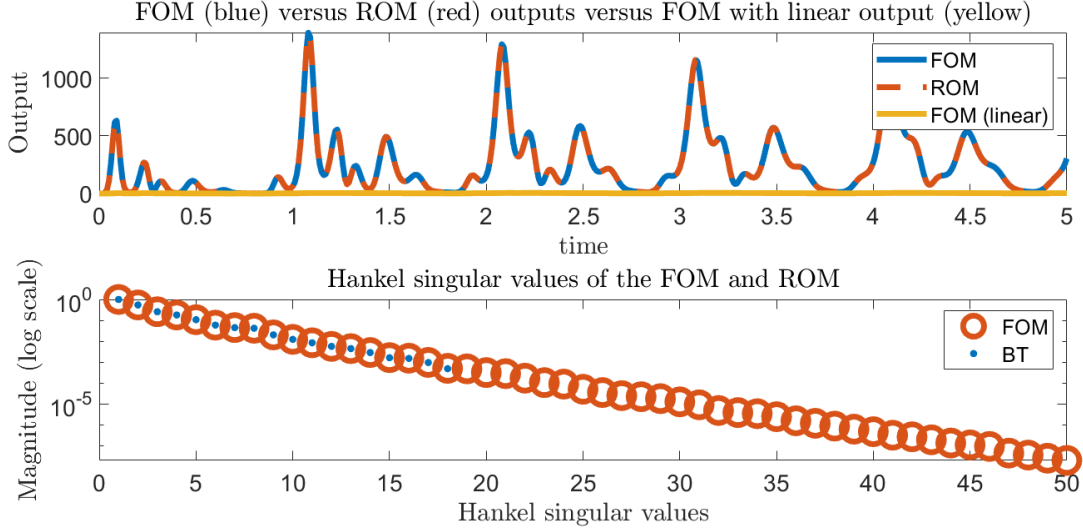


Figure 3.1: Comparison of the output response and Hankel singular values of the FOM and ROM obtained by BT

gramians as a means of avoiding such methods. We notice that the gramians in Def. 3.3.2 are defined in a recursive manner. Moreover, the matrix equation satisfied by  $\mathbf{Q}_{ij}$  depends on  $\mathbf{Q}_{i-1,j}$  or  $\mathbf{P}_j^B$  depending on the value of  $i$ . We can make use of this nestedness and define the truncated gramians  $\mathbf{P}_\tau$  and  $\mathbf{Q}_\tau$  for the BQO system.

**Definition 3.7.1.** We define the truncated gramians  $\mathbf{P}_\tau$  and  $\mathbf{Q}_\tau$  as,

$$\begin{aligned}\mathbf{P}_\tau &= \mathbf{P}_1 + \mathbf{P}_2 \\ \mathbf{Q}_\tau &= \mathbf{Q}_1^B + \mathbf{Q}_2^B + \mathbf{Q}_{11} + \mathbf{Q}_{12} + \mathbf{Q}_{21} + \mathbf{Q}_{22}.\end{aligned}$$

where the terms appearing on the RHS are as defined in Defn. 3.3.1 and Defn. 3.3.2.

**Corollary 3.7.1.** The truncated gramians satisfy the following equations,

$$\begin{aligned}\mathbf{A}\mathbf{P}_\tau + \mathbf{P}_\tau\mathbf{A}^\top + \mathbf{N}\mathbf{P}_1\mathbf{N}^\top + \mathbf{B}\mathbf{B}^\top &= 0 \\ \mathbf{A}^\top\mathbf{Q}_\tau + \mathbf{Q}_\tau\mathbf{A} + \mathbf{C}^\top\mathbf{C} + \mathbf{N}^\top\mathbf{Q}_1^B\mathbf{N} + \mathbf{M}(\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{Q}_{11} + \mathbf{Q}_{12})\mathbf{M} &= 0.\end{aligned}$$

We solve for the terms -  $\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}_1^B, \mathbf{Q}_2^B, \mathbf{Q}_{11}, \mathbf{Q}_{12}$  in that order to ensure that we only need to solve Lyapunov equations. Now we can proceed in the exact same way to construct  $\mathbf{L}, \mathbf{U}$  from the truncated gramians  $\mathbf{Q}_\tau$  and  $\mathbf{P}_\tau$ . The new algorithm for balanced truncation

of BQO systems remains the same as in Algorithm 3 with the only difference being that we construct the square root factors for  $\mathbf{P}_\tau$  and  $\mathbf{Q}_\tau$  instead of  $\mathbf{P}$  and  $\mathbf{Q}$ .

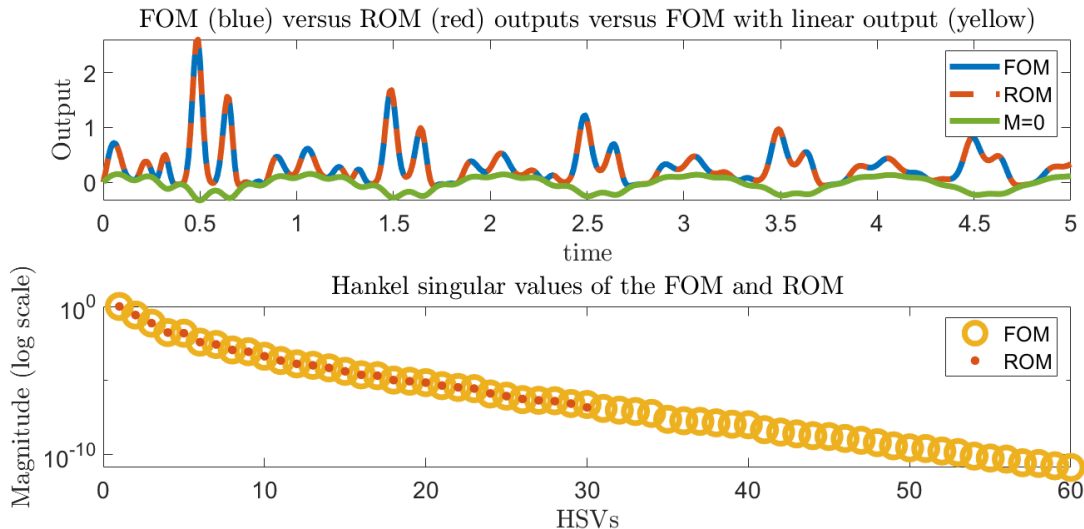


Figure 3.2: Comparison of the output response and Hankel singular values of the FOM and ROM obtained by BT using truncated gramians

In Fig. 3.2 we use the balanced truncation algorithm using the truncated gramians to obtain a reduced model of the heat transfer model mentioned in Ref. [23]. The state dimension of the full-order model is 225. We plot the output response of the FOM and ROM to input signal  $\mathbf{u}(t) = 5(\cos(2\pi t) + \sin(12\pi t)e^{-0.4t})$  in the upper plot and the Hankel singular values of the FOM and ROM ( $r = 30$ ) in the bottom plot. We only show the first 60 HSVs of the FOM since we are only interested in the dominant HSVs and the latter HSVs are close to machine precision. Despite the fact that we used an approximate gramian (truncated gramian), we observe that the ROM has HSVs which closely match the 30 dominant singular values of the original system. This example supports the idea of using appropriate truncated gramians to avoid numerical issues encountered while solving the matrix equations Eq. 3.5 and 3.6.



# Chapter 4

## Quad BT for LQO systems

We start by considering the extension of Quad BT to linear systems with quadratic output (LQO). This class is a sub-case of the BQO systems discussed in Chap. 3 and is obtained by setting  $\mathbf{N} = 0$  as shown in Eq. 4.1. Thus, we will be using many of the results from the previous chapter in this chapter without deriving them again for the LQO sub-case.

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{x}(t)^\top \mathbf{M}\mathbf{x}(t)\end{aligned}\tag{4.1}$$

The state equations for these systems are linear, but the output expression is a quadratic form in  $\mathbf{x}$ . These systems arise in examples where one is interested in observing quantities like the variance of a state from a reference level. This is the case in random vibration analysis [27] and problems concerning the observation of quantities like energy or power. The problem of MOR for LQO systems has been explored in the past. Lifting transforms have been one approach to deal with these systems. Ref. [28, 29] use lifting to rewrite the (SISO) LQO system as a (SIMO) linear system (with linear output). Following this, they employ classical techniques like Balanced truncation and moment-matching methods on the lifted linear system. However, lifting methods suffer from the problem of increasing the system state dimension of the original system, making reduction tasks computationally demanding. Ref. [9] introduces the balanced truncation (BT) algorithm specifically for LQO systems. The BT algorithm is however intrusive i.e. it requires information on the system matrices. In this chapter, we attempt to extend the data-driven method - Quadrature-based balanced

truncation (Quad BT) to LQO systems. The presence of the quadratic non-linearity in the expression for  $\mathbf{y}$  and not in  $\mathbf{x}$  makes this case mild but a step towards extending Quad BT to quadratic-bilinear systems.

## 4.1 Kernels and gramians for LQO systems

We note that since LQO systems are a sub-case of BQO systems with  $\mathbf{N} = 0$ , we do not repeat the details about the solutions, kernels or gramians of these systems in this section. Instead, we simply present them as definitions. For more details on the derivation please refer to Chap. 3 while setting  $\mathbf{N} = 0$ .

**Definition 4.1.1.** *The kernel functions for  $\Sigma$  are defined as follows*

$$\mathbf{h}_1(\tau) = \mathbf{C}e^{\mathbf{A}\tau}\mathbf{B} \quad (4.2)$$

$$\mathbf{h}_2(\tau_1, \tau_2) = \mathbf{K}(e^{\mathbf{A}\tau_1}\mathbf{B} \otimes e^{\mathbf{A}\tau_2}\mathbf{B}) \quad (4.3)$$

$$= \mathbf{B}^\top e^{\mathbf{A}^\top\tau_1}\mathbf{M}e^{\mathbf{A}\tau_2}\mathbf{B}. \quad (4.4)$$

where  $\mathbf{K} = (\text{vec } \mathbf{M})^\top$ .

It is worth mentioning that  $\mathbf{M}$  can be assumed to be a symmetric matrix without loss of generality. This follows by observing that replacing  $\mathbf{x}^\top \mathbf{M} \mathbf{x}$  with  $\mathbf{x}^\top \frac{(\mathbf{M} + \mathbf{M}^\top)}{2} \mathbf{x}$  doesn't change the output of the system. So we take the liberty of assuming  $\mathbf{M}$  is symmetric for the rest of the section.

**Definition 4.1.2.** *The gramians for LQO system  $\Sigma$  are defined as*

$$\mathcal{P} = \int_0^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top\tau} d\tau = \mathbf{U}\mathbf{U}^\top \quad (4.5)$$

$$\mathcal{Q}_1 = \int_0^\infty e^{\mathbf{A}^\top\tau}\mathbf{C}^\top\mathbf{C}e^{\mathbf{A}\tau} d\tau = \mathbf{L}_1\mathbf{L}_1^\top. \quad (4.6)$$

$$\mathcal{Q}_2 = \int_0^\infty e^{\mathbf{A}^\top t}\mathbf{M}^\top\mathbf{P}\mathbf{M}e^{\mathbf{A}t} dt = \mathbf{L}_2\mathbf{L}_2^\top \quad (4.7)$$

$$\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2 \quad (4.8)$$

where  $\mathbf{U}$  and  $\mathbf{L} = [\mathbf{L}_1 \ \mathbf{L}_2]$  are the square root factors of the gramians  $\mathcal{P}$  and  $\mathcal{Q}$  respectively.

**Remark 4.1.1.** *Assuming that  $\mathbf{A}$  has eigenvalues in the left half plane, the gramians  $\mathbf{P}$  and  $\mathbf{Q}$  satisfy the following matrix equations,*

$$\begin{aligned}\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top &= 0 \\ \mathbf{A}^\top\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{M}\mathbf{P}\mathbf{M} + \mathbf{C}^\top\mathbf{C} &= 0\end{aligned}$$

It is shown in Chap. 3 that we can use these gramians to define balanced truncation for LQO systems. The square root terms  $\mathbf{U}$  and  $\mathbf{L}$  are calculated by taking the Cholesky factors of the solutions of the gramian equations given in Eq. 4.1.1. We quickly summarise the balanced truncation algorithm for LQO systems. We begin by calculating the SVD of the product  $\mathbf{L}^\top\mathbf{U}$  and picking a truncation index  $r$ . The singular values of  $\mathbf{L}^\top\mathbf{U}$  encode the system's Hankel singular values, just like in the LTI case.

$$\mathbf{L}^\top\mathbf{U} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1^\top \\ \mathbf{Y}_2^\top \end{bmatrix}. \quad (4.9)$$

The BT ROM is obtained by truncating the system and dropping off the smallest  $n - r$  singular values. The matrices  $\mathbf{Y}_1, \mathbf{Z}_1$  and  $\mathbf{S}_1$  obtained by partitioning the SVD of  $\mathbf{L}^\top\mathbf{U}$  in Eq. 4.9 are then used to construct projection matrices  $\mathcal{V}_r = \mathbf{L}\mathbf{Z}_1\mathbf{S}_1^{-1/2}$  and  $\mathcal{W}_r = \mathbf{U}\mathbf{Y}_1\mathbf{S}_1^{-1/2}$ . The reduced matrices are then given by,

$$\begin{aligned}\mathbf{A}_r &= \mathbf{S}_1^{-1/2}\mathbf{Z}_1^\top (\mathbf{L}^\top\mathbf{A}\mathbf{U}) \mathbf{Y}_1\mathbf{S}_1^{-1/2}, \\ \mathbf{B}_r &= \mathbf{S}_1^{-1/2}\mathbf{Z}_1^\top (\mathbf{L}^\top\mathbf{B}), \\ \mathbf{C}_r &= (\mathbf{C}\mathbf{U})\mathbf{Y}_1\mathbf{S}_1^{-1/2}, \\ \mathbf{M}_r &= \mathbf{S}_1^{-1/2}\mathbf{Y}_1^\top (\mathbf{U}^\top\mathbf{M}\mathbf{U})\mathbf{Y}_1\mathbf{S}_1^{-1/2}.\end{aligned} \quad (4.10)$$

In Quad BT, we deal with data-driven quantities - samples of the kernels and their derivatives since we don't have access to the matrices of the FOM. In the linear case we circumvented this problem by approximating the gramians with quadrature-based approximations. We proceed in the same way by approximating the square root factors  $\mathbf{U}$  and  $\mathbf{L} = [\mathbf{L}_1 \ \mathbf{L}_2]$  with

quadrature points to get  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}} = [\tilde{\mathbf{L}}_1 \tilde{\mathbf{L}}_2]$ .

$$\begin{aligned}
\tilde{\mathbf{U}}\tilde{\mathbf{U}}^\top &= \sum_{i=1}^{N_p} (\rho_i e^{\mathbf{A}\mu_j} \mathbf{B}) (\rho_i e^{\mathbf{A}\mu_j} \mathbf{B})^\top && \Rightarrow \tilde{\mathbf{U}} = [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{B}] \\
\tilde{\mathbf{L}}_1 \tilde{\mathbf{L}}_1^\top &= \sum_{j=1}^{N_q} (\phi_j e^{\mathbf{A}^\top \omega_j} \mathbf{C}^\top) (\phi_j e^{\mathbf{A}^\top \omega_j} \mathbf{C}^\top)^\top && \Rightarrow \tilde{\mathbf{L}}_1 = [\phi_1 e^{\mathbf{A}^\top \omega_1} \mathbf{C}^\top \dots \phi_{N_q} e^{\mathbf{A}^\top \omega_{N_q}} \mathbf{C}^\top] \\
\tilde{\mathbf{L}}_2 \tilde{\mathbf{L}}_2^\top &= \sum_{j=1}^{N_q} (\phi_j e^{\mathbf{A}^\top \omega_j} \mathbf{M}^\top \tilde{\mathbf{U}}) (\phi_j e^{\mathbf{A}^\top \omega_j} \mathbf{M}^\top \tilde{\mathbf{U}})^\top && \Rightarrow \tilde{\mathbf{L}}_2 = [\phi_1 e^{\mathbf{A}^\top \omega_1} \mathbf{M}^\top \tilde{\mathbf{U}} \dots \phi_{N_q} e^{\mathbf{A}^\top \omega_{N_q}} \mathbf{M}^\top \tilde{\mathbf{U}}]
\end{aligned} \tag{4.11}$$

Recall the terms in Eq. 4.10. The terms  $\mathbf{S}_1, \mathbf{Y}_1, \mathbf{Z}_1$  are obtained from the SVD of  $\mathbf{L}^\top \mathbf{U}$  which leaves us with 5 terms to deal with -  $\mathbf{L}^\top \mathbf{U}, \mathbf{L}^\top \mathbf{A} \mathbf{U}, \mathbf{L}^\top \mathbf{B}, \mathbf{C} \mathbf{U}$  and  $\mathbf{U}^\top \mathbf{M} \mathbf{U}$ . In order to obtain a reduced model in a non-intrusive fashion, we need to substitute these terms with data-driven quantities. The obvious candidates are  $\tilde{\mathbb{H}} = \tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}, \tilde{\mathbb{M}} = \tilde{\mathbf{L}}^\top \mathbf{A} \tilde{\mathbf{U}}, \tilde{\mathbf{h}} = \tilde{\mathbf{L}}^\top \mathbf{B}, \tilde{\mathbf{g}} = \mathbf{C} \tilde{\mathbf{U}}$  and  $\tilde{\mathbb{K}} = \tilde{\mathbf{U}}^\top \mathbf{M} \tilde{\mathbf{U}}$  respectively which are constructed from the quadrature based approximate gramians defined in Eq. 4.11. In the following section, we proceed to show that these quadrature-based matrices can be exactly written as products of quadrature weights and appropriate kernels (and their derivatives) of the LQO system.

## 4.2 Replacing intrusive terms with data matrices

In this section, we show some of the calculations to replace the quadrature-based matrices  $\tilde{\mathbb{H}}, \tilde{\mathbb{M}}, \tilde{\mathbf{h}}, \tilde{\mathbf{g}}$  with data entries (samples of kernels and their derivatives). We show some simple examples to better explain the structure of these matrices. The first term we consider is the  $\tilde{\mathbb{H}} = \tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}$  term.

**Theorem 4.2.1.** *Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 4.11, define the matrix  $\tilde{\mathbb{H}} = \tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}$ , then we have,*

$$\tilde{\mathbb{L}} = \begin{cases} \rho_i \phi_j \mathbf{h}_1(\mu_i + \omega_j), & \text{for } \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}} \in \mathbb{R}^{N_q \times N_p}, \\ \rho_i \phi_j \rho_k \mathbf{h}_2(\mu_k, \omega_j + \mu_i), & \text{for } \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}} \in \mathbb{R}^{N_q N_p \times N_p}. \end{cases} \tag{4.12}$$

where  $1 \leq i, k \leq N_p, 1 \leq j \leq N_q$ .

**Proof.** The result follows by simply multiplying the terms in Eq. 4.11.

$$\begin{aligned}\tilde{\mathbf{L}}^\top \tilde{\mathbf{U}} &= \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}} \\ \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} \rho_i \phi_j \mathbf{C} e^{\mathbf{A}\omega_j} e^{\mathbf{A}\mu_i} \mathbf{B} \\ \phi_j \rho_i \tilde{\mathbf{U}}^\top \mathbf{M} e^{\mathbf{A}\omega_j} e^{\mathbf{A}\mu_i} \mathbf{B} \end{bmatrix}_{i,j} \in \mathbb{R}^{(N_p N_q + N_q) \times N_p} \\ &= \begin{bmatrix} \rho_i \phi_j \mathbf{h}_1(\mu_i + \omega_j) \\ \phi_j \rho_i \begin{bmatrix} \vdots \\ \rho_k \mathbf{h}_2(\mu_k, \omega_j + \mu_i) \\ \vdots \end{bmatrix} \end{bmatrix}_{i,j}.\end{aligned}$$

**Example 1.** Let  $\tilde{\mathbf{L}}$  and  $\tilde{\mathbf{U}}$  be defined as in Eq. 4.11 and  $N_p = N_q = 2$  with unity weights. Then using Thm. 4.2.1 we have,

$$\tilde{\mathbb{H}} = \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}} \\ \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1(\mu_1 + \omega_1) & \mathbf{h}_1(\mu_2 + \omega_1) \\ \mathbf{h}_1(\mu_1 + \omega_2) & \mathbf{h}_1(\mu_2 + \omega_2) \\ \mathbf{h}_2(\mu_1, \mu_1 + \omega_1) & \mathbf{h}_2(\mu_1, \mu_2 + \omega_1) \\ \mathbf{h}_2(\mu_1, \mu_1 + \omega_2) & \mathbf{h}_2(\mu_1, \mu_2 + \omega_2) \\ \mathbf{h}_2(\mu_2, \mu_1 + \omega_1) & \mathbf{h}_2(\mu_2, \mu_2 + \omega_1) \\ \mathbf{h}_2(\mu_2, \mu_1 + \omega_2) & \mathbf{h}_2(\mu_2, \mu_2 + \omega_2) \end{bmatrix}$$

**Theorem 4.2.2.** Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 4.11, define the matrix  $\tilde{\mathbb{M}} = \tilde{\mathbf{L}}^\top \mathbf{A} \tilde{\mathbf{U}}$ , then we have,

$$\tilde{\mathbb{M}} = \begin{cases} \rho_i \phi_j \mathbf{h}'_1(\mu_i + \omega_j), & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}} \in \mathbb{R}^{N_q \times N_p}, \\ \rho_i \phi_j \rho_k \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_k, \omega_j + \mu_i), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}} \in \mathbb{R}^{N_p N_q \times N_p}. \end{cases} \quad (4.13)$$

where  $1 \leq i, k \leq N_p, 1 \leq j \leq N_q$ .

**Proof.** The result follows by a simple multiplication of the terms in Eq. 4.11.

$$\begin{aligned}\tilde{\mathbb{M}} &= \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}} \\ \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} \rho_i \phi_j \mathbf{C} \mathbf{A} e^{\mathbf{A}\omega_j} e^{\mathbf{A}\mu_i} \mathbf{B} \\ \phi_j \rho_i \tilde{\mathbf{U}}^\top \mathbf{M} \mathbf{A} e^{\mathbf{A}\omega_j} e^{\mathbf{A}\mu_i} \mathbf{B} \end{bmatrix}_{i,j} = \begin{bmatrix} \rho_i \phi_j \mathbf{h}'_1(\mu_i + \omega_j) \\ \phi_j \rho_i \begin{bmatrix} \vdots \\ \rho_k \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_k, \omega_j + \mu_i) \\ \vdots \end{bmatrix} \end{bmatrix}_{i,j} \\ \tilde{\mathbb{M}} &\in \mathbb{R}^{(N_p N_q + N_q) \times N_p}\end{aligned}$$

**Example 2.** Let  $\tilde{\mathbf{L}}$  and  $\tilde{\mathbf{U}}$  be defined as in Eq. 4.11 and  $N_p = N_q = 2$  with unity weights. Then using Thm. 4.2.2 we have,

$$\tilde{\mathbf{M}} = \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}} \\ \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{h}_1}{\partial t}(\mu_1 + \omega_1) & \frac{\partial \mathbf{h}_1}{\partial t}(\mu_2 + \omega_1) \\ \frac{\partial \mathbf{h}_1}{\partial t}(\mu_1 + \omega_2) & \frac{\partial \mathbf{h}_1}{\partial t}(\mu_2 + \omega_2) \\ \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_1, \mu_1 + \omega_1) & \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_1, \mu_2 + \omega_1) \\ \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_1, \mu_1 + \omega_2) & \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_1, \mu_2 + \omega_2) \\ \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_2, \mu_1 + \omega_1) & \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_2, \mu_2 + \omega_1) \\ \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_2, \mu_1 + \omega_2) & \frac{\partial \mathbf{h}_2}{\partial t_2}(\mu_2, \mu_2 + \omega_2) \end{bmatrix}$$

**Theorem 4.2.3.** Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 4.11, define the matrices  $\tilde{\mathbf{h}} = \tilde{\mathbf{L}}^\top \mathbf{B}$ ,  $\tilde{\mathbf{g}} = \mathbf{C} \tilde{\mathbf{U}}$  and  $\tilde{\mathbf{K}} = \tilde{\mathbf{U}}^\top \mathbf{M} \tilde{\mathbf{U}}$  then we have,

$$\tilde{\mathbf{h}} = \begin{cases} \phi_j \mathbf{h}_1(\omega_j), & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{B} \in \mathbb{R}^{N_q \times 1}. \\ \phi_j \rho_k \mathbf{h}_2(\mu_k, \omega_j), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{B} \in \mathbb{R}^{N_q N_p \times 1}. \end{cases} \quad (4.14)$$

$$\tilde{\mathbf{g}} = \begin{cases} \rho_i \mathbf{h}_1(\mu_i), & \text{for } \mathbf{C} \tilde{\mathbf{U}} \in \mathbb{R}^{1 \times N_p}, \end{cases} \quad (4.15)$$

$$\tilde{\mathbf{K}} = \begin{cases} \rho_i \rho_k \mathbf{h}_2(\mu_i, \mu_k), & \text{for } \tilde{\mathbf{U}}^\top \mathbf{M} \tilde{\mathbf{U}} \in \mathbb{R}^{N_p \times N_p}. \end{cases} \quad (4.16)$$

where  $1 \leq i, k \leq N_p, 1 \leq j \leq N_q$ .

**Example 3.** Let  $\tilde{\mathbf{L}}$  and  $\tilde{\mathbf{U}}$  be defined as in Eq. 4.11 and  $N_p = N_q = 2$  with unity weights. Then using Thm. 4.2.3 we have,

$$\tilde{\mathbf{h}} = \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \mathbf{B} \\ \tilde{\mathbf{L}}_2^\top \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1(\omega_1) \\ \mathbf{h}_1(\omega_2) \\ \mathbf{h}_2(\mu_1, \omega_1) \\ \mathbf{h}_2(\mu_1, \omega_2) \\ \mathbf{h}_2(\mu_2, \omega_1) \\ \mathbf{h}_2(\mu_2, \omega_2) \end{bmatrix}, \quad \tilde{\mathbf{K}} = \begin{bmatrix} \mathbf{h}_2(\mu_1, \mu_1) & \mathbf{h}_2(\mu_1, \mu_2) \\ \mathbf{h}_2(\mu_2, \mu_1) & \mathbf{h}_2(\mu_2, \mu_2) \end{bmatrix}, \quad \tilde{\mathbf{g}} = [\mathbf{h}_1(\mu_1) \ \mathbf{h}_1(\mu_2)]$$

Using Thm. 4.2.1, 4.2.2 and 4.2.3 we have replaced the intrusive terms appearing in the BT algorithm by equivalent data-driven quantities. Finally, we provide the algorithm - Quad

BT for linear systems with quadratic output.

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**Algorithm 4** Quadrature based Balanced Truncation for LQO systems

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**Input:** Samples of  $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}'_1$  and  $\mathbf{h}'_2$ , quadrature weights  $\rho_1, \dots, \rho_{N_p}, \phi_1, \dots, \phi_{N_q}$  and nodes  $\mu_1, \dots, \mu_{N_p}, \omega_1, \dots, \omega_{N_q}$

**Output:** The reduced order matrices  $\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{M}_r$

1. Construct the data matrices  $\tilde{\mathbf{H}}, \tilde{\mathbf{M}}, \tilde{\mathbf{h}}, \tilde{\mathbf{g}}, \tilde{\mathbf{K}}$  using the appropriate quadrature nodes  $\rho_i, \phi_j$  as per Thm. 4.2.1, 4.2.2 and 4.2.3. Choose an appropriate truncation index  $r$  while ensuring that  $r \leq N_p$ .
2. Compute SVD of  $\tilde{\mathbf{L}}$  and partition the matrices as follows,

$$\tilde{\mathbf{L}} = \begin{bmatrix} \tilde{\mathbf{Z}}_1 & \tilde{\mathbf{Z}}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{S}}_1 & \\ & \tilde{\mathbf{S}}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{Y}}_1^\top \\ \tilde{\mathbf{Y}}_2^\top \end{bmatrix}$$

where  $\tilde{\mathbf{Z}}_1 \in \mathbb{R}^{(N_q N_p + N_q) \times r}$ ,  $\tilde{\mathbf{S}}_1 \in \mathbb{R}^{r \times r}$ , and  $\tilde{\mathbf{Y}}_1^\top \in \mathbb{R}^{r \times N_p}$ .

3. Compute the reduced matrices as follows

$$\begin{aligned} \mathbf{A}_r &= \mathbf{S}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{M}} \tilde{\mathbf{Y}}_1 \mathbf{S}_1^{-1/2} & \mathbf{B}_r &= \mathbf{S}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{h}} \\ \mathbf{C}_r &= \tilde{\mathbf{g}}^\top \tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{S}}_1^{-1/2} & \mathbf{M}_r &= \tilde{\mathbf{S}}_1^{1/2} \tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{K}} \tilde{\mathbf{Y}}_1 \mathbf{S}_1^{-1/2} \end{aligned}$$


---

### 4.3 Computational Issues and Improvements

The size of the matrix  $\tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}$  in Quad BT for LQO systems scales as  $N_p \times (N_q N_p + N_q)$ . Assuming  $N_p = N_q = N$ , the size increases as a cubic in  $N$  making the method computationally demanding even for moderate systems. This problem is especially bad for systems with oscillatory impulse responses because we need a large number of nodes to capture the response. Even in the simpler linear case, we sometimes require  $N$  of order 100 to obtain reasonable approximations. Such a high number of nodes is infeasible in the LQO case due to the cubic scaling. This is especially a problem for approximating the second kernel response since the number of nodes used in  $\tilde{\mathbf{L}}_2$  is multiplied by the number of nodes in  $\tilde{\mathbf{U}}$ . We address these issues by taking advantage of the fact that we have flexibility in the choice of

the quadrature nodes and their number. We start by constructing two approximations of  $\tilde{\mathbf{U}}_1$  - a coarse one (with  $n_1$  nodes) and a fine one (with  $N_q$  nodes) with  $n_1 \ll N_q$  as shown below.

$$\begin{aligned}\mathbf{U}_1 &\approx \tilde{\mathbf{U}}_1 = [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{B}], \\ \mathbf{U}_1 &\approx \tilde{\mathbf{U}}'_1 = [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{n_1} e^{\mathbf{A}\mu_{n_1}} \mathbf{B}]\end{aligned}$$

Now we use the coarse  $\tilde{\mathbf{U}}'_1$  in the expression for  $\tilde{\mathbf{L}}_2$  as shown below.

$$\tilde{\mathbf{L}}_2 = \left[ \phi_1 e^{\mathbf{A}^\top \omega_1} \mathbf{M}^\top \tilde{\mathbf{U}}'_1 \dots \phi_{N'_q} e^{\mathbf{A}^\top \omega_{N'_q}} \mathbf{M}^\top \tilde{\mathbf{U}}'_1 \right]$$

The advantage of doing this is that now the matrix  $\tilde{\mathbf{L}}_2$  is of dimension  $N_p \times (N'_q n_1 + N_q)$ . This allows us to pick many more quadrature points to approximate the linear response since we can choose  $N_q$  such that  $N_q \approx N'_q n_1$  without affecting the order of the matrix. Moreover, it offers us flexibility in the number of nodes used in the approximation of  $\tilde{\mathbf{L}}_2$ . This improved approximation of the linear response however comes at the cost of compromising on the approximation of the second kernel since we are using a low-quality approximation  $\tilde{\mathbf{U}}'_1$ . Using adaptive schemes to cleverly choose quadrature nodes might be a way to overcome this and is an interesting avenue for future research.

Thus, for the QuadBT method, we have the following parameters that must be set beforehand -  $N_p, N_q, N'_q, n_1$ , quadrature points and weights. For all the examples discussed, we only consider equidistant nodes with unity weights for simplicity.

## 4.4 Extension to the MIMO case

The results in Thm. 4.2.1, 4.2.2 and 4.2.3 also hold in the multiple input multiple output (MIMO) LQO case given by Eq. 4.17 below.

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \begin{bmatrix} \mathbf{y}_1(t) \\ \vdots \\ \mathbf{y}_p(t) \end{bmatrix} &= \mathbf{C}\mathbf{x} + \begin{bmatrix} \mathbf{x}^\top(t) \mathbf{M}_1 \mathbf{x}(t) \\ \vdots \\ \mathbf{x}^\top(t) \mathbf{M}_p \mathbf{x}(t) \end{bmatrix}\end{aligned}\tag{4.17}$$

where  $\mathbf{u}(t) \in \mathbb{R}^m, \mathbf{y}(t) \in \mathbb{R}^p, \mathbf{M}_i \in \mathbb{R}^{n \times n}$  for  $1 \leq i \leq m$  and the other terms are the same as in the SISO case. Now, the kernels appearing in the theorems 4.2.1, 4.2.2 and 4.2.3 are no longer scalar quantities but a matrix of dimension  $p \times m$ . We present an example to illustrate the structure of  $\tilde{\mathbb{H}}$  in the MIMO case.

**Example 4.** Let  $\tilde{\mathbf{L}}$  and  $\tilde{\mathbf{U}}$  be defined as in Eq. 4.11 and  $N_p = N_q = 2$  with unity weights. Then we have,

$$\tilde{\mathbb{H}} = \begin{bmatrix} \mathbf{h}_1(\mu_1 + \omega_1) \in \mathbb{R}^{p \times m} & \mathbf{h}_1(\mu_2 + \omega_1) \in \mathbb{R}^{p \times m} \\ \mathbf{h}_1(\mu_1 + \omega_2) \in \mathbb{R}^{p \times m} & \mathbf{h}_1(\mu_2 + \omega_2) \in \mathbb{R}^{p \times m} \\ \mathbf{h}_2(\mu_1, \mu_1 + \omega_1) \in \mathbb{R}^{p \times m} & \mathbf{h}_2(\mu_1, \mu_2 + \omega_1) \in \mathbb{R}^{p \times m} \\ \mathbf{h}_2(\mu_1, \mu_1 + \omega_2) \in \mathbb{R}^{p \times m} & \mathbf{h}_2(\mu_1, \mu_2 + \omega_2) \in \mathbb{R}^{p \times m} \\ \mathbf{h}_2(\mu_2, \mu_1 + \omega_1) \in \mathbb{R}^{p \times m} & \mathbf{h}_2(\mu_2, \mu_2 + \omega_1) \in \mathbb{R}^{p \times m} \\ \mathbf{h}_2(\mu_2, \mu_1 + \omega_2) \in \mathbb{R}^{p \times m} & \mathbf{h}_2(\mu_2, \mu_2 + \omega_2) \in \mathbb{R}^{p \times m} \end{bmatrix}$$

The data matrices  $\tilde{\mathbb{H}}, \tilde{\mathbb{M}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}}$  and  $\tilde{\mathbb{K}}$  now have kernel blocks as their entries. The definition of these matrices doesn't change; the only difference is that their dimensions are scaled by  $p$  or  $m$  as applicable. The other detail is that for each  $\mathbf{M}_i$  matrix, an appropriate  $\tilde{\mathbb{K}}_i$  would have to be constructed using Thm. 4.2.3. We proceed with the larger matrices with no changes to Algorithm 4.

## 4.5 Numerical tests

In this section, we test the efficiency of the proposed Quad BT algorithm for LQO systems given in Algorithm. 4 and compare it with that of classical balanced truncation for LQO systems. The first system we use for testing is the International Space Station benchmark (referred to as [iss1r]) [22] modelling the 1r component of the International Space Station. It has a system dimension of 270, an input dimension of 3 and an output dimension of 3. We modify the system by constructing a tri-diagonal  $\mathbf{M}$  matrix and using only the first components of the input and output vectors to obtain a SISO LQO system. We construct reduced matrices of order  $r = 30$  using BT and QBT. We pick  $N_p = N_q = N'_q = 1000, n_1 = 30$

equidistant nodes with unity weights between  $[0,50]$  for the QBT ROM. Fig. 4.1 shows the performance of Quad BT (QBT) when compared with BT. We observe that the absolute error of QBT ROM is comparable to that of the BT ROM.

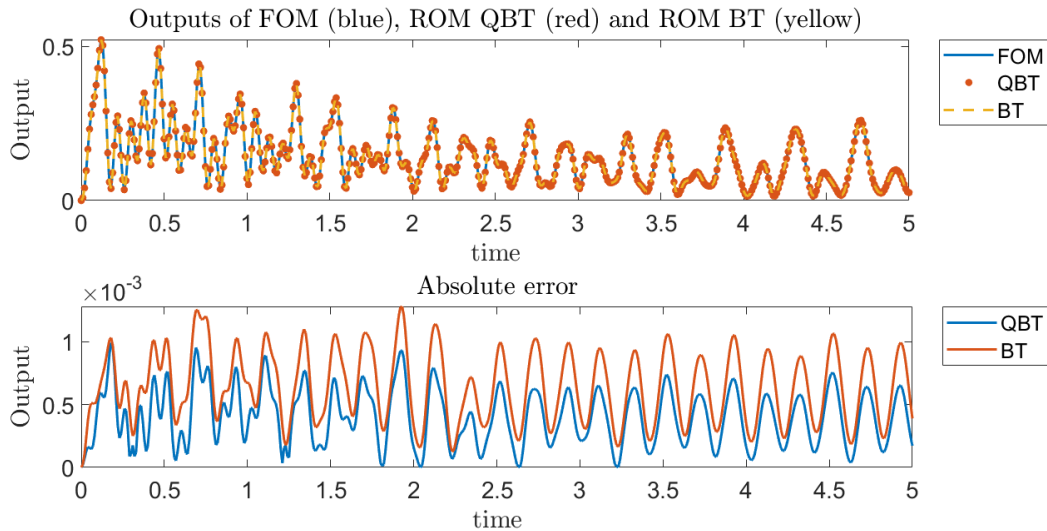


Figure 4.1: Outputs of the FOM, ROM (QBT) and ROM (BT) to input  $\mathbf{u}(t) = 5(\cos(5\pi t) + \sin(12\pi t)e^{-0.4t})$

#### 4.5.1 How to choose $N_p, N_q$ and quadrature rules?

There are no hard and fast rules on how to pick the number of quadrature points -  $N_p, N_q$ . Since we are using quadrature approximations, the more nodes we pick the better! However, we are limited by computational bottlenecks which is why one would have to use tricks like the one mentioned in Sec. 4.3. Throughout this section, we use equidistant nodes with unity weights (easiest quadrature rule). Using fancier quadrature rules might be a way to reduce the number of quadrature points required to obtain good reduced models. Another useful recipe in picking quadrature points would be to study the impulse response of the system and note the time  $t_e$  at which the response falls below a certain threshold. The quadrature rule would then be employed between time 0 and  $t_e$ .

## 4.5.2 Comparisons with Balanced truncation

In the following plots, we compare Quad BT with BT using the  $\mathcal{H}_2$  error metric used in Ref. [9]. The  $\mathcal{H}_2$  error between system  $\mathbf{H} = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{M})$  and  $\hat{\mathbf{H}} = (\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{M}})$  we use is,

$$\|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathcal{H}_2} = \sqrt{\text{tr}(\mathbf{B}^T \mathbf{Q} \mathbf{B} + \hat{\mathbf{B}} \hat{\mathbf{Q}} \hat{\mathbf{B}} - 2\mathbf{B}^T \mathbf{Z} \hat{\mathbf{B}})},$$

where  $\mathbf{Q}$  and  $\hat{\mathbf{Q}}$  are the observability Gramians for the systems  $\mathbf{H}$  and  $\hat{\mathbf{H}}$ , respectively, and  $\mathbf{Z}$  is the cross-Gramian, which is the solution of

$$\mathbf{A} \mathbf{Z} + \mathbf{Z} \hat{\mathbf{A}} + \mathbf{B} \hat{\mathbf{B}}^T = 0.$$

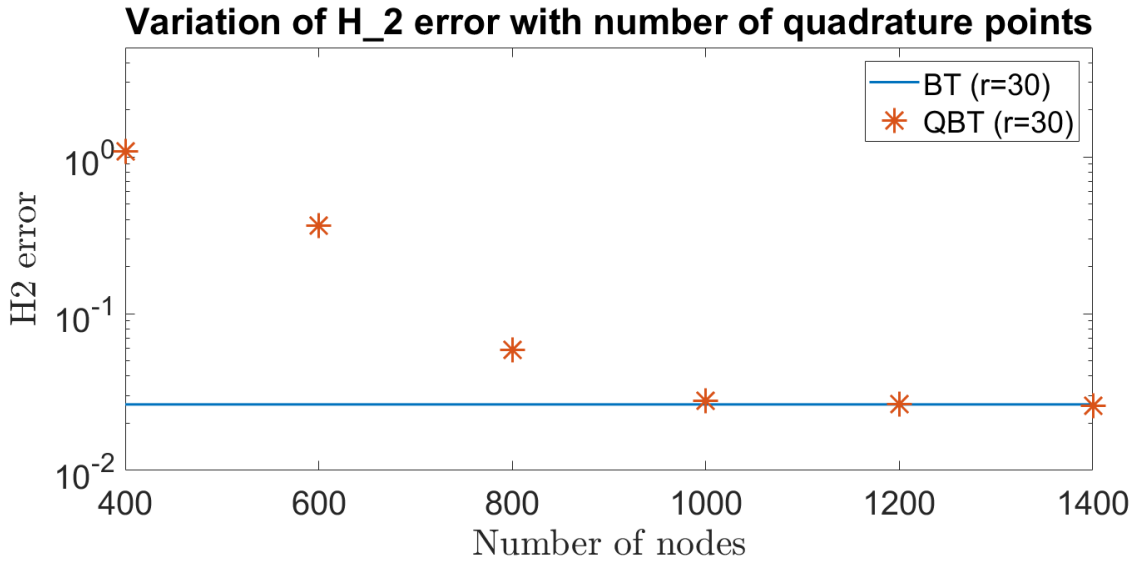


Figure 4.2: Variation of the  $\mathcal{H}_2$  error with the number of quadrature points in Quad BT

In Fig. 4.2 we show the  $\mathcal{H}_2$  error between the reduced order model and full order model as a function of the number of quadrature points used in the Quad BT ROM. The blue line serves as a reference which denotes the  $\mathcal{H}_2$  error of the ROM obtained from balanced truncation. The value of the truncation index is set at  $r = 30$  for all ROMs. We notice that the performance of Quad BT (while setting  $n_1 = 30$ ) is comparable to that of BT when we choose a sufficient number of quadrature points.

In Fig. 4.3, we plot the  $\mathcal{H}_2$  error comparison of the performance of BT and Quad BT

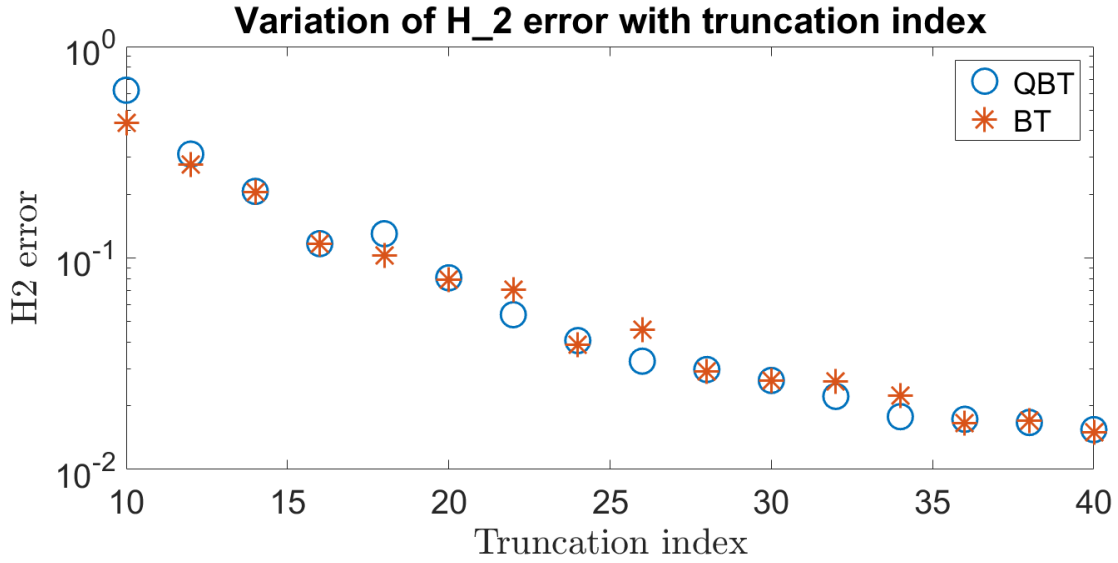


Figure 4.3: Variation of the  $\mathcal{H}_2$  error with truncation order for BT and QuadBT

reduced models as a function of the truncation index  $r$ . Naturally, the  $\mathcal{H}_2$  error decreases as we increase the value of  $r$  both for BT and Quad BT. The QuadBT ROM performance is comparable to that of BT in this case. In this case, we have used equidistant nodes with unity nodes between  $[0,50]$  to construct the Quad BT ROM with parameters  $N_p = N_q = N'_q = 1200, n_1 = 40$ .

# Chapter 5

## Quad BT for quadratic bilinear systems (QB)

Quadratic systems (QB) systems are denoted by the following equations,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{H}(\mathbf{x}(t) \otimes \mathbf{x}(t)) + \sum_{k=1}^p \mathbf{N}_k \mathbf{x}(t) \mathbf{u}_k(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t).\end{aligned}$$

where  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times p}$ ,  $\mathbf{C} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{N}_k \in \mathbb{R}^{n \times n}$ ,  $\mathbf{H} \in \mathbb{R}^{n \times n^2}$ ,  $\mathbf{u}(t) \in \mathbb{R}^p$ ,  $\mathbf{x}(t) \in \mathbb{R}^n$  and  $\mathbf{y}(t) \in \mathbb{R}^m$ . Considering  $\mathbf{H}$  to be the 1-matricization (see Sec. 2.2) of a tensor  $\mathcal{H} \in \mathbb{R}^{n \times n \times n}$ , it has been shown in Ref. [12, 16] that without loss of generality,

$$\mathbf{H}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{H}(\mathbf{v} \otimes \mathbf{u}) \quad \forall \mathbf{v}, \mathbf{u} \in \mathbb{R}^n,$$

and this also leads to the equality  $\mathbf{H}^{(2)} = \mathbf{H}^{(3)}$  where  $\mathbf{H}^{(2)}$  and  $\mathbf{H}^{(3)}$  denote the 2-matricization and 3-matricization of  $\mathcal{H}$  respectively. In this chapter, we assume that the input is scalar ( $p = 1$ ) to make the calculations less tedious. Under such an assumption, we would have a single bilinear term -  $\mathbf{N}$  instead of multiple  $\mathbf{N}_k$  terms. The results shown in this chapter can be easily extended to vector inputs. QB systems are an important class of systems since a large class of nonlinear systems can be written in this form by lifting.

The theory for general nonlinear systems has been studied in Ref. [24, 30, 31, 32]. How-

ever, these methods are based on the calculation of the energy functionals which are often difficult to compute. Lifting [33] offers a solution to this problem by allowing us to transform nonlinear systems containing exponentials, trigonometric functions, and polynomial terms into QB systems by introducing additional variables as shown in Example 5. The advantage of such methods is that the ‘lifting’ is exact and involves no approximation of the original system. In fact, the input-output response of the two systems are identical.

**Example 5.** *Consider the scalar nonlinear system*

$$\dot{\mathbf{x}}(t) = \mathbf{x}(t)e^{\mathbf{x}} + \mathbf{u}(t).$$

*Define the variables  $\mathbf{y} = e^{\mathbf{x}}$  and  $\mathbf{z} = \mathbf{x}e^{\mathbf{x}}$ . Then rewriting the system using  $\mathbf{y}$  and  $\mathbf{z}$  we get,*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{z} + \mathbf{u} \\ \dot{\mathbf{y}} &= e^{\mathbf{x}}\dot{\mathbf{x}} = e^{\mathbf{x}}(\mathbf{z} + \mathbf{u}) \\ &= \mathbf{yz} + \mathbf{yu} \\ \dot{\mathbf{z}} &= (e^{\mathbf{x}} + \mathbf{x}e^{\mathbf{x}})\dot{\mathbf{x}} = e^{\mathbf{x}}\mathbf{z} + \mathbf{zz} + e^{\mathbf{x}}\mathbf{u} + \mathbf{zu} \\ &= \mathbf{yz} + \mathbf{z}^2 + (\mathbf{y} + \mathbf{z})\mathbf{u} \end{aligned}$$

*Thus, by introducing new variables  $\mathbf{y}, \mathbf{z}$  (lifting) we have written a nonlinear system as a QB system.*

There have been several methods developed for the reduction of QB systems, see [12, 34, 35, 36]. The authors in Ref. [12] define the gramians and truncated gramians for QB systems and introduce balanced truncation based on the truncated gramians. The method they propose is intrusive and requires information on the state or the system matrices.

In this chapter, we formulate the data-driven Quad BT algorithm for QB systems. We begin by deriving the QB kernels and enumerating them in Sec. 5.1. Sec. 5.2 summarises some of the results from Ref. [12] which we build on in Sec. 5.3 to define the quadrature-based gramian approximations. In Sec. 5.4 we replace the approximate gramian terms with samples of the QB kernels (and their derivatives) and present the Quad BT algorithm for QB systems in Alg. 6. We address some of the computational challenges of our proposed algorithm in Sec. 5.5 and demonstrate its performance on some benchmark datasets in Sec. 5.6.

## 5.1 Kernels for QB systems

Consider a quadratic bilinear system  $\Sigma^{\text{QB}}$ . The idea is to rewrite the bilinear system as a sum of infinite subsystems. We consider the response to inputs of the form  $\alpha \mathbf{u}(t)$  where  $\alpha$  is an arbitrary scalar. Since the system  $\Sigma^{\text{QB}}$  is analytic, we can write the solution  $\mathbf{x}(t)$  as a power series in  $\alpha$ ,  $\mathbf{x}(t) = \sum_{k=1}^{\infty} \alpha^k \mathbf{x}_k(t)$ . Using these two statements, the solution is given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \alpha \mathbf{B}\mathbf{u}(t) + \alpha \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{H}(\mathbf{x}(t) \otimes \mathbf{x}(t)) \\ \sum_{k=1}^{\infty} \alpha^k \dot{\mathbf{x}}_k(t) &= \mathbf{A} \left( \sum_{k=1}^{\infty} \alpha^k \mathbf{x}_k(t) \right) + \alpha \mathbf{B}\mathbf{u}(t) + \alpha \mathbf{N} \left( \sum_{k=1}^{\infty} \alpha^k \mathbf{x}_k(t) \right) \mathbf{u}(t) \\ &\quad + \mathbf{H} \left( \left( \sum_{i=1}^{\infty} \alpha^i \mathbf{x}_i(t) \right) \otimes \left( \sum_{j=1}^{\infty} \alpha^j \mathbf{x}_j(t) \right) \right) \end{aligned}$$

Since the choice of  $\alpha$  is arbitrary, we must have that the terms corresponding to  $\alpha^k$  must be 0 for all values of  $k$ . Then, by collecting the corresponding terms we have  $\mathbf{x}(t) = \sum_{k=1}^{\infty} \mathbf{x}_k(t)$  with the equations for  $\mathbf{x}_k$  given by -

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{A}\mathbf{x}_1(t) + \mathbf{B}\mathbf{u}(t) \\ \dot{\mathbf{x}}_k(t) &= \mathbf{A}\mathbf{x}_k(t) + \mathbf{N}\mathbf{x}_{k-1}(t)\mathbf{u}(t) + \sum_{i=1}^{k-1} \mathbf{H}(\mathbf{x}_i(t) \otimes \mathbf{x}_{k-i}(t)) \quad \text{for } (k > 1) \end{aligned} \quad (5.1)$$

Let  $z(t) = e^{-\mathbf{A}t}\mathbf{x}(t)$ . Then computing  $\dot{z}(t)$  we get,

$$\dot{z}(t) = -\mathbf{A}e^{-\mathbf{A}t}\mathbf{x}(t) + e^{-\mathbf{A}t}\dot{\mathbf{x}}(t)$$

For  $k = 1$ , we have  $\mathbf{x}_1(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$ . For  $k > 1$ ,

$$\begin{aligned} \dot{\mathbf{z}}_k(\tau) &= -e^{\mathbf{A}\tau}\mathbf{A}\mathbf{x}_k(\tau) + e^{-\mathbf{A}\tau} \left[ \mathbf{A}\mathbf{x}_k(\tau) + \mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau) + \sum_{i=1}^{k-1} \mathbf{H}(\mathbf{x}_i(\tau) \otimes \mathbf{x}_{k-i}(\tau)) \right] \\ \mathbf{z}_k(t) &= \int_0^t e^{-\mathbf{A}\tau}\mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)d\tau + \sum_{i=1}^{k-1} \int_0^t e^{-\mathbf{A}\tau}\mathbf{H}(\mathbf{x}_i(\tau) \otimes \mathbf{x}_{k-i}(\tau))d\tau \\ \mathbf{x}_k(t) &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{N}\mathbf{x}_{k-1}(\tau)\mathbf{u}(\tau)d\tau + \sum_{i=1}^{k-1} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{H}(\mathbf{x}_i(\tau) \otimes \mathbf{x}_{k-i}(\tau))d\tau \end{aligned}$$

Plugging in the expression for  $k = 2$  we get,

$$\begin{aligned}
\mathbf{x}_2(t) &= \int_0^t e^{\mathbf{A}(t-t_1)} \mathbf{N} \left( \int_0^{t_1} e^{\mathbf{A}(t_1-t_2)} \mathbf{B} \mathbf{u}(t_2) dt_2 \right) \mathbf{u}(t_1) dt_1 + \\
&\int_0^t e^{\mathbf{A}(t-t_1)} \mathbf{H} \left[ \left( \int_0^{t_1} e^{\mathbf{A}(t_1-t_2)} \mathbf{B} \mathbf{u}(t_2) dt_2 \right) \otimes \left( \int_0^{t_1} e^{\mathbf{A}(t_1-t_3)} \mathbf{B} \mathbf{u}(t_3) dt_3 \right) \right] dt_1 \\
&= \int_0^t \int_0^{t_1} e^{\mathbf{A}(t-t_1)} \mathbf{N} e^{\mathbf{A}(t_1-t_2)} \mathbf{B} \mathbf{u}(t_1) \mathbf{u}(t_2) dt_2 dt_1 + \\
&\int_0^t \int_0^{t_1} \int_0^{t_1} e^{\mathbf{A}(t-t_1)} \mathbf{H} [e^{\mathbf{A}(t_1-t_2)} \mathbf{B} \mathbf{u}(t_2) \otimes e^{\mathbf{A}(t_1-t_3)} \mathbf{B} \mathbf{u}(t_3)] dt_2 dt_3 dt_1 \\
&= \int_0^t \int_0^{t_1} \mathbf{h}_2^{\mathbf{N}} \mathbf{u}(t_1) \mathbf{u}(t_2) dt_2 dt_1 + \int_0^t \int_0^{t_1} \int_0^{t_1} \mathbf{h}_2^{\mathbf{H}} (\mathbf{u}(t_2) \otimes \mathbf{u}(t_3)) dt_2 dt_3 dt_1
\end{aligned}$$

Now we call the two kernels  $\mathbf{h}_2^{\mathbf{N}}$  and  $\mathbf{h}_2^{\mathbf{H}}$  for brevity. Before writing the kernels, we explicitly write another term  $\mathbf{x}_3$ .

$$\begin{aligned}
\mathbf{x}_3(t) &= \int_0^t e^{\mathbf{A}(t-t_1)} \mathbf{N} \left[ \int_0^{t_1} \int_0^{t_2} e^{\mathbf{A}(t_1-t_2)} \mathbf{N} e^{\mathbf{A}(t_2-t_3)} \mathbf{B} \mathbf{u}(t_2) \mathbf{u}(t_3) dt_3 dt_2 \right] \mathbf{u}(t_1) dt_1 \\
&+ \int_0^t e^{\mathbf{A}(t-t_1)} \mathbf{N} \left[ \int_0^{t_1} \int_0^{t_2} \int_0^{t_2} e^{\mathbf{A}(t_1-t_2)} \mathbf{H} [e^{\mathbf{A}(t_2-t_3)} \mathbf{B} \mathbf{u}(t_3) \otimes e^{\mathbf{A}(t_2-t_4)} \mathbf{B} \mathbf{u}(t_4)] \mathbf{u}(t_1) dt_3 dt_4 dt_2 dt_1 \right. \\
&+ \int_0^t e^{\mathbf{A}(t-t_1)} \mathbf{H} \left[ \left( \int_0^{t_1} e^{\mathbf{A}(t_1-t_2)} \mathbf{B} \mathbf{u}(t_2) dt_2 \right) \otimes \int_0^{t_1} \int_0^{t_3} \mathbf{h}_2^{\mathbf{N}} \mathbf{u}(t_3) \mathbf{u}(t_4) dt_4 dt_3 \right] \\
&+ \int_0^t e^{\mathbf{A}(t-t_1)} \mathbf{H} \left[ \left( \int_0^{t_1} e^{\mathbf{A}(t_1-t_2)} \mathbf{B} \mathbf{u}(t_2) dt_2 \right) \otimes \int_0^{t_1} \int_0^{t_3} \int_0^{t_3} \mathbf{h}_2^{\mathbf{H}} (\mathbf{u}(t_4) \otimes \mathbf{u}(t_5)) dt_4 dt_5 dt_3 \right] \\
&+ \int_0^t e^{\mathbf{A}(t-t_1)} \mathbf{H} \left[ \int_0^{t_1} \int_0^{t_3} \mathbf{h}_2^{\mathbf{N}} \mathbf{u}(t_3) \mathbf{u}(t_4) dt_4 dt_3 \otimes \left( \int_0^{t_1} e^{\mathbf{A}(t_1-t_2)} \mathbf{B} \mathbf{u}(t_2) dt_2 \right) \right] dt_1 \\
&+ \int_0^t e^{\mathbf{A}(t-t_1)} \mathbf{H} \left[ \int_0^{t_1} \int_0^{t_3} \int_0^{t_3} \mathbf{h}_2^{\mathbf{H}} (\mathbf{u}(t_4) \otimes \mathbf{u}(t_5)) dt_4 dt_5 dt_3 \otimes \left( \int_0^{t_1} e^{\mathbf{A}(t_1-t_2)} \mathbf{B} \mathbf{u}(t_2) dt_2 \right) \right] dt_1
\end{aligned}$$

Now the first two terms in the expression for  $\mathbf{x}_3$  correspond to  $\int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{N} \mathbf{x}_2(\tau) \mathbf{u}(\tau) d\tau$  term while the last four correspond to the  $\sum_{i=1}^{k-1} \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{H} (\mathbf{x}_i(\tau) \otimes \mathbf{x}_{k-i}(\tau)) d\tau$ . Now let's enumerate all the kernels classified by the sublevels/subsystems they appear in. Each level has contributions from the bilinear term and from the quadratic part (terms of the form  $\mathbf{H}(\mathbf{x}_i \otimes \mathbf{x}_j)$  where  $i + j = k$ ). This allows us to define the kernels recursively -

**Definition 5.1.1.** Consider the partition of the QB system into infinite subsystems like in

Eq. 5.1. We define the set  $\mathbf{S}_k$  for each sublevel as follows,

$$\mathbf{S}_1 = \{e^{\mathbf{A}t}\mathbf{B}\}$$

$$\mathbf{S}_2 = \{e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{B}, e^{\mathbf{A}t}\mathbf{H}(e^{\mathbf{A}t}\mathbf{B} \otimes e^{\mathbf{A}t}\mathbf{B})\} \text{ and for } k > 2,$$

$$\mathbf{S}_k = \{e^{\mathbf{A}t}\mathbf{N}_{s_{k-1}}, e^{\mathbf{A}t}\mathbf{H}(s_i \otimes s_j)\}, \text{ where } s_k \in \mathbf{S}_{k-1} \ \& \ s_i \in \mathbf{S}_i, s_j \in \mathbf{S}_j \text{ such that } i + j = k$$

The kernels for the QB system are then defined as -

$$\mathbf{K}_1 = \{\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\}$$

$$\mathbf{K}_2 = \{\mathbf{C}e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{B}, \mathbf{C}e^{\mathbf{A}t}\mathbf{H}(e^{\mathbf{A}t}\mathbf{B} \otimes e^{\mathbf{A}t}\mathbf{B})\}$$

$$\mathbf{K}_3 = \{\mathbf{C}e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{B}, \mathbf{C}e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{H}(e^{\mathbf{A}t}\mathbf{B} \otimes e^{\mathbf{A}t}\mathbf{B}), \mathbf{C}e^{\mathbf{A}t}\mathbf{H}(e^{\mathbf{A}t}\mathbf{B} \otimes \mathbf{C}e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{B}), \dots\}$$

$$\mathbf{K}_k = \{\mathbf{C}e^{\mathbf{A}t}\mathbf{N}_{s_{k-1}}, \dots, \mathbf{C}e^{\mathbf{A}t}\mathbf{H}(s_i \otimes s_j), \dots\} \text{ where } s_k \in \mathbf{S}_{k-1} \ \& \ s_i \in \mathbf{S}_i, s_j \in \mathbf{S}_j \text{ such that } i + j = k$$

**Notation 5.1.1.** Define the set  $\Gamma = \{\mathbf{H}(-, -), \mathbf{N}\}$ . We denote the kernel  $\mathbf{h}_k^{(\gamma_1, \dots, \gamma_{k-1})}$  as the kernel associated with the  $k$ th subsystem with terms from  $\Gamma$  occurring in the order  $(\gamma_1, \dots, \gamma_{k-1})$  where  $\gamma_i \in \Gamma$ . Some examples:

$$\begin{aligned} \mathbf{h}_3^{\mathbf{N}, \mathbf{N}} &= \mathbf{C}e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{B} \\ \mathbf{h}_3^{\mathbf{N}, \mathbf{H}(-, -)} &= \mathbf{C}e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{H}(e^{\mathbf{A}t}\mathbf{B} \otimes e^{\mathbf{A}t}\mathbf{B}) \\ \mathbf{h}_3^{\mathbf{H}(-, \mathbf{N})} &= \mathbf{C}e^{\mathbf{A}t}\mathbf{H}(e^{\mathbf{A}t}\mathbf{B} \otimes e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{B}) \\ \mathbf{h}_4^{\mathbf{H}(\mathbf{H}(-, \mathbf{N}), -)} &= \mathbf{C}e^{\mathbf{A}t}\mathbf{H}(e^{\mathbf{A}t}\mathbf{H}(e^{\mathbf{A}t}\mathbf{B} \otimes e^{\mathbf{A}t}\mathbf{N}e^{\mathbf{A}t}\mathbf{B}) \otimes e^{\mathbf{A}t}\mathbf{B}) \end{aligned}$$

**Remark 5.1.1.** The superscript in the kernel notation not only describes the nested structure of the kernel but also allows one to calculate the number of arguments of the kernel. Number of arguments of the kernel = number of  $\mathbf{N}$ s +  $2 \times$  number of  $\mathbf{H}$ s + 1.

The kernels for level  $\mathbf{K}_k$  are obtained by multiplying the elements of  $\mathbf{S}_k$  by  $\mathbf{C}$ . The kernel  $\mathbf{h}_4^{\mathbf{H}(\mathbf{H}(-, \mathbf{N}), -)}$  above shows the kind of kernels that arise due to the nested structure of  $\mathbf{x}_k$  in Eq. 5.1. The superscript basically contains the left-to-right order of  $\mathbf{N}$  and  $\mathbf{H}$  appearing in the kernel. The notation ignores the  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  terms appearing in the kernel. The number of kernels in sublevel  $k$  grows quickly with  $k$  as shown below.

**Remark 5.1.2.** Let us denote the number of kernels at level  $k$  as  $n_k$ . Then we have

$$n_k = n_{k-1} + \sum_{i=1}^{k-1} n_i n_{k-i} \text{ where } n_1 = 1, n_2 = 2.$$

## 5.2 Gramians for QB systems

The definitions of the reachability and observability gramians for QB systems were first given in Ref. [12].

**Definition 5.2.1.** [12] *We can define the gramians for QB systems as follows:*

$$\begin{aligned}\bar{\mathbf{P}}_1(t_1) &= e^{\mathbf{A}t_1} \mathbf{B} \\ \bar{\mathbf{P}}_2(t_1, t_2) &= e^{\mathbf{A}t_2} \mathbf{N} (\mathbf{I}_m \otimes \bar{\mathbf{P}}_1(t_1)) \\ \bar{\mathbf{P}}_i(t_1, \dots, t_i) &= e^{\mathbf{A}t_i} [\mathbf{H} [\bar{\mathbf{P}}_1(t_1) \otimes \bar{\mathbf{P}}_{i-2}(t_2, \dots, t_{i-1}), \bar{\mathbf{P}}_2(t_1, t_2) \otimes \bar{\mathbf{P}}_{i-3}(t_3, \dots, t_{i-1}), \\ &\quad \dots, \bar{\mathbf{P}}_{i-2}(t_1, \dots, t_{i-2}) \otimes \bar{\mathbf{P}}_1(t_{i-1})], \\ &\quad \mathbf{N} (\mathbf{I}_m \otimes \bar{\mathbf{P}}_{i-1}(t_1, \dots, t_{i-1}))], \forall i \geq 3\end{aligned}$$

$$\mathbf{P} = \sum_{i=1}^{\infty} \mathbf{P}_i \quad \text{with} \quad \mathbf{P}_i = \int_0^{\infty} \dots \int_0^{\infty} \bar{\mathbf{P}}_i(t_1, \dots, t_i) \bar{\mathbf{P}}_i^T(t_1, \dots, t_i) dt_1 \dots dt_i.$$

$$\begin{aligned}\bar{\mathbf{Q}}_1(t_1) &= e^{\mathbf{A}^\top t_1} \mathbf{C}^\top, \\ \bar{\mathbf{Q}}_2(t_1, t_2) &= e^{\mathbf{A}^\top t_2} \mathbf{N}^\top (\mathbf{I}_m \otimes \bar{\mathbf{Q}}_1(t_1)), \\ &\quad \vdots \\ \bar{\mathbf{Q}}_i(t_1, \dots, t_i) &= e^{\mathbf{A}^\top t_i} [\mathbf{H}^{(2)} [\bar{\mathbf{P}}_1(t_1) \otimes \bar{\mathbf{Q}}_{i-2}(t_2, \dots, t_{i-1}), \\ &\quad \dots, \bar{\mathbf{P}}_{i-2}(t_1, \dots, t_{i-2}) \otimes \bar{\mathbf{Q}}_1(t_{i-1})], \\ &\quad \mathbf{N}^\top (\mathbf{I}_m \otimes \bar{\mathbf{Q}}_{i-1}(t_1, \dots, t_{i-1}))], \forall i \geq 3.\end{aligned}$$

$$\mathbf{Q} = \sum_{i=1}^{\infty} \mathbf{Q}_i \quad \text{with} \quad \mathbf{Q}_i = \int_0^{\infty} \dots \int_0^{\infty} \bar{\mathbf{Q}}_i(t_1, \dots, t_i) \bar{\mathbf{Q}}_i^T(t_1, \dots, t_i) dt_1 \dots dt_i.$$

These definitions are however difficult to work with since the associated Lyapunov equations are quadratic in  $\mathbf{P}$  and  $\mathbf{Q}$ . Therefore, we define the truncated gramians for the system which are easier to calculate.

**Definition 5.2.2.** [12] *Truncated gramians of a QB system as the sum of the first three*

components of the QB system

$$\mathbf{P}_{\mathcal{T}} = \sum_{i=1}^3 \mathbf{P}_i, \quad \text{where} \quad \mathbf{P}_i = \int_0^{\infty} \bar{\mathbf{P}}_i(t_1, \dots, t_i) \bar{\mathbf{P}}_i^T(t_1, \dots, t_i) dt_1 \cdots dt_i \quad (5.2)$$

$$\mathbf{Q}_{\mathcal{T}} = \sum_{i=1}^3 \mathbf{Q}_i, \quad \text{where} \quad \mathbf{Q}_i = \int_0^{\infty} \bar{\mathbf{Q}}_i(t_1, \dots, t_i) \bar{\mathbf{Q}}_i^T(t_1, \dots, t_i) dt_1 \cdots dt_i, \quad (5.3)$$

**Proposition 5.2.1.** *The truncated gramians of the QB system  $P_{\tau}, Q_{\tau}$  satisfy the following equations,*

$$\begin{aligned} \mathbf{A}\mathbf{P}_{\mathcal{T}} + \mathbf{P}_{\mathcal{T}}\mathbf{A}^{\top} + \mathbf{H}(\mathbf{P}_1 \otimes \mathbf{P}_1)\mathbf{H}^{\top} + \mathbf{N}\mathbf{P}_1\mathbf{N}^{\top} + \mathbf{B}\mathbf{B}^{\top} &= 0 \\ \mathbf{A}^{\top}\mathbf{Q}_{\mathcal{T}} + \mathbf{Q}_{\mathcal{T}}\mathbf{A}^{\top} + \mathbf{H}^{(2)}(\mathbf{Q}_1 \otimes \mathbf{Q}_1)\mathbf{H}^{(2)\top} + \mathbf{N}^{\top}\mathbf{Q}_1\mathbf{N} + \mathbf{C}^{\top}\mathbf{C} &= 0 \end{aligned}$$

**Remark 5.2.1.** *The advantage of using truncated gramians is evident from Prop. 5.2.1. Instead of having to solve a quadratic matrix equation, we have to solve two linear equations - first for  $\mathbf{P}_1$  (or  $\mathbf{Q}_1$ ) followed by substituting it into the equation for  $\mathbf{P}_{\mathcal{T}}$  (or  $\mathbf{Q}_{\mathcal{T}}$ ).*

The truncated gramians can be used to define balanced truncation for QB systems. The procedure remains the same as in the LTI or BQO case and is summarised below.

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**Algorithm 5** Balanced truncation for QB systems using truncated gramians

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**Input:** QB system given by matrices  $\mathbf{A}, \mathbf{H}, \mathbf{N}, \mathbf{B}, \mathbf{C}$  and truncation order  $r$

**Output:** BT ROM given by matrices  $\hat{\mathbf{A}}, \hat{\mathbf{H}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ .

1. Determine low rank approximation of  $\mathbf{P}_{\tau} \approx \mathbf{U}\mathbf{U}^{\top}$  and  $\mathbf{Q}_{\tau} \approx \mathbf{L}\mathbf{L}^{\top}$ .
2. Compute the SVD of the matrix  $\mathbf{L}^{\top}\mathbf{U}$ , partitioned as follows

$$\mathbf{L}^{\top}\mathbf{U} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 & \\ & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1^{\top} \\ \mathbf{Y}_2^{\top} \end{bmatrix},$$

where  $\mathbf{Z}_1 \in \mathbb{R}^{n \times r}$ ,  $\mathbf{S}_1 \in \mathbb{R}^{r \times r}$  and  $\mathbf{Y}_1^{\top} \in \mathbb{R}^{r \times n}$ .

3. Construct projection matrices  $\mathcal{W} = \mathbf{L}\mathbf{Z}_1\mathbf{S}_1^{-1/2}$  and  $\mathcal{V} = \mathbf{U}\mathbf{Y}_1\mathbf{S}_1^{-1/2}$  to obtain,  $\hat{\mathbf{A}} = \mathcal{W}^{\top}\mathbf{A}\mathcal{V}$ ,  $\hat{\mathbf{H}} = \mathcal{W}^{\top}\mathbf{H}(\mathbf{U} \otimes \mathbf{U})(\mathbf{Y}_1\mathbf{S}_1^{-1/2} \otimes \mathbf{Y}_1\mathbf{S}_1^{-1/2})$ ,  $\hat{\mathbf{N}} = \mathcal{W}^{\top}\mathbf{N}\mathcal{V}$ ,  $\hat{\mathbf{B}} = \mathcal{W}^{\top}\mathbf{B}$ ,  $\hat{\mathbf{C}} = \mathbf{C}\mathcal{V}$
-

### 5.3 Quadrature approximations of the gramians

Having defined the truncated gramians, we proceed by computing quadrature-based approximations to each component in typical Quad BT spirit! Just like in the linear and LQO case, the truncated QB gramians and their square root factors are approximated as -

$$\begin{aligned}
\tilde{\mathcal{P}}_1 &= \sum_{i=1}^{N_p} \rho_i^2 e^{\mathbf{A}\mu_i} \mathbf{B} \mathbf{B}^\top e^{\mathbf{A}^\top \mu_i} & \tilde{\mathbf{U}}_1 &= [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{B}] \\
\tilde{\mathcal{P}}_2 &= \sum_{i=1}^{N_p} \rho_i^2 e^{\mathbf{A}\mu_i} \mathbf{N} \tilde{\mathbf{U}}_1 \tilde{\mathbf{U}}_1^\top \mathbf{N}^\top e^{\mathbf{A}^\top \mu_i} & \tilde{\mathbf{U}}_2 &= [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{N} \tilde{\mathbf{U}}_1 \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{N} \tilde{\mathbf{U}}_1] \\
\tilde{\mathcal{P}}_3 &= \sum_{i=1}^{N_p} e^{\mathbf{A}\mu_i} \mathbf{H} (\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_1) \mathbf{H}^\top e^{\mathbf{A}^\top \mu_i} & \tilde{\mathbf{U}}_3 &= [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{H} (\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H} (\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1)]
\end{aligned}$$

We replace the  $\mathbf{U}_1$  terms in the definition for  $\mathbf{U}_2$  and  $\mathbf{U}_3$  with  $\tilde{\mathbf{U}}_1$  to obtain  $\tilde{\mathbf{U}}_2$  and  $\tilde{\mathbf{U}}_3$ . We follow this exact process for  $\mathbf{Q}_\tau$ .

$$\begin{aligned}
\tilde{\mathcal{Q}}_1 &= \sum_{j=1}^{N_q} \phi_j^2 e^{\mathbf{A}^\top \omega_j} \mathbf{C}^\top \mathbf{C} e^{\mathbf{A} \omega_j} & \tilde{\mathbf{L}}_1 &= [\phi_1 e^{\mathbf{A}^\top \omega_1} \mathbf{C}^\top \dots \phi_{N_q} e^{\mathbf{A}^\top \omega_{N_q}} \mathbf{C}^\top] \\
\tilde{\mathcal{Q}}_2 &= \sum_{j=1}^{N_q} \phi_j^2 e^{\mathbf{A}^\top \omega_j} \mathbf{N}^\top \tilde{\mathcal{Q}}_1 \mathbf{N} e^{\mathbf{A} \omega_j} & \tilde{\mathbf{L}}_2 &= [\phi_1 e^{\mathbf{A}^\top \omega_1} \mathbf{N}^\top \tilde{\mathbf{L}}_1 \dots \phi_{N_q} e^{\mathbf{A}^\top \omega_{N_q}} \mathbf{N}^\top \tilde{\mathbf{L}}_1] \\
\tilde{\mathcal{Q}}_3 &= \sum_{j=1}^{N_q} \phi_j^2 e^{\mathbf{A}^\top \omega_j} \mathbf{H}^{(2)} [\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{Q}}_1] \mathbf{H}^{(2)\top} e^{\mathbf{A} \omega_j} & \tilde{\mathbf{L}}_3 &= [\dots \phi_j e^{\mathbf{A}^\top \omega_j} \mathbf{H}^{(2)} [\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1] \dots]
\end{aligned}$$

Finally we obtain the quadrature-based square root terms  $\tilde{\mathbf{L}}$  and  $\tilde{\mathbf{U}}$  -

$$\tilde{\mathcal{Q}}_\tau = \tilde{\mathbf{L}} \tilde{\mathbf{L}}^\top = \begin{bmatrix} \tilde{\mathbf{L}}_1 & \tilde{\mathbf{L}}_2 & \tilde{\mathbf{L}}_3 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \\ \tilde{\mathbf{L}}_2^\top \\ \tilde{\mathbf{L}}_3^\top \end{bmatrix} \quad (5.4)$$

$$\tilde{\mathcal{P}}_\tau = \tilde{\mathbf{U}} \tilde{\mathbf{U}}^\top = \begin{bmatrix} \tilde{\mathbf{U}}_1 & \tilde{\mathbf{U}}_2 & \tilde{\mathbf{U}}_3 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{U}}_1^\top \\ \tilde{\mathbf{U}}_2^\top \\ \tilde{\mathbf{U}}_3^\top \end{bmatrix} \quad (5.5)$$

## 5.4 Replacing intrusive terms with data matrices

Recall the terms in Eq. 3. The terms  $\mathbf{S}_1, \mathbf{Y}_1, \mathbf{Z}_1$  are obtained from the SVD of  $\mathbf{L}^\top \mathbf{U}$ . This leaves us with the following terms to deal with -  $\mathbf{L}^\top \mathbf{U}$ ,  $\mathbf{L}^\top \mathbf{A} \mathbf{U}$ ,  $\mathbf{L}^\top \mathbf{N} \mathbf{U}$ ,  $\mathbf{L}^\top \mathbf{B}$ ,  $\mathbf{C} \mathbf{U}$  and  $\mathbf{L}^\top \mathbf{H}(\mathbf{U} \otimes \mathbf{U})$ . In order to obtain a reduced model in a non-intrusive fashion, we need to substitute these terms with data-driven quantities. The terms  $\tilde{\mathbb{H}} = \tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}$ ,  $\tilde{\mathbb{M}} = \tilde{\mathbf{L}}^\top \mathbf{A} \tilde{\mathbf{U}}$ ,  $\tilde{\mathbb{N}} = \tilde{\mathbf{L}}^\top \mathbf{N} \tilde{\mathbf{U}}$ ,  $\tilde{\mathbf{h}} = \tilde{\mathbf{L}}^\top \mathbf{B}$ ,  $\tilde{\mathbf{g}} = \mathbf{C} \tilde{\mathbf{U}}$  and  $\tilde{\mathbb{K}} = \tilde{\mathbf{L}}^\top \mathbf{H}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}})$  are constructed from the quadrature based approximate gramians defined in Eq. 4.11. In the following section, we proceed to show that these quadrature-based matrices can be exactly written as products of quadrature weights and appropriate kernels (and their derivatives) of the QB system. The first term we consider is the  $\tilde{\mathbb{H}} = \tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}$  term.

**Theorem 5.4.1.** *Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 5.4 and Eq. 5.5, define the matrix  $\tilde{\mathbb{H}} = \tilde{\mathbf{L}}^\top \tilde{\mathbf{U}}$ , then we have,*

$$\tilde{\mathbb{H}} = \begin{cases} \phi_i \rho_j \mathbf{h}_1(\omega_i + \mu_j), & \text{for } \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}}_1, 1 \leq i \leq N_q \ \& \ 1 \leq j \leq N_p \\ \phi_i \rho_{j,k} \mathbf{h}_2^{\mathbf{N}}(\omega_i + \mu_j, \mu_k), & \text{for } \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}}_2, 1 \leq i \leq N_q \ \& \ 1 \leq j, k \leq N_p \\ \phi_i \rho_{j,k,l} \mathbf{h}_2^{\mathbf{H}}(\omega_i + \mu_j, \mu_k, \mu_l), & \text{for } \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}}_3, 1 \leq i \leq N_q \ \& \ 1 \leq j, k, l \leq N_p \\ \phi_{i,j} \rho_k \mathbf{h}_2^{\mathbf{N}}(\omega_i, \omega_j + \mu_k), & \text{for } \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}}_1, 1 \leq i, j \leq N_q, 1 \leq k \leq N_p \\ \phi_{i,j} \rho_{k,l} \mathbf{h}_3^{\mathbf{N},\mathbf{N}}(\omega_i, \omega_j + \mu_k, \mu_l), & \text{for } \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}}_2, 1 \leq i, j \leq N_q, 1 \leq k, l \leq N_p \\ \phi_{i,j} \rho_{k,l,m} \mathbf{h}_3^{\mathbf{N},\mathbf{H}}(\omega_i, \omega_j + \mu_k, \mu_l, \mu_m), & \text{for } \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}}_3, 1 \leq i, j \leq N_q, 1 \leq k, l, m \leq N_p \\ \phi_{i,j} \rho_{k,l} \mathbf{h}_2^{\mathbf{H}}(\omega_i, \mu_k, \omega_j + \mu_l)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_1, 1 \leq i, j \leq N_q, 1 \leq k, l \leq N_p \\ \phi_{i,j} \rho_{k,l,m} \mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}(\omega_i, \mu_k, \omega_j + \mu_l, \mu_m)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_2, 1 \leq i, j \leq N_q, 1 \leq k, l, m \leq N_p \\ \phi_{i,j} \rho_{k,l,m,n} \mathbf{h}_3^{\mathbf{H}(-,\mathbf{H})}(\omega_i, \mu_k, \omega_j + \mu_l, \mu_m, \omega_n)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_3, 1 \leq i, j \leq N_q, 1 \leq k, l, m, n \leq N_p \end{cases} \quad (5.6)$$

$\tilde{\mathbb{H}}$  has the structure of a Hankel matrix (see [13]) and has a block structure,

$$\tilde{\mathbb{H}} = \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}}_3 \\ \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}}_3 \\ \tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_3 \end{bmatrix} \in \mathbb{R}^{(N_q + N_q^2 + N_q^2 N_p) \times (N_p + N_p^2 + N_p^3)}$$

**Proof.** This follows from the direct multiplication of the terms. The calculation for the terms  $\tilde{\mathbf{L}}_i^\top \tilde{\mathbf{U}}_j$  where  $i, j \leq 2$  are straightforward and shown in Ref. [8]. We focus on the terms containing  $\tilde{\mathbf{L}}_3$  or  $\tilde{\mathbf{U}}_3$  since they are slightly more involved. First, we consider the terms  $\tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}}_3$

and  $\tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}}_3$  since they are simplest.

$$\begin{aligned}
\tilde{\mathbf{L}}_1^\top \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_j \mathbf{C} e^{\mathbf{A}\omega_1} \\ \vdots \end{bmatrix} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \right] \\
&= \phi_i \rho_j \rho_k \mathbf{C} e^{\mathbf{A}\omega_i} e^{\mathbf{A}\mu_j} \mathbf{H} (\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \rho_l e^{\mathbf{A}\mu_l} \mathbf{B}) \\
&= [\phi_i \rho_{j,k,l} \mathbf{h}_2^{\mathbf{H}}(\omega_i + \mu_j, \mu_k, \mu_l)] \\
\tilde{\mathbf{L}}_2^\top \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_j \tilde{\mathbf{L}}_1^\top & \mathbf{N} e^{\mathbf{A}\omega_1} \\ \vdots & \vdots \end{bmatrix} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \right] \\
&= \phi_i \phi_j \rho_k \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{N} e^{\mathbf{A}\omega_j} e^{\mathbf{A}\mu_k} \mathbf{H} (\rho_l e^{\mathbf{A}\mu_l} \mathbf{B} \otimes \rho_m e^{\mathbf{A}\mu_m} \mathbf{B}) \\
&= [\phi_{i,j} \rho_{k,l,m} \mathbf{h}_3^{\mathbf{N},\mathbf{H}}(\omega_i, \omega_j + \mu_k, \mu_l, \mu_m)]
\end{aligned}$$

We now reach the  $\tilde{\mathbf{L}}_3 \tilde{\mathbf{U}}$  terms. The procedure for handling these terms is the same. We use the properties of matricizations and transform the second matricization  $\mathbf{H}^{(2)}$  term to its first matricization  $\mathbf{H}$  using Eq. 1.

$$\begin{aligned}
\tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_1 &= \begin{bmatrix} \phi_1 e^{\mathbf{A}^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ \vdots & \vdots \end{bmatrix} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{B} \right] \\
&= \rho_l \phi_j [\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \phi_i e^{\mathbf{A}^\top \omega_i} \mathbf{C}^\top]^\top \mathbf{H}^{(2)\top} e^{\mathbf{A}\omega_j} e^{\mathbf{A}\mu_l} \mathbf{B} \\
&= \rho_l \phi_j \left[ e^{\mathbf{A}^\top (\omega_j + \mu_l)} \mathbf{B} \otimes \rho_k \mathbf{B}^\top e^{\mathbf{A}^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{\mathbf{A}^\top \omega_l} \mathbf{C}^\top \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\
&= [\phi_{i,j} \rho_{k,l} \mathbf{h}_2^{\mathbf{H}}(\omega_i, \mu_k, \omega_j + \mu_l)^\top]
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_2 &= \begin{bmatrix} \phi_1 e^{\mathbf{A}^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ \vdots & \vdots \end{bmatrix} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{N} \tilde{\mathbf{U}}_1 \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{N} \tilde{\mathbf{U}}_1 \right] \\
&= \rho_l \phi_j [\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \phi_i e^{\mathbf{A}^\top \omega_i} \mathbf{C}^\top]^\top \mathbf{H}^{(2)\top} \underbrace{e^{\mathbf{A}\omega_j} e^{\mathbf{A}\mu_l} \mathbf{N} e^{\mathbf{A}\mu_m} \mathbf{B}}_{\text{Let's call this } \mathbf{\Gamma}} \\
&= \rho_l \phi_j \left[ \mathbf{\Gamma}^\top \otimes \rho_k \mathbf{B}^\top e^{\mathbf{A}^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{\mathbf{A}^\top \omega_l} \mathbf{C}^\top \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\
&= [\phi_{i,j} \rho_{k,l,m} \mathbf{h}_3^{\mathbf{H}^{(-,\mathbf{N})}}(\omega_i, \mu_k, \omega_j + \mu_l, \mu_m)^\top]
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_1 e^{A^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ & \vdots \end{bmatrix} \left[ \rho_1 e^{A\mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \dots \rho_{N_p} e^{A\mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \right] \\
&= \rho_l \phi_j \left[ \rho_k e^{A\mu_k} \mathbf{B} \otimes \phi_i e^{A^\top \omega_i C^\top} \right]^\top \mathbf{H}^{(2)\top} \underbrace{e^{A\omega_j} e^{A\mu_l} \mathbf{H} (\rho_m e^{A\mu_m} \mathbf{B} \otimes \rho_n e^{A\mu_n} \mathbf{B})}_{\text{Let's call this } \mathbf{\Gamma}} \\
&= \rho_l \phi_j \left[ \mathbf{\Gamma}^\top \otimes \rho_k \mathbf{B}^\top e^{A^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{A^\top \omega_l C^\top} \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\
&= \left[ \phi_{i,j} \rho_{k,l,m,n} \mathbf{h}_3^{\mathbf{H}(-,\mathbf{H})} (\omega_i, \mu_k, \omega_j + \mu_l, \mu_m, \mu_n)^\top \right]
\end{aligned}$$

□

We move on to the  $\tilde{\mathbf{L}}^\top \mathbf{A} \mathbf{U}$  term which can be approximated with derivatives of the QB kernels. Unlike the linear case, we have partial derivatives of the kernels as elements of the matrix  $\tilde{\mathbf{M}}$ .

**Theorem 5.4.2.** *Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 5.4 and Eq. 5.5, define the matrix  $\tilde{\mathbf{M}} = \tilde{\mathbf{L}}^\top \mathbf{A} \tilde{\mathbf{U}}$ , then we have,*

$$\tilde{\mathbf{M}} = \begin{cases} \phi_i \rho_j \frac{\partial \mathbf{h}_1}{\partial t_1} (\omega_i + \mu_j), & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}}_1, 1 \leq i \leq N_q \ \& \ 1 \leq j \leq N_p \\ \phi_i \rho_{j,k} \frac{\partial \mathbf{h}_2^{\mathbf{N}}}{\partial t_1} (\omega_i + \mu_j, \mu_k), & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}}_2, 1 \leq i \leq N_q \ \& \ 1 \leq j, k \leq N_p \\ \phi_i \rho_{j,k,l} \frac{\partial \mathbf{h}_2^{\mathbf{H}}}{\partial t_1} (\omega_i + \mu_j, \mu_k, \mu_l), & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}}_3, 1 \leq i \leq N_q \ \& \ 1 \leq j, k, l \leq N_p \\ \phi_{i,j} \rho_k \frac{\partial \mathbf{h}_2^{\mathbf{N}}}{\partial t_2} (\omega_i, \omega_j + \mu_k), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}}_1, 1 \leq i, j \leq N_q, 1 \leq k \leq N_p \\ \phi_{i,j} \rho_{k,l} \frac{\partial \mathbf{h}_3^{\mathbf{N},\mathbf{N}}}{\partial t_2} (\omega_i, \omega_j + \mu_k, \mu_l), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}}_2, 1 \leq i, j \leq N_q, 1 \leq k, l \leq N_p \\ \phi_{i,j} \rho_{k,l,m} \frac{\partial \mathbf{h}_3^{\mathbf{N},\mathbf{H}}}{\partial t_2} (\omega_i, \omega_j + \mu_k, \mu_l, \mu_m), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}}_3, 1 \leq i, j \leq N_q, 1 \leq k, l, m \leq N_p \\ \phi_{i,j} \rho_{k,l} \frac{\partial \mathbf{h}_2^{\mathbf{H}}}{\partial t_3} (\omega_i, \mu_k, \omega_j + \mu_l)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_1, 1 \leq i, j \leq N_q, 1 \leq k, l \leq N_p \\ \phi_{i,j} \rho_{k,l,m} \frac{\partial \mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}}{\partial t_3} (\omega_i, \mu_k, \omega_j + \mu_l, \mu_m)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_2, 1 \leq i, j \leq N_q, 1 \leq k, l, m \leq N_p \\ \phi_{i,j} \rho_{k,l,m,n} \frac{\partial \mathbf{h}_3^{\mathbf{H}(-,\mathbf{H})}}{\partial t_3} (\omega_i, \mu_k, \omega_j + \mu_l, \mu_m, \omega_n)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_3, 1 \leq i, j \leq N_q, 1 \leq k, l, m, n \leq N_p \end{cases} \quad (5.7)$$

$\tilde{\mathbf{M}}$  has the structure of a shifted Hankel matrix (see [13]) with the following block structure,

$$\tilde{\mathbf{M}} = \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}}_3 \\ \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}}_3 \\ \tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_3 \end{bmatrix} \in \mathbb{R}^{(N_q + N_q^2 + N_q^3) \times (N_p + N_p^2 + N_p^3)}$$

**Proof.** The calculations for  $\tilde{\mathbf{M}}$  are exactly the same as in  $\tilde{\mathbf{H}}$  but with an  $\mathbf{A}$  term appearing between the two terms. The  $\tilde{\mathbf{M}}$  term is composed of derivatives of the kernels appearing in

$\tilde{\mathbb{H}}$ . Just like  $\tilde{\mathbb{H}}$  we only show the calculations for terms involving  $\tilde{\mathbf{L}}_3^\top$  and  $\tilde{\mathbf{U}}_3$ .

$$\begin{aligned}\tilde{\mathbf{L}}_1^\top \mathbf{A} \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_1 & \mathbf{C}e^{\mathbf{A}\omega_1} \\ & \vdots \end{bmatrix} \mathbf{A} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1) \right] \\ &= \phi_i \rho_j \rho_k \mathbf{C} \mathbf{A} e^{\mathbf{A}\omega_i} e^{\mathbf{A}\mu_j} \mathbf{H}(\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \rho_l e^{\mathbf{A}\mu_l} \mathbf{B}) \\ &= \left[ \phi_i \rho_{j,k,l} \frac{\partial \mathbf{h}_2^{\mathbf{H}}}{\partial t_1}(\omega_i + \mu_j, \mu_k, \mu_l) \right]\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{L}}_2^\top \mathbf{A} \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_1 \tilde{\mathbf{L}}_1^\top & \mathbf{N}e^{\mathbf{A}\omega_1} \\ & \vdots \end{bmatrix} \mathbf{A} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1) \right] \\ &= \phi_i \phi_j \rho_k \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{N} e^{\mathbf{A}\omega_j} \mathbf{A} e^{\mathbf{A}\mu_k} \mathbf{H}(\rho_l e^{\mathbf{A}\mu_l} \mathbf{B} \otimes \rho_m e^{\mathbf{A}\mu_m} \mathbf{B}) \\ &= \left[ \phi_{i,j} \rho_{k,l,m} \frac{\partial \mathbf{h}_3^{\mathbf{N},\mathbf{H}}}{\partial t_2}(\omega_i, \omega_j + \mu_k, \mu_l, \mu_m) \right]\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_1 &= \begin{bmatrix} \phi_1 e^{\mathbf{A}^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ & \vdots \end{bmatrix} \mathbf{A} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{B} \right] \\ &= \rho_l \phi_j [\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \phi_i e^{\mathbf{A}^\top \omega_i} \mathbf{C}^\top]^\top \mathbf{H}^{(2)\top} e^{\mathbf{A}\omega_j} \mathbf{A} e^{\mathbf{A}\mu_l} \mathbf{B} \\ &= \rho_l \phi_j \left[ \mathbf{A}^\top e^{\mathbf{A}^\top (\omega_j + \mu_l)} \mathbf{B} \otimes \rho_k \mathbf{B}^\top e^{\mathbf{A}^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{\mathbf{A}^\top \omega_l} \mathbf{C}^\top \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\ &= \left[ \phi_{i,j} \rho_{k,l} \frac{\partial \mathbf{h}_2^{\mathbf{H}}}{\partial t_3}(\omega_i, \mu_k, \omega_j + \mu_l)^\top \right]\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_2 &= \begin{bmatrix} \phi_1 e^{\mathbf{A}^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ & \vdots \end{bmatrix} \mathbf{A} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{N} \tilde{\mathbf{U}}_1 \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{N} \tilde{\mathbf{U}}_1 \right] \\ &= \rho_l \phi_j [\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \phi_i e^{\mathbf{A}^\top \omega_i} \mathbf{C}^\top]^\top \mathbf{H}^{(2)\top} \underbrace{e^{\mathbf{A}\omega_j} \mathbf{A} e^{\mathbf{A}\mu_l} \mathbf{N} e^{\mathbf{A}\mu_m} \mathbf{B}}_{\text{Let's call this } \mathbf{\Gamma}} \\ &= \rho_l \phi_j \left[ \mathbf{\Gamma}^\top \otimes \rho_k \mathbf{B}^\top e^{\mathbf{A}^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{\mathbf{A}^\top \omega_l} \mathbf{C}^\top \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\ &= \left[ \phi_{i,j} \rho_{k,l,m} \frac{\partial \mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}}{\partial t_3}(\omega_i, \mu_k, \omega_j + \mu_l, \mu_m)^\top \right]\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{L}}_3^\top \mathbf{A} \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_1 e^{A^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ \vdots & \vdots \end{bmatrix} \mathbf{A} \left[ \rho_1 e^{A \mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \dots \rho_{N_p} e^{A \mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \right] \\
&= \rho_l \phi_j \left[ \rho_k e^{A \mu_k} \mathbf{B} \otimes \phi_i e^{A^\top \omega_i C^\top} \right]^\top \mathbf{H}^{(2)\top} \underbrace{\mathbf{A} e^{A \omega_j} e^{A \mu_i} \mathbf{H} \left( \rho_m e^{A \mu_m} \mathbf{B} \otimes \rho_n e^{A \mu_n} \mathbf{B} \right)}_{\text{Let's call this } \Gamma} \\
&= \rho_l \phi_j \left[ \Gamma^\top \otimes \rho_k \mathbf{B}^\top e^{A^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{A^\top \omega_l C^\top} \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\
&= \left[ \phi_{i,j} \rho_{k,l,m,n} \frac{\partial \mathbf{h}_3^{\mathbf{H}(-,\mathbf{H})}}{\partial t_3} (\omega_i, \mu_k, \omega_j + \mu_l, \mu_m, \mu_n)^\top \right]
\end{aligned}$$

**Theorem 5.4.3.** Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 5.4 and Eq. 5.5, define the matrix  $\tilde{\mathbf{N}} = \tilde{\mathbf{L}}^\top \mathbf{N} \tilde{\mathbf{U}}$ , then we have,

$$\tilde{\mathbf{N}} = \begin{cases} \phi_i \rho_j \mathbf{h}_2^{\mathbf{N}}(\omega_i, \mu_j), & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{N} \tilde{\mathbf{U}}_1, 1 \leq i \leq N_q \ \& \ 1 \leq j \leq N_p \\ \phi_i \rho_{j,k} \mathbf{h}_3^{\mathbf{N},\mathbf{N}}(\omega_i, \mu_j, \mu_k), & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{N} \tilde{\mathbf{U}}_2, 1 \leq i \leq N_q \ \& \ 1 \leq j, k \leq N_p \\ \phi_i \rho_{j,k,l} \mathbf{h}_3^{\mathbf{N},\mathbf{H}}(\omega_i, \mu_j, \mu_k, \mu_l), & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{N} \tilde{\mathbf{U}}_3, 1 \leq i \leq N_q \ \& \ 1 \leq j, k, l \leq N_p \\ \phi_{i,j} \rho_k \mathbf{h}_3^{\mathbf{N},\mathbf{N}}(\omega_i, \omega_j, \mu_k), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{N} \tilde{\mathbf{U}}_1, 1 \leq i, j \leq N_q, 1 \leq k \leq N_p \\ \phi_{i,j} \rho_{k,l} \mathbf{h}_4^{\mathbf{N},\mathbf{N},\mathbf{N}}(\omega_i, \omega_j, \mu_k, \mu_l), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{N} \tilde{\mathbf{U}}_2, 1 \leq i, j \leq N_q, 1 \leq k, l \leq N_p \\ \phi_{i,j} \rho_{k,l,m} \mathbf{h}_4^{\mathbf{N},\mathbf{N},\mathbf{H}}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{N} \tilde{\mathbf{U}}_3, 1 \leq i, j \leq N_q, 1 \leq k, l, m \leq N_p \\ \phi_{i,j} \rho_{k,l} \mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}(\omega_i, \mu_k, \omega_j, \mu_l)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_1, 1 \leq i, j \leq N_q, 1 \leq k, l \leq N_p \\ \phi_{i,j} \rho_{k,l,m} \mathbf{h}_4^{\mathbf{H}(-,\mathbf{N},\mathbf{N})}(\omega_i, \mu_k, \omega_j, \mu_l, \mu_m)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_2, 1 \leq i, j \leq N_q, 1 \leq k, l, m \leq N_p \\ \phi_{i,j} \rho_{k,l,m,n} \mathbf{h}_4^{\mathbf{H}(-,\mathbf{N},\mathbf{H})}(\omega_i, \mu_k, \omega_j, \mu_l, \mu_m, \mu_n)^\top, & \text{for } \tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_3, 1 \leq i, j \leq N_q, 1 \leq k, l, m, n \leq N_p \end{cases} \quad (5.8)$$

$\tilde{\mathbf{N}}$  can also be written as a block matrices as follows,

$$\tilde{\mathbf{N}} = \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \mathbf{N} \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_1^\top \mathbf{N} \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_1^\top \mathbf{N} \tilde{\mathbf{U}}_3 \\ \tilde{\mathbf{L}}_2^\top \mathbf{N} \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_2^\top \mathbf{N} \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_2^\top \mathbf{N} \tilde{\mathbf{U}}_3 \\ \tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_1 & \tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_2 & \tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_3 \end{bmatrix} \in \mathbb{R}^{(N_q + N_q^2 + N_q^3) \times (N_p + N_p^2 + N_p^3)}$$

**Proof.** The calculations for  $\tilde{\mathbf{N}}$  are exactly the same as in  $\tilde{\mathbf{H}}$  but with an  $\mathbf{N}$  term appearing between the two terms. Just like  $\tilde{\mathbf{H}}$  we only show the calculations for terms involving  $\tilde{\mathbf{L}}_3^\top$

and  $\tilde{\mathbf{U}}_3$ .

$$\begin{aligned}
\tilde{\mathbf{L}}_1^\top \mathbf{N} \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_1 & \mathbf{C}e^{\mathbf{A}\omega_1} \\ & \vdots \end{bmatrix} \mathbf{N} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \right] \\
&= \phi_i \rho_j \rho_k \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{N} e^{\mathbf{A}\mu_j} \mathbf{H} (\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \rho_l e^{\mathbf{A}\mu_l} \mathbf{B}) \\
&= \left[ \phi_i \rho_{j,k,l} \mathbf{h}_3^{\mathbf{N},\mathbf{H}}(\omega_i, \mu_j, \mu_k, \mu_l) \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{L}}_2^\top \mathbf{N} \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_1 \tilde{\mathbf{L}}_1^\top & \mathbf{N} e^{\mathbf{A}\omega_1} \\ & \vdots \end{bmatrix} \mathbf{N} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \right] \\
&= \phi_i \phi_j \rho_k \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{N} e^{\mathbf{A}\omega_j} \mathbf{N} e^{\mathbf{A}\mu_k} \mathbf{H} (\rho_l e^{\mathbf{A}\mu_l} \mathbf{B} \otimes \rho_m e^{\mathbf{A}\mu_m} \mathbf{B}) \\
&= \left[ \phi_{i,j} \rho_{k,l,m} \mathbf{h}_4^{\mathbf{N},\mathbf{N},\mathbf{H}}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m)^\top \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_1 &= \begin{bmatrix} \phi_1 e^{\mathbf{A}^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ & \vdots \end{bmatrix} \mathbf{N} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{B} \right] \\
&= \rho_l \phi_j [\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \phi_i e^{\mathbf{A}^\top \omega_i} \mathbf{C}^\top]^\top \mathbf{H}^{(2)\top} e^{\mathbf{A}\omega_j} \mathbf{N} e^{\mathbf{A}\mu_l} \mathbf{B} \\
&= \rho_l \phi_j \left[ \mathbf{A}^\top e^{\mathbf{A}^\top \omega_j} \mathbf{N} e^{\mathbf{A}^\top \mu_l} \mathbf{B} \otimes \rho_k \mathbf{B}^\top e^{\mathbf{A}^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{\mathbf{A}^\top \omega_l} \mathbf{C}^\top \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\
&= \left[ \phi_{i,j} \rho_{k,l} \mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}(\omega_i, \mu_k, \omega_j, \mu_l)^\top \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_2 &= \begin{bmatrix} \phi_1 e^{\mathbf{A}^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ & \vdots \end{bmatrix} \mathbf{N} \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{N} \tilde{\mathbf{U}}_1 \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{N} \tilde{\mathbf{U}}_1 \right] \\
&= \rho_l \phi_j [\rho_k e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \phi_i e^{\mathbf{A}^\top \omega_i} \mathbf{C}^\top]^\top \mathbf{H}^{(2)\top} \underbrace{e^{\mathbf{A}\omega_j} \mathbf{N} e^{\mathbf{A}\mu_l} \mathbf{N} e^{\mathbf{A}\mu_m} \mathbf{B}}_{\text{Let's call this } \mathbf{\Gamma}} \\
&= \rho_l \phi_j \left[ \mathbf{\Gamma}^\top \otimes \rho_k \mathbf{B}^\top e^{\mathbf{A}^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{\mathbf{A}^\top \omega_l} \mathbf{C}^\top \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\
&= \left[ \phi_{i,j} \rho_{k,l,m} \mathbf{h}_4^{\mathbf{H}(-,\mathbf{NN})}(\omega_i, \mu_k, \omega_j, \mu_l, \mu_m)^\top \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{L}}_3^\top \mathbf{N} \tilde{\mathbf{U}}_3 &= \begin{bmatrix} \phi_1 e^{A^\top \omega_1} & H^{(2)} \left[ \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{L}}_1 \right] \\ \vdots & \end{bmatrix} \mathbf{N} \left[ \rho_1 e^{A \mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \dots \rho_{N_p} e^{A \mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1 \right) \right] \\
&= \rho_l \phi_j \left[ \rho_k e^{A \mu_k} \mathbf{B} \otimes \phi_i e^{A^\top \omega_i} \mathbf{C}^\top \right]^\top \mathbf{H}^{(2)\top} \underbrace{e^{A \omega_j} \mathbf{N} e^{A \mu_l} \mathbf{H} \left( \rho_m e^{A \mu_m} \mathbf{B} \otimes \rho_n e^{A \mu_n} \mathbf{B} \right)}_{\text{Let's call this } \Gamma} \\
&= \rho_l \phi_j \left[ \Gamma^\top \otimes \rho_k \mathbf{B}^\top e^{A^\top \mu_k} \right] \mathbf{H}^\top \phi_l e^{A^\top \omega_l} \mathbf{C}^\top \quad (\text{Using property of } \mathbf{H} \text{ Eq. 1}) \\
&= \left[ \phi_{i,j} \rho_{k,l,m,n} \mathbf{h}_4^{\mathbf{H}(-,\mathbf{NH})}(\omega_i, \mu_k, \omega_j, \mu_l, \mu_m, \mu_n)^\top \right]
\end{aligned}$$

**Remark 5.4.1.** In the case where we have an input vector (not scalar), each of the  $\mathbf{L}^\top \mathbf{N}_k \mathbf{U}$  terms can be replaced in exactly the same way.

**Theorem 5.4.4.** Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 5.4 and Eq. 5.5, define the matrix  $\tilde{\mathbb{K}} = \tilde{\mathbf{L}}^\top \mathbf{H}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}})$ , then we have,

$$\tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}) = \begin{cases} \phi_i \rho_{j-k} \mathbf{h}_2^{\mathbf{H}}(\omega_i, \mu_j, \mu_k) \\ \phi_i \rho_{j-l} \mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}(\omega_i, \mu_j, \mu_k, \mu_l) \\ \phi_i \rho_{j-m} \mathbf{h}_3^{\mathbf{H}(-,\mathbf{H})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m) \\ \phi_i \rho_{j-l} \mathbf{h}_3^{\mathbf{H}(\mathbf{N},-)}(\omega_i, \mu_j, \mu_k, \mu_l) \\ \phi_i \rho_{j-m} \mathbf{h}_4^{\mathbf{H}(\mathbf{N},\mathbf{N})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m) \\ \phi_i \rho_{j-n} \mathbf{h}_4^{\mathbf{H}(\mathbf{N},\mathbf{H})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m, \mu_n) \\ \phi_i \rho_{j-m} \mathbf{h}_3^{\mathbf{H}(\mathbf{H},-)}(\omega_i, \mu_j, \mu_j, \mu_k, \mu_l, \mu_m) \\ \phi_i \rho_{j-n} \mathbf{h}_4^{\mathbf{H}(\mathbf{H},\mathbf{N})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m, \mu_n) \\ \phi_i \rho_{j-o} \mathbf{h}_4^{\mathbf{H}(\mathbf{H},\mathbf{H})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o) \end{cases} \quad (5.9)$$

$$\tilde{\mathbf{L}}_2^\top \mathbf{H}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}) = \begin{cases} \phi_{i,j} \rho_{k,l} \mathbf{h}_3^{\mathbf{NH}}(\omega_i, \omega_j, \mu_k, \mu_l) \\ \phi_{i,j} \rho_{k-m} \mathbf{h}_4^{\mathbf{NH}(-,\mathbf{N})}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m) \\ \phi_{i,j} \rho_{k-n} \mathbf{h}_4^{\mathbf{NH}(-,\mathbf{H})}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n) \\ \phi_{i,j} \rho_{k-m} \mathbf{h}_4^{\mathbf{NH}(\mathbf{N},-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m) \\ \phi_{i,j} \rho_{k-n} \mathbf{h}_5^{\mathbf{NH}(\mathbf{N},\mathbf{N})}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n) \\ \phi_{i,j} \rho_{k-o} \mathbf{h}_5^{\mathbf{NH}(\mathbf{N},\mathbf{H})}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o) \\ \phi_{i,j} \rho_{k-n} \mathbf{h}_4^{\mathbf{NH}(\mathbf{H},-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n) \\ \phi_{i,j} \rho_{k-o} \mathbf{h}_5^{\mathbf{NH}(\mathbf{H},\mathbf{N})}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o) \\ \phi_{i,j} \rho_{k-p} \mathbf{h}_5^{\mathbf{NH}(\mathbf{H},\mathbf{H})}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o, \mu_p) \end{cases} \quad (5.10)$$

$$\tilde{\mathbf{L}}_3^\top \mathbf{H}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}) = \begin{cases} \phi_{i,j} \rho_{k-m} \mathbf{h}_3^{\mathbf{H}(\mathbf{H},-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m) \\ \phi_{i,j} \rho_{k-n} \mathbf{h}_4^{\mathbf{H}(\mathbf{H}(-,\mathbf{N}),-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n) \\ \phi_{i,j} \rho_{k-o} \mathbf{h}_4^{\mathbf{H}(\mathbf{H}(-,\mathbf{H}),-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o) \\ \phi_{i,j} \rho_{k-n} \mathbf{h}_4^{\mathbf{H}(\mathbf{H}(\mathbf{N},-),-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n) \\ \phi_{i,j} \rho_{k-o} \mathbf{h}_5^{\mathbf{H}(\mathbf{H}(\mathbf{N},\mathbf{N}),-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o) \\ \phi_{i,j} \rho_{k-p} \mathbf{h}_5^{\mathbf{H}(\mathbf{H}(\mathbf{N},\mathbf{H}),-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o, \mu_p) \\ \phi_{i,j} \rho_{k-o} \mathbf{h}_4^{\mathbf{H}(\mathbf{H}(\mathbf{H},-),-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o) \\ \phi_{i,j} \rho_{k-p} \mathbf{h}_5^{\mathbf{H}(\mathbf{H}(\mathbf{H},\mathbf{N}),-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o, \mu_p) \\ \phi_{i,j} \rho_{k-q} \mathbf{h}_5^{\mathbf{H}(\mathbf{H}(\mathbf{H},\mathbf{H}),-)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o, \mu_p, \mu_q) \end{cases} \quad (5.11)$$

The term  $\tilde{\mathbb{K}}$  term can similarly be written as a block matrix as follows,

$$\tilde{\mathbb{K}} = \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}) & \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_2 \otimes \tilde{\mathbf{U}}) & \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_3 \otimes \tilde{\mathbf{U}}) \\ \tilde{\mathbf{L}}_2^\top \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}) & \tilde{\mathbf{L}}_2^\top \mathbf{H}(\tilde{\mathbf{U}}_2 \otimes \tilde{\mathbf{U}}) & \tilde{\mathbf{L}}_2^\top \mathbf{H}(\tilde{\mathbf{U}}_3 \otimes \tilde{\mathbf{U}}) \\ \tilde{\mathbf{L}}_3^\top \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}) & \tilde{\mathbf{L}}_3^\top \mathbf{H}(\tilde{\mathbf{U}}_2 \otimes \tilde{\mathbf{U}}) & \tilde{\mathbf{L}}_3^\top \mathbf{H}(\tilde{\mathbf{U}}_3 \otimes \tilde{\mathbf{U}}) \end{bmatrix}$$

**Proof.** The calculation of the  $\tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}})$  and  $\tilde{\mathbf{L}}_2^\top \mathbf{H}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}})$  terms are similar so we only provide the steps for the former.

$$\begin{aligned} \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j e^{\mathbf{A}\mu_j} \mathbf{B} \otimes \rho_k e^{\mathbf{A}\mu_k} \mathbf{B}) \\ &= \phi_i \rho_j \rho_k \mathbf{h}_2^{\mathbf{H}}(\omega_i, \mu_j, \mu_k) \\ \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_2) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j e^{\mathbf{A}\mu_j} \mathbf{B} \otimes \rho_k \rho_l e^{\mathbf{A}\mu_k} \mathbf{N} e^{\mathbf{A}\mu_l} \mathbf{B}) \\ &= \phi_i \rho_j \rho_l \mathbf{h}_3^{\mathbf{H}(-,\mathbf{N})}(\omega_i, \mu_j, \mu_k, \mu_l) \\ \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_3) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j e^{\mathbf{A}\mu_j} \mathbf{B} \otimes \rho_k \rho_l \rho_m e^{\mathbf{A}\mu_k} \mathbf{H}(e^{\mathbf{A}\mu_l} \mathbf{B}, e^{\mathbf{A}\mu_m} \mathbf{B})) \\ &= \phi_i \rho_j \rho_m \mathbf{h}_3^{\mathbf{H}(-,\mathbf{H})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m) \\ \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_2 \otimes \tilde{\mathbf{U}}_1) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j \rho_k e^{\mathbf{A}\mu_j} \mathbf{N} e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \rho_l e^{\mathbf{A}\mu_l} \mathbf{B}) \\ &= \phi_i \rho_j \rho_l \mathbf{h}_3^{\mathbf{H}(\mathbf{N},-)}(\omega_i, \mu_j, \mu_k, \mu_l) \\ \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_2 \otimes \tilde{\mathbf{U}}_2) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j \rho_k e^{\mathbf{A}\mu_j} \mathbf{N} e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \rho_l e^{\mathbf{A}\mu_l} \mathbf{N} e^{\mathbf{A}\mu_m} \mathbf{B}) \\ &= \phi_i \rho_j \rho_m \mathbf{h}_4^{\mathbf{H}(\mathbf{N},\mathbf{N})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m) \\ \tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_2 \otimes \tilde{\mathbf{U}}_3) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j \rho_k e^{\mathbf{A}\mu_j} \mathbf{N} e^{\mathbf{A}\mu_k} \mathbf{B} \otimes \rho_l e^{\mathbf{A}\mu_l} \mathbf{H}(e^{\mathbf{A}\mu_m} \mathbf{B} \otimes e^{\mathbf{A}\mu_n} \mathbf{B})) \\ &= \phi_i \rho_j \rho_n \mathbf{h}_4^{\mathbf{H}(\mathbf{N},\mathbf{H})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m, \mu_n) \end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_3 \otimes \tilde{\mathbf{U}}_1) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j \rho_k \rho_l e^{\mathbf{A}\mu_j} \mathbf{H}(e^{\mathbf{A}\mu_k} \mathbf{B}, e^{\mathbf{A}\mu_l} \mathbf{B}) \otimes \rho_m e^{\mathbf{A}\mu_m} \mathbf{B}) \\
&= \phi_i \rho_{j-m} \mathbf{h}_3^{\mathbf{H}(\mathbf{H}, -)}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m) \\
\tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_3 \otimes \tilde{\mathbf{U}}_2) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j \rho_k \rho_l e^{\mathbf{A}\mu_j} \mathbf{H}(e^{\mathbf{A}\mu_k} \mathbf{B}, e^{\mathbf{A}\mu_l} \mathbf{B}) \otimes \rho_m \rho_n e^{\mathbf{A}\mu_m} \mathbf{N} e^{\mathbf{A}\mu_n} \mathbf{B}) \\
&= \phi_i \rho_{j-n} \mathbf{h}_3^{\mathbf{H}(\mathbf{H}, \mathbf{N})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m, \mu_n) \\
\tilde{\mathbf{L}}_1^\top \mathbf{H}(\tilde{\mathbf{U}}_3 \otimes \tilde{\mathbf{U}}_3) &= \phi_i \mathbf{C} e^{\mathbf{A}\omega_i} \mathbf{H}(\rho_j \rho_k \rho_l e^{\mathbf{A}\mu_j} \mathbf{H}(e^{\mathbf{A}\mu_k} \mathbf{B}, e^{\mathbf{A}\mu_l} \mathbf{B}) \otimes \rho_m e^{\mathbf{A}\mu_m} \mathbf{H}(e^{\mathbf{A}\mu_n} \mathbf{B} \otimes e^{\mathbf{A}\mu_o} \mathbf{B})) \\
&= \phi_i \rho_{j-o} \mathbf{h}_3^{\mathbf{H}(\mathbf{H}, \mathbf{H})}(\omega_i, \mu_j, \mu_k, \mu_l, \mu_m, \mu_n, \mu_o)
\end{aligned}$$

The calculation for  $\tilde{\mathbf{L}}_3^\top \mathbf{H}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}})$  is similar to that of  $\tilde{\mathbf{L}}_3^\top \tilde{\mathbf{U}}_3$ . We show the calculation for one of the terms, following the same procedure gives us the result in the theorem.

$$\begin{aligned}
\tilde{\mathbf{L}}_3^\top \mathbf{H}(\tilde{\mathbf{U}}_1 \otimes \tilde{\mathbf{U}}_1) &= \left[ \rho_k \mathbf{B}^\top e^{\mathbf{A}^\top \mu_k} \otimes \phi_i e^{\mathbf{A}^\top \omega_i} \mathbf{C}^\top \right] \mathbf{H}^{(2)\top} \underbrace{e^{\mathbf{A}\omega_j} \phi_j \mathbf{H}(\rho_l e^{\mathbf{A}\mu_l} \mathbf{B} \otimes \rho_m e^{\mathbf{A}\mu_m} \mathbf{B})}_{\text{Let's call this } \Gamma} \\
&= \left[ \Gamma^\top \otimes \rho_k \mathbf{B}^\top e^{\mathbf{A}^\top \mu_k} \right] \mathbf{H}^\top \phi_i e^{\mathbf{A}^\top \omega_i} \mathbf{C}^\top \\
&= \phi_{i,j} \rho_{k-m} \mathbf{h}_3^{\mathbf{H}(\mathbf{H}, -)}(\omega_i, \omega_j, \mu_k, \mu_l, \mu_m)
\end{aligned}$$

**Theorem 5.4.5.** *Let  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{L}}$  be defined as in Eqn. 5.4 and Eq. 5.5, define the matrix  $\tilde{\mathbf{h}} = \tilde{\mathbf{L}}^\top \mathbf{B}$  and  $\tilde{\mathbf{g}} = \mathbf{C}\tilde{\mathbf{U}}$ , then we have,*

$$\tilde{\mathbf{h}} = \begin{cases} \phi_i \mathbf{h}_1(\omega_i) & \text{for } \tilde{\mathbf{L}}_1^\top \mathbf{B}, 1 \leq i \leq N_q \\ \phi_i \phi_j \mathbf{h}_2^{\mathbf{N}}(\omega_i, \omega_j), & \text{for } \tilde{\mathbf{L}}_2^\top \mathbf{B}, 1 \leq i, j \leq N_q \\ \phi_i \phi_j \rho_k \mathbf{h}_2^{\mathbf{H}}(\omega_i, \mu_k, \omega_j), & \text{for } \tilde{\mathbf{L}}_3^\top \mathbf{B}, 1 \leq i, j \leq N_q \text{ \& } 1 \leq k \leq N_p \end{cases} \quad (5.12)$$

$$\tilde{\mathbf{g}} = \begin{cases} \rho_i \mathbf{h}_1(\mu_i) & \text{for } \mathbf{C}\tilde{\mathbf{U}}_1, 1 \leq i \leq N_p \\ \rho_i \rho_j \mathbf{h}_2^{\mathbf{N}}(\mu_i, \mu_j), & \text{for } \mathbf{C}\tilde{\mathbf{U}}_2, 1 \leq i, j \leq N_p \\ \rho_i \rho_j \rho_k \mathbf{h}_2^{\mathbf{H}}(\mu_i, \mu_j, \mu_k), & \text{for } \mathbf{C}\tilde{\mathbf{U}}_3, 1 \leq i, j, k \leq N_p \end{cases} \quad (5.13)$$

The matrices  $\tilde{\mathbf{g}}$  and  $\tilde{\mathbf{h}}$  can be written as,

$$\tilde{\mathbf{h}} = \begin{bmatrix} \tilde{\mathbf{L}}_1^\top \mathbf{B} \\ \tilde{\mathbf{L}}_2^\top \mathbf{B} \\ \tilde{\mathbf{L}}_3^\top \mathbf{B} \end{bmatrix} \in \mathbb{R}^{(N_q + N_q^2 + N_q^3) \times 1}, \quad \tilde{\mathbf{g}} = \begin{bmatrix} \mathbf{C}\tilde{\mathbf{U}}_1 & \mathbf{C}\tilde{\mathbf{U}}_2 & \mathbf{C}\tilde{\mathbf{U}}_3 \end{bmatrix} \in \mathbb{R}^{1 \times (N_p + N_p^2 + N_p^3)}$$

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**Algorithm 6** Quadrature based Balanced Truncation for QB systems
 

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**Input:** Samples of kernel functions for the QB system, quadrature weights  $\rho_1, \dots, \rho_{N_p}, \phi_1, \dots, \phi_{N_q}$  and nodes  $\mu_1, \dots, \mu_{N_p}, \omega_1, \dots, \omega_{N_q}$

**Output:** The reduced order matrices  $\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{N}_r, \mathbf{H}_r$

1. Construct the data matrices  $\tilde{\mathbf{H}}, \tilde{\mathbf{M}}, \tilde{\mathbf{h}}, \tilde{\mathbf{g}}, \tilde{\mathbf{K}}$  using the appropriate quadrature nodes  $\rho_i, \phi_j$  as per Thm. 5.4.1, 5.4.2, 5.4.3, 5.4.5 and 5.4.4. Choose an appropriate truncation index  $r$  while ensuring that  $r \leq \min(N_p^3 + N_p^2 + N_p, N_q^2 N_p + N_q^2 + N_q)$ .
2. Compute SVD of  $\tilde{\mathbf{L}}$  and partition the matrices as follows,

$$\tilde{\mathbf{L}} = \begin{bmatrix} \tilde{\mathbf{Z}}_1 & \tilde{\mathbf{Z}}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{S}}_1 & \\ & \tilde{\mathbf{S}}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{Y}}_1^\top \\ \tilde{\mathbf{Y}}_2^\top \end{bmatrix}$$

where  $\tilde{\mathbf{Z}}_1 \in \mathbb{R}^{(N_q^2 N_p + N_q^2 + N_q) \times r}$ ,  $\tilde{\mathbf{S}}_1 \in \mathbb{R}^{r \times r}$  and  $\tilde{\mathbf{Y}}_1^\top \in \mathbb{R}^{r \times (N_p^3 + N_p^2 + N_p)}$ .

3. Compute the reduced matrices as follows

$$\begin{aligned} \mathbf{A}_r &= \tilde{\mathbf{S}}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{M}} \tilde{\mathbf{Y}}_1 \tilde{\mathbf{S}}_1^{-1/2} & \mathbf{B}_r &= \tilde{\mathbf{S}}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{h}} \\ \mathbf{C}_r &= \tilde{\mathbf{g}} \tilde{\mathbf{Y}}_1 \tilde{\mathbf{S}}_1^{-1/2} & \mathbf{N}_r &= \tilde{\mathbf{S}}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{N}} \tilde{\mathbf{Y}}_1 \tilde{\mathbf{S}}_1^{-1/2} \\ \mathbf{H}_r &= \tilde{\mathbf{S}}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{K}} (\tilde{\mathbf{Y}}_1 \tilde{\mathbf{S}}_1^{-1/2} \otimes \tilde{\mathbf{Y}}_1 \tilde{\mathbf{S}}_1^{-1/2}) \end{aligned}$$


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## 5.5 Computational issues and improvements

The first practical issue we encounter is that the quality of the approximation obtained by Quad BT depends on the number of quadrature nodes used to compute  $\tilde{\mathbf{L}}$  and  $\tilde{\mathbf{U}}$ . Unfortunately, the size of the matrix  $\tilde{\mathbf{H}}$  scales as  $N_p^4 N_q^2$ . This is a computational issue since we have to calculate the SVD of this matrix. A way out is to construct a lower order approximation of  $\tilde{\mathbf{U}}_1$  and use it in the expression for  $\tilde{\mathbf{U}}_2$  and  $\tilde{\mathbf{U}}_3$ . In other words,

$$\begin{aligned} \mathbf{U}_1 &\approx \tilde{\mathbf{U}}_1 = [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{B}] \\ \mathbf{U}_1 &\approx \tilde{\mathbf{U}}'_1 = [\rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{n_1} e^{\mathbf{A}\mu_{n_1}} \mathbf{B}] \end{aligned}$$

where  $N_p \gg n_1$ . The  $\tilde{\mathbf{U}}_1$  term is going to be a better approximation however, since  $\tilde{\mathbf{U}}'_1$  is a smaller matrix it allows us to use more nodes in approximating  $\tilde{\mathcal{P}}_2$ .

$$\begin{aligned}\tilde{\mathcal{P}}_2 &= \sum_{i=1}^{N_p} \rho_i^2 e^{\mathbf{A}\mu_i} \mathbf{N} \tilde{\mathcal{P}}'_1 \mathbf{N}^\top e^{\mathbf{A}^\top \mu_i} & \tilde{\mathbf{U}}_2 &= \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{N} \tilde{\mathbf{U}}'_1 \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{N} \tilde{\mathbf{U}}'_1 \right] \\ \tilde{\mathcal{P}}_3 &= \int_0^\infty e^{\mathbf{A}t_3} \mathbf{H} \left( \tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_1 \right) \mathbf{H}^\top e^{\mathbf{A}^\top t_3} dt_3 & \tilde{\mathbf{U}}_3 &= \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}'_1 \otimes \tilde{\mathbf{U}}'_1 \right) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}'_1 \otimes \tilde{\mathbf{U}}'_1 \right) \right]\end{aligned}$$

Now the advantage of this is that the dimension of the matrix  $\tilde{\mathbb{H}}$  is  $(N_p + N_p n_1 + N_p n_1^2)(N_q + N_q n_1 + N_q n_1^2)$ , which is actually quadratic in  $N - N_p N_q = N^2$  just like the linear case! Of course, we need to ensure that  $n_1$  is of small order. In fact, we have a lot of flexibility in choosing the number of nodes being used to approximate  $\tilde{\mathbf{U}}_2$  and  $\tilde{\mathbf{U}}_3$ . For example,

$$\begin{aligned}\tilde{\mathbf{U}}'_1 &= \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{n_1} e^{\mathbf{A}\mu_{n_1}} \mathbf{B} \right] \\ \tilde{\mathbf{U}}''_1 &= \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{B} \dots \rho_{n_2} e^{\mathbf{A}\mu_{n_2}} \mathbf{B} \right] \\ \tilde{\mathbf{U}}_2 &= \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{N} \tilde{\mathbf{U}}'_1 \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{N} \tilde{\mathbf{U}}'_1 \right] \\ \tilde{\mathbf{U}}_3 &= \left[ \rho_1 e^{\mathbf{A}\mu_1} \mathbf{H} \left( \tilde{\mathbf{U}}''_1 \otimes \tilde{\mathbf{U}}''_1 \right) \dots \rho_{N_p} e^{\mathbf{A}\mu_{N_p}} \mathbf{H} \left( \tilde{\mathbf{U}}''_1 \otimes \tilde{\mathbf{U}}''_1 \right) \right].\end{aligned}$$

As long as we ensure  $N_p \approx N'_p n_1 \approx N''_p n_2^2$ , the runtime (practically) won't be affected. This allows us a way to include more quadrature nodes.

Another issue is that of computing the  $\tilde{\mathbb{K}}$  term. Forming the Kronecker product  $(\tilde{\mathbf{Y}}_1 \mathbf{S}_1^{-1/2} \otimes \tilde{\mathbf{Y}}_1 \mathbf{S}_1^{-1/2})$  is a challenge even for moderate systems. The solution to this problem is by cleverly using the symmetric properties of  $\mathbf{H}$ . We have the term  $\mathbf{H}_r = \mathbf{S}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top \tilde{\mathbb{K}} (\tilde{\mathbf{Y}}_1 \mathbf{S}_1^{-1/2} \otimes \tilde{\mathbf{Y}}_1 \mathbf{S}_1^{-1/2})$ . Let  $\mathbf{W} = \mathbf{S}_1^{-1/2} \tilde{\mathbf{Z}}_1^\top$  and  $\mathbf{V} = \tilde{\mathbf{Y}}_1 \mathbf{S}_1^{-1/2}$ . Now we follow the procedure given in Ref. [37]:

1. Compute  $\mathcal{Y} \in \mathbb{R}^{r \times n \times n}$  via  $\mathcal{Y}^{(1)} = \mathbf{W}^\top \mathbf{H}$ .
2. Compute  $\mathcal{Z} \in \mathbb{R}^{r \times r \times n}$  via  $\mathcal{Z}^{(2)} = \mathbf{V}^\top \mathcal{Y}^{(2)}$ .
3. Compute  $\mathcal{H}_r \in \mathbb{R}^{r \times r \times r}$  via  $\mathcal{H}_r^{(3)} = \mathbf{V}^\top \mathcal{Z}^{(3)}$ .

This allows to compute the product  $\mathbf{H}_r = \mathbf{W} \tilde{\mathbb{K}} (\mathbf{V} \otimes \mathbf{V})$  in  $\mathcal{O}(n^2 r)$  instead of  $\mathcal{O}(n^3)$ .

## 5.6 Numerical tests

In this section, we test the performance of our algorithm on two benchmark datasets - the build dataset [22] and the burgers quadratic equation.

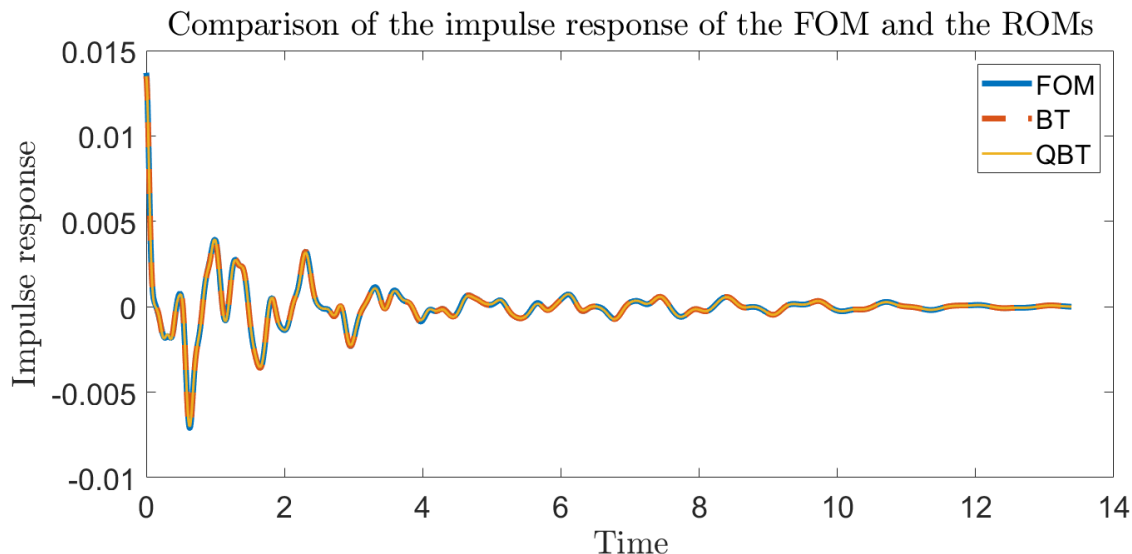


Figure 5.1: Comparing the impulse response of the FOM (build) with that of the ROM obtained from BT and Quad BT

Fig. 5.1 plots the impulse responses of the Quad BT ROM ( $r = 42$ ) and the FOM (of dimension 48). We also show the response of the BT ROM ( $r = 42$ ) for comparison. The oscillatory behaviour of the impulse response makes the ‘build’ system difficult to handle despite the small dimension. However, by using the procedure mentioned in Sec. 5.5, we are able to pick enough nodes to capture this oscillatory behaviour.

In Fig. 5.2 we plot the Hankel singular values of the build FOM, Quad BT ROM ( $r = 42$ ) and BT ROM ( $r = 42$ ). The HSVs of the Quad BT ROM approximate that of the FOM nearly as well as the BT ROM. In Fig. 5.3 we consider a discretization of the burgers quadratic equation which yields a QB system of dimension 150. We construct ROM using Quad BT and BT with truncation index  $r = 31$  and observe that the Quad BT HSVs are close to that of the FOM until a certain point but begin to deviate after the first 20 singular values. However, the singular values that differ are of order  $10^{-10}$ . Using more nodes with other quadrature rules to approximate the gramians might be a way to improve on this. Balanced truncation naturally outperforms Quad BT in this case, firstly because it is intrusive and we

haven't used a sufficient number of quadrature nodes.

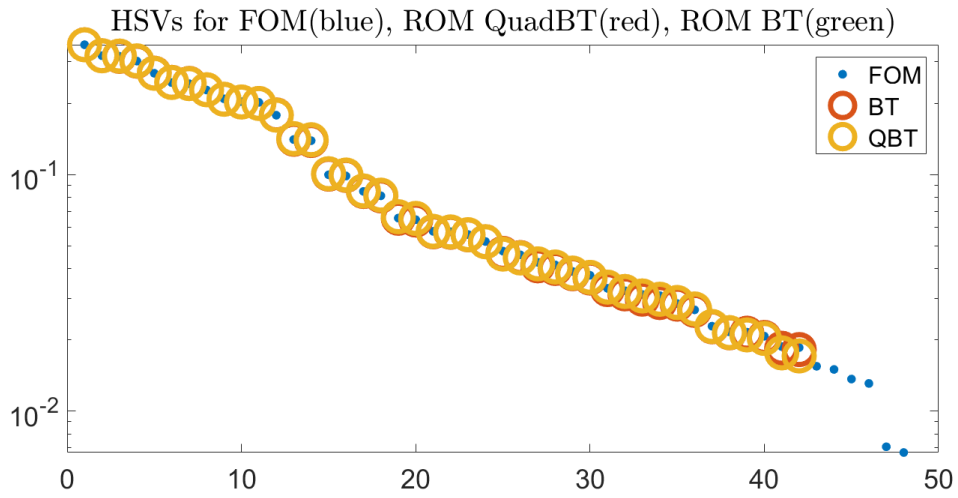


Figure 5.2: Comparing the HSVs of the FOM (build) with BT and Quad BT ROMs

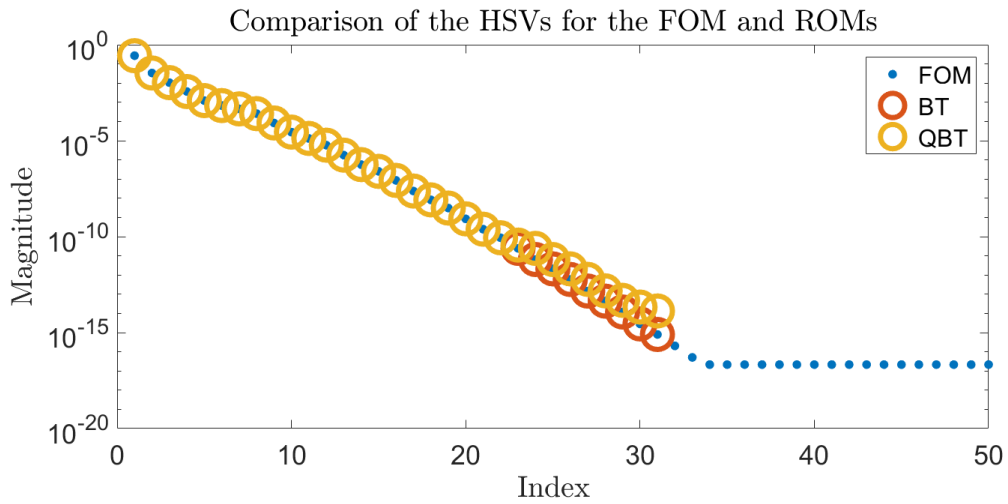


Figure 5.3: Comparing the HSVs of the FOM (burgers quadratic) with BT and Quad BT ROMs



# Chapter 6

## Conclusion and outlook

In this thesis, we extended two popular model reduction frameworks to more general classes of systems.

In chapter 3, we defined the gramians, kernels and balanced truncation for BQO systems. We are close to proving a result connecting the definition of the gramians with the energy functionals of the BQO system however, due to the unfinished nature of the work, it has not been added to this thesis. In the future, we also hope to be able to provide certain a priori  $\mathcal{H}_\infty$  error bounds like in the linear case.

It would be an interesting question to study how many gramian components in the truncated gramian definition are necessary for good approximation of the FOM. Naturally, the more components we take, the better the approximation. For the few systems I studied, it seems that a truncation index of 2 seems to be enough but this may not be true for slightly more complicated systems.

In Chapter 4 we extend the data-driven method Quad BT to linear systems with quadratic output. We introduce a way to increase the computational speed of the algorithm by using ‘course’ Gramian approximations. A deeper study of this ‘turbo’ trick of Quad BT for nonlinear systems would help us understand how to make our algorithm run fast while at the same time giving good results. Using adaptive schemes to cleverly pick quadrature nodes might be an alternative way out without having to work with coarse/low-quality approximates.

Finally, in Chapter 5 we present the Quad BT algorithm for quadratic bilinear systems. Due to the time complexity of the algorithm, using the turbo extension becomes even more important than in the LQO case. Adaptive schemes might also provide a way out of this problem. We do not have a clear understanding of how many nodes or what quadrature scheme to pick. The practical aspects of the method need to be explored further especially for the QB system case. It would also be interesting to see if Quad SPA [38], an algorithm also based on quadrature-based approximations, can be extended to slightly nonlinear systems just like Quad BT.

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