An expository report on the paper "From Goeritz matrices to quasi alternating links" by Józef H. Przyticki

A thesis submitted to
Indian Institute of Science Education and Research Pune
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Mathematics M.Sc Degree Program
under the supervision of
Prof. Rama Mishra

by
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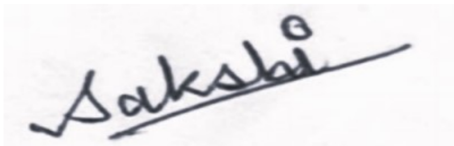
This is to certify that this thesis entitled "*An expository report on the paper "From Goeritz matrices to quasi alternating links" by Józef H. Przyticki*" submitted towards the partial fulfilment of the Mathematics M.Sc Degree Program at the Indian Institute of Science Education and Research Pune represents work carried out by *Sakshi Suresh Manmode* under the supervision of *Prof. Rama Mishra*.

Prof. Rama Mishra
Master's Thesis Supervisor

DECLARATION

I declare that the work presented in this report is an expository report on the paper by Józef H. Przytycki. I've studied the concepts in the paper and from the references mentioned and written it in my own words. To the best of my knowledge, it has not been previously submitted for a degree, diploma, or other qualification at this or any other institution. I further declare that this thesis adheres to all applicable rules and regulations of IISER Pune for MSc thesis. The information presented has been reported truthfully, without falsification or misinterpretation.

Dated, Nagpur.
Monday, May 20, 2024.



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Contents

1 Introduction

([Prz07])

The knot theory revolves around studying the problem of distinguishing knots upto ambient isotopy. The simplest case of differentiating the unknot with the trefoil knot, was not achieved until some tools from algebraic topology were utilized in "Analysis Situs" paper ([Poi95]) by Jules Henri Poincaré (1854-1912). Here, in this project, we are using matrices and graphs related to the links and define the invariants of links with the help of matrices.

2 Complexity of a graph

The complexity of a circuit was defined by Gustav Robert Kirchhoff (1824-1887) in his fundamental paper published in 1847 on electrical circuits([Kir47]). This complexity of a graph H in graph theory is defined as the number of spanning trees of H, and denoted by $\tau(D)$. Let e be an edge of H that is not a loop then

$$\tau(D) = \tau(D - e) + \tau(D/e),$$

where $H - e$ is the graph obtained from H by the deletion of the edge e, and H/e is the graph obtained from H by the contraction of the edge e.
Now we move ahead to define the Kirchoff matrix of a graph, G, the determinant of which is the complexity $\tau(D)$.

Definition 2.1. Let H be a graph with vertices $\{v_0, v_1, \dots v_k\}$.

1. Let $A(H)$ be the $(k+1) \times (k+1)$ matrix whose entries, a_{ij} are the number of edges joining v_i to v_j when $i \neq j$, and $a_{ii} = 0$. Then $A(H)$ is said to be the *adjacency matrix* of H.

2. Let $\Delta(H)$ be the $(k+1) \times (k+1)$ diagonal matrix whose entries, a_{ij} are the degree of the vertex v_i when $i = j$, and $a_{ij} = 0$ when $i \neq j$. Then $\Delta(H)$ is said to be The *degree matrix* of H.

3. Let $Q'(H)$ be the matrix that is defined as $\Delta(H) - A(H)$. Then $Q'(H)$ is said to be the *Laplacian matrix* of H.

4. If we delete the first row and first column from $Q'(H)$, then the resulting matrix is said to be the *Kirchhoff matrix* of H, denoted by $Q(H)$.

Theorem 2.1. $\det(Q(H)) = \tau(H)$.

Example 2.1. Consider the graph H given below.

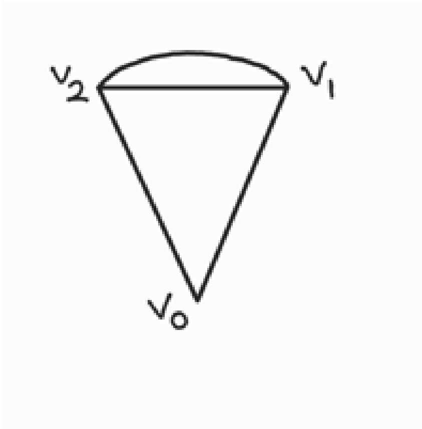


Figure 1: The graph H

For this graph, we have:

$$A(H) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \ \Delta(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$Q'(H) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix}, \ Q(H) = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$$

$$\det(Q(H)) = \det(\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}) = 5 = \tau(H).$$

2.1 Relation between knots and graphs

P.G.Tait was the first one to recognise the relation between planar graphs and links. The graph he constructed for a given knot was called as Tait's graph for the knot. The graph is constructed by checkerboard coloring the regions of knot diagram, and placing a vertex inside each white region, and then connecting the vertices by edges going through crossings of the knot diagram.

Example of Tait's construction of graphs from link diagrams is given below:

Let H be a signed planar graph. For the Tait's construction from H to a link diagram $D(H)$, we will replace every signed edge of the graph with a crossing according to the below convention for over and under crossings depending on the sign of the edge (see figure below):

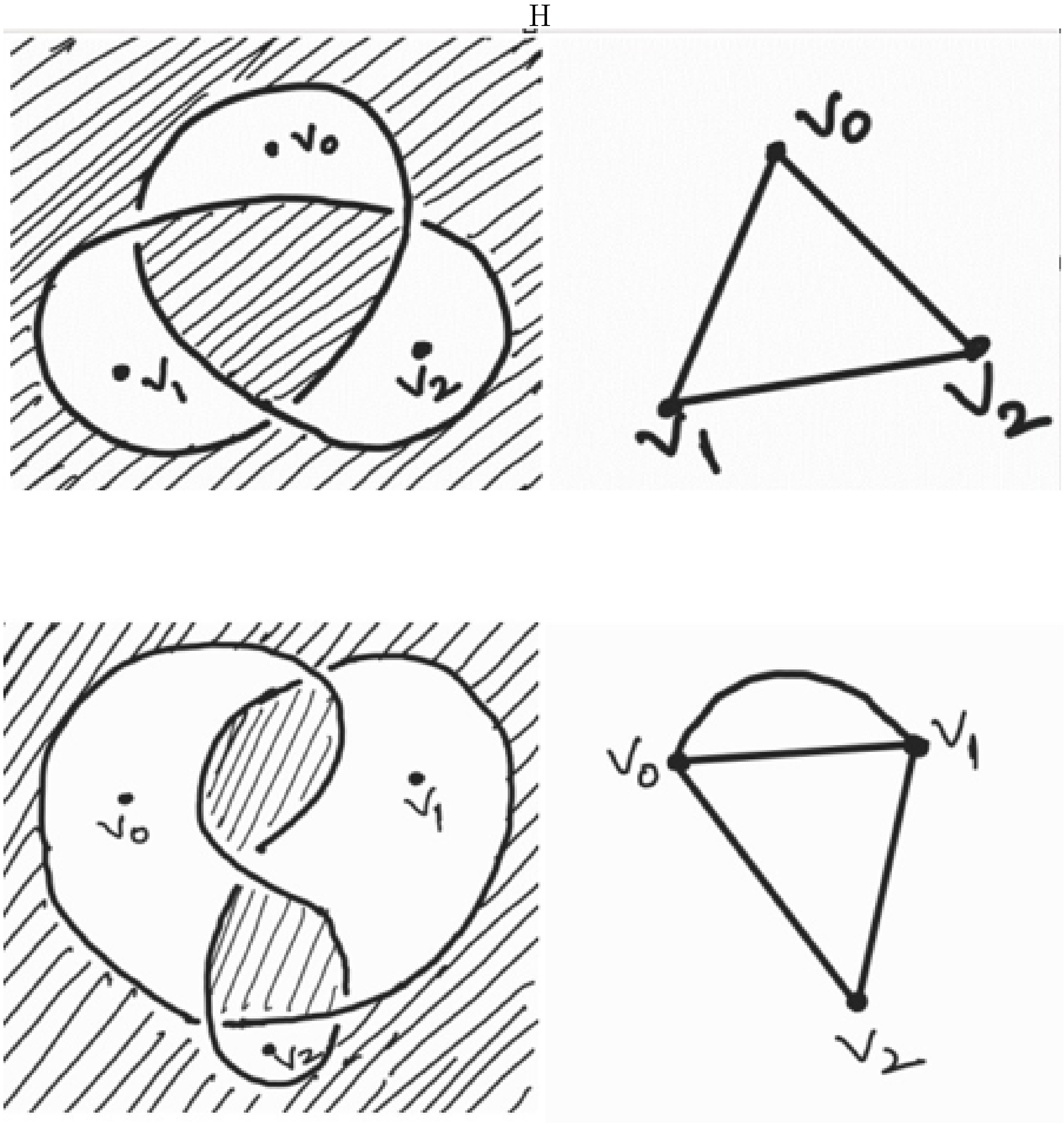


Figure 2: From link diagrams to Tait's graph

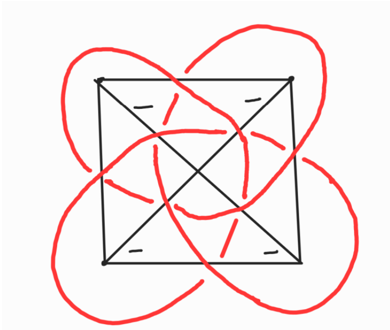


Figure 3: Knot 8_{19} and it's Tait graph

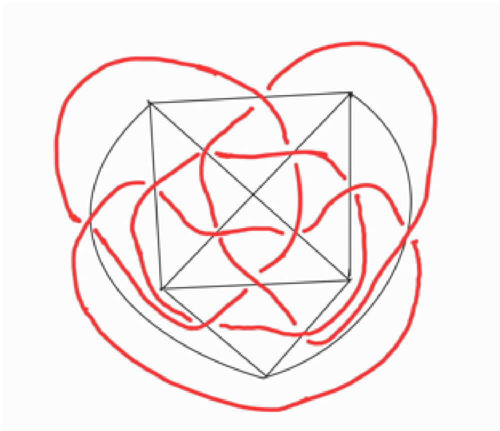


Figure 4: link associated with Octahedral graph with all the edges positive.

Proposition 2.1. The diagram $D(H)$ of a connected graph H is alternating if and only of H is either a positive graph or a negative graph.

2.2 Link Diagram and Reidemeister moves

Definition 2.2 (Polygonal knots and Δ-equivalence)**.**

1. A simple closed polygonal curve in $\mathbb{R}^3$ is called a polygonal knot.

2. Let us assume that u is a line segment (edge) in a polygonal knot P in $\mathbb{R}^3$. Let $\Delta \in \mathbb{R}^3$ be a triangle with three line segments u, v, w as the boundary. whose and such that $\Delta \cap L = u$. The polygonal curve defined as $P' = (P - u) \cup u \cup w$ is a new polygonal knot in $\mathbb{R}^3$. The knot P' is said to be obtained from P by a Δ-move. Conversely, say that the knot P was obtained from P' by a Δ^{-1}-move. The triangle Δ can be degenerate, which means the subdivision of u is allowed.

3. Two polygonal knots P and P' are said to be Δ-equivalent if one can be obtained from the other by applying a finite number of Δ and Δ^{-1} moves.

We present the polygonal link by the projection onto a plane. Let $P \subset \mathbb{R}^3$ be a link, and $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection of P. Then a multiple point $m \in \pi(P)$ is a point such that $\pi^{-1}(m)$ contains more than one point. Moreover, the multiplicity of m is defined as the number of points in $\pi^{-1}(m)$.

Definition 2.3. The projection π is said to be regular if:

1. there are only finite number of multiple points of π and all of them have multiplicity two,

2. the inverse image of a multiple point of p is never a vertex.

The question that when are two projections represent equivalent knots was first considered by Maxwell. The elementary moves he had considered reminded of Reidemeister moves in the future, although he never published his findings. Reidemeister, and Alexander and Briggs did the formal interpretation of Δ-equivalence of knots using diagrams in 1927.

Theorem 2.2 (Reidemeister theorem)**.** Two link diagrams $D_1(L)$ and $D_2(L)$ are Δ-equivalent if and only if they are connected by a finite sequence of Reidemeister moves $R_i^{\pm}, i = 1, 2, 3$ as in fig below, and isotopy of the plane of the diagram.
This theorem also holds for oriented links and diagrams.

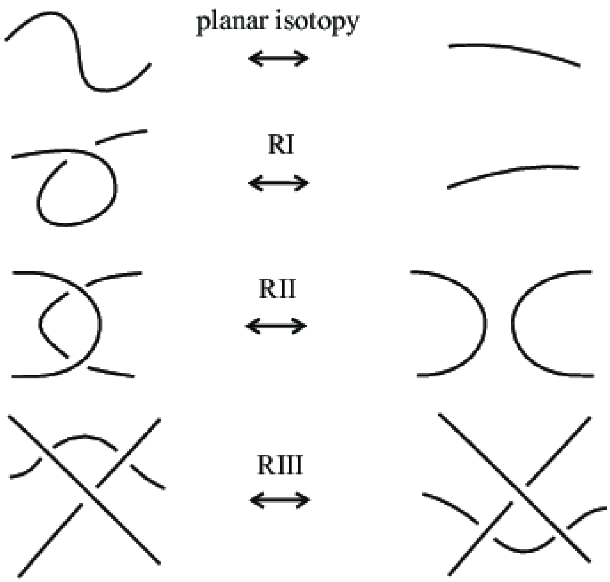


Figure 5: Reidemeister moves (source: internet)

3 Some definitions

Definition 3.1 (Simplicial Complex)**.** ([Kaw96]) A set S of simplices in $\mathbb{R}^n$ for some n is called as simplicial complex if it satisfies the following conditions:

1. For each $A_i, A_j \in S$, $A_i \cap A_j$ is either ϕ or is a face of A_1 and A_2.

2. Let $A \in S$, then all the faces of A are in S.

3. For each $A \in S$, $S \cap A$ is finite.

The union of all simplices in S is called the *polyhedron* of S and denoted by $|S|$.

Definition 3.2 (Triangulation)**.** For a topological space X, the triangulation of X is the pair (S, t) where S is a simplicial complex, and $t\colon |S| \cong X$ is a homeomorphism.

4 Signature of a link and its association to Goeritz matrix

Definition 4.1. Let D be a diagram of the link P.Checkerboard colour the complement of the diagram D is such a way that the unbounded region of $\mathbb{R}^2$ P is coloured white and denote it by v_0, and denote the other white regions by $v_1, v_2 \dots v_n$. Let q be a crossing of D associate $\eta(p)$ to each such q such that $\eta(q)$ is either 1 or -1 according to the convention described below:

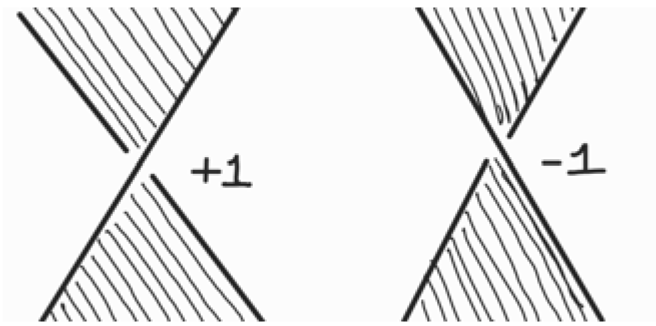


Figure 6: convention

Define $G' = \{a_{j,k}\}_{j,k=0}^{n}$, where

$$a_{j,k} = \begin{cases} -\sum_q \eta(q) & \text{for } j \neq k, \text{ where } q \text{ is the crossing which connects } v_j \text{ and } v_k \\ -\sum_{l=0,1,\dots n; l\neq k} g_{j,l} & \text{for } j = k \end{cases}$$

This matrix $G' = G'(D)$ is called as the unreduced Goeritz matrix of the diagram D of link P. By removing the first row and first column of G' we obtain the Goeritz matrix for D.

Theorem 4.1. ([Kaw96]) Let L_1 and L_2 are two diagrams of a given link. Then the matrices $G(L_1)$ and $G(L_2)$ can be obtained from the other by a finite number of elementary operation on matrices as follows:

1. $G \Leftrightarrow BGB^T$, where B is a matrix with integer entries and $\det(B) = \pm 1$.
2. $$G \Leftrightarrow \begin{pmatrix} G & 0 \\ 0 & \pm 1 \end{pmatrix}$$
3. $$G \Leftrightarrow \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$$

Moreover, if L is a diagram of knot, then (1) and (2) are sufficient.

Proof. To prove the theorem, we will check how Goeritz matrix varies when Reidemeister moves are applied on a link. Consider an oriented link diagram L for some link. Using Tait's construction, but with taking vertices in the black region, we construct a graph corresponding to L.
Let this graph has $B(L)$ number of components, R be a Reidemeister move. We denote the Goeritz matrix $G(L)$ for L as G_1, and $R(L)$ as G_2.

Figure 7: Type I and tyoe II crossings

Let μ is defined as $\mu(L) = \sum \eta(q)$, where q is a crossing of type II according to the convention given in above figure. Set $\mu_1 = \mu(L)$, $\mu_2 = \mu(R(L))$; and $\beta_1 = B(L)$, $\beta_2 = B(R(L))$. The convention we will be using is $G_1 \approx G_2$ if G_1 and G_2 are related by (1), and $G_1 \sim G_2$ if we can obtain G_2 from G_2 by using finite sequence of relations from (1), (2) and (3).

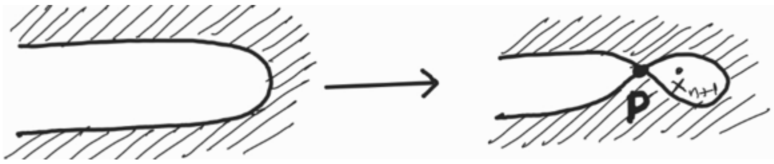
1. R_1, the first Reidemeister move.

   - Case (i):

   From the above figure, we can observe that $\mu_2 = \mu_1$, $\beta_2 = \beta_1$ and $G_2 \approx G_1$

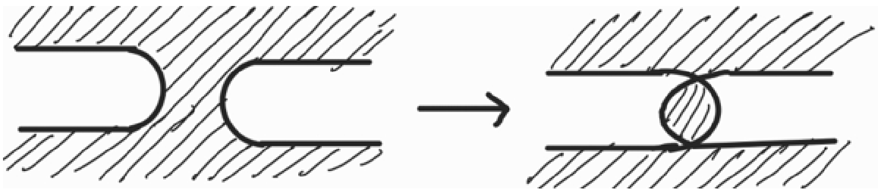
   - Case (ii):

   From the above figure, we can observe that $\mu_2 = \mu_1 + \eta(p)$, $\beta_2 = \beta_1$ and $G_2 = \begin{pmatrix} G_1 & 0 \\ 0 & \eta(p) \end{pmatrix}$.

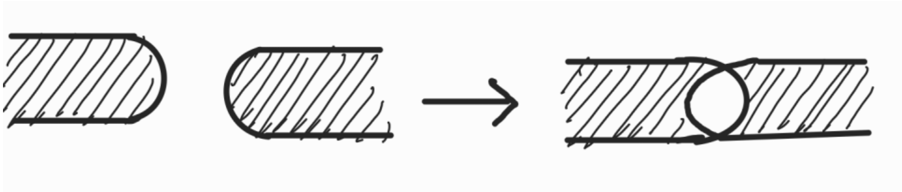
2. R_2, the second Reidemeister move.

   - Case (i):

   In the above figure, let us name the new crossings p_1 and p_2, we can observe that both p_1 and p_2 will either be of type I, or both will be of type II, and $\eta(p_1) = -\eta(p_2)$, this gives us $\mu_2 = \mu_1 + \eta(p_1) + \eta(p_2) = \mu_1 + \eta(p_1) - \eta(p_1) = \mu_1$ thus $\mu_2 = \mu_1$, $\beta_2 = \beta_1$ and hence $G_2 \approx G_1$

   - Case (ii):



In the above figure, let us name the new crossings p_1 and p_2, we can observe that both p_1 and p_2 will either be of type I, or both will be of type II, and $\eta(p_1) = -\eta(p_2)$, this gives us $\mu_2 = \mu_1 + \eta(p_1) + \eta(p_2) = \mu_1 + \eta(p_1) - \eta(p_1) = \mu_1$ thus $\mu_2 = \mu_1$. There will be 2 subcases considering this move either keeps the number of components as it is or decreases it.

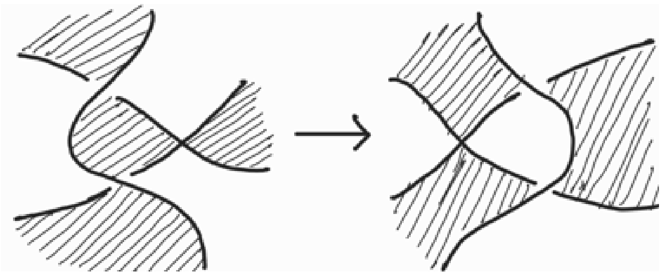
(i) Let $\beta_2 = \beta_1$, then $G_2 \approx \begin{pmatrix} G_1 & & \bigcirc \\ & 1 & \\ \bigcirc & & 1 \end{pmatrix}$

which implies $G2 \sim G_1$.

(ii) Let $\beta_2 = \beta_1 - 1$, then $G_2 \approx \begin{pmatrix} G_1 & \bigcirc \\ \bigcirc & 0 \end{pmatrix}$

This is the third relation.

3. R_3, the third Reidemeister move.



From the above figure, we can observe clearly that $\beta_2 = \beta_1$, considering different orientations of the diagram, and two possible types for the crossing p_0, we will have $\mu_2 = \mu_1 + \eta(p_0)$. This will give us $G_2 \approx \begin{pmatrix} G_1 & 0 \\ 0 & \eta(p_0) \end{pmatrix}$ Also we know $\eta(p_0)$ is either $+1$ and -1. Thus this move clearly corresponds to the relation (2) in the theorem.

□

Corollary 4.1. $|\det(G)|$ is called the determinant of a knot and it is an invariant of knots.

Corollary 4.2. Let K be a link and $\mu(K) = \sum \eta(q)$, where the summation is taken over the crossings of type 2.

1. The signature of the link, $\sigma(K)$ is defined as $\sigma(K) = \sigma(G(K)) - \mu(K)$, where $\sigma(G(K))$ is the signature of Goeritz matrix of K. $\sigma(K)$ is an invariant of the link K.

2. The nullity of the link, $N(K)$ is defined as $N(L) = N(G(K)) - \beta(K) - 1$, where $N(G(K))$ is the nullity of Goeritz matrix of K. $N(K)$ is an invariant of the link K.

For oriented links, L. Traldi introduced a modified matrix for which the signature and nullity are invariants of the link.

Definition 4.2. Let K be the diagram of an oriented link. Then we define the generalized Goeritz matrix.

$$H(K) = \begin{pmatrix} G & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & Q \end{pmatrix},$$

where G is the Goeritz matrix of K, M is the diagonal matrix of dimension equal to the number of type II crossings, with the diagonal entries equal to $-\eta(q)$, where q's are the crossings of type II, and Q is a zero matrix of dimension $\beta(K) - 1$.

Lemma 4.1. ([Kaw96]) If D_1 and D_2 are diagrams of two isotopic oriented links then $H(D_1)$ can be obtained from $H(D_2)$ by sequence of the following elementary equivalence operations:

1. $H \Leftrightarrow BHB^T$, where B is a matrix with integer entries and $\det(B) = \pm 1$.

2. $$H \Leftrightarrow \begin{pmatrix} H & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Proof. The proof will follow from the proof of Theorem 3.1. □

Corollary 4.3. The determinant the link K, denoted by Det_K which is equal to $\det(iH(K))$ is an isotopy invariant of link. Morever, $\sigma(H(K)) = \sigma(K)$ and $N(H(K)) = N(K)$.

Example 4.1. Consider the Torus link $T_{2,k}$. For odd k, $T_{2,k}$ is a knot and for even k, $T_{2,k}$ is a link with 2 components; see Fig below:

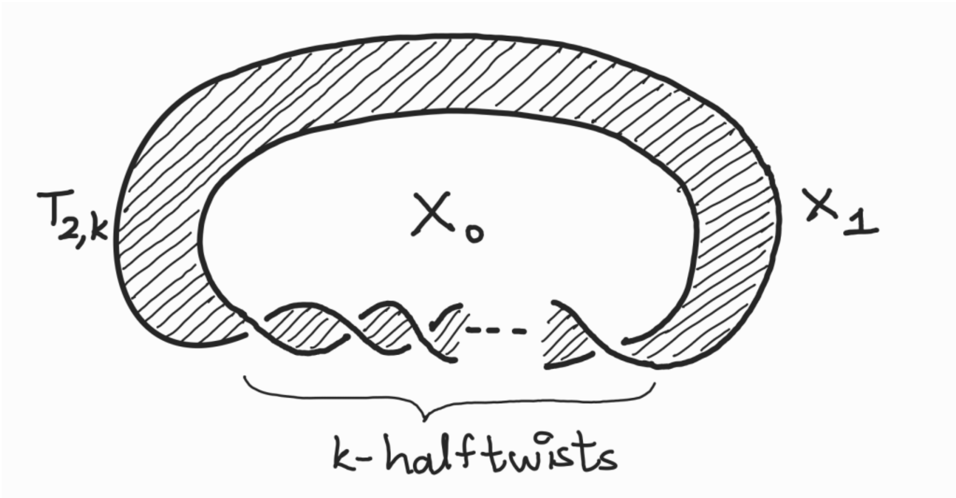


Figure 8: Torus link $T_{2,k}$

We compute the matrices G', G and H for $T_{2,k}$ as:

$$G'(T_{2,k}) = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

Thus the Goeritz matrix will be $G = [k]$.
Moreover, $\beta = 1$ and $\mu = k$ as all the crossings of $T_{2,k}$ are of type II, Thus for $k \neq 0$,

$$\sigma(T_{2,k}) = \sigma(G) - \mu = 1 - k, \text{ and } N(T_{2,k}) = N(G) = 0.$$

This the generalized Goeritz matrix of the knot $T_{2,k}$ will have dimension $k+1$ and is equal to

$$H(T_{2,k}) = \begin{pmatrix} k & & & & \bigcirc \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ \bigcirc & & & & -1 \end{pmatrix}$$

Thus $\mathrm{Det}_L = \det(iH) = (-1)^k i^{k+1} k = i^{1-k} k$. Note that, $i^{\sigma}(T_{2,k}) = \frac{\mathrm{Det}_{T_{2,k}}}{|\mathrm{Det}_{T_{2,k}}|}$

If in the checkerboard colouring of diagram of a link , we connect the black regions by half-twisted bands as shown in below figure, we will get a surface in $\mathbb{R}^3$, for which the boundary will be the given link. We call the surface as Tait surface of the link and denote it by F_b.

Definition 4.3 (Special diagram)**.** If the constructed surface from the construction mentioned above has an orientation that results into the given orientation of the link for some checkerboard colouring of the plane, then this oriented diagram is called a special diagram.

Remark 4.1. An oriented diagram of a link is special if and only if all crossings of the link are of type I for some checkerboard colouring of the plane. Moreover, for a special diagram S_d, we have $\sigma(S_d) = \sigma(G(S_d))$.

Remark 4.2. Any oriented link has a special diagram. Also, for any oriented link K, $\text{Det}_K = i^{\sigma(K)}|\text{Det}_K|$.

Proposition 4.1. As a result of above corollaries, we get

1. $\text{Det}_K = -1^{\text{lk}(K-K_0,K_0)}\text{Det}_K$.
2. $\sigma(K') = \sigma(K) + 2\text{lk}(K - K_0, K_0)$.
3. $\sigma(K) + \text{lk}(K)$ is independent on orientation of K.

4.1 n-move on a link

A local change of an unoriented link diagram described as follows is called an n-move:

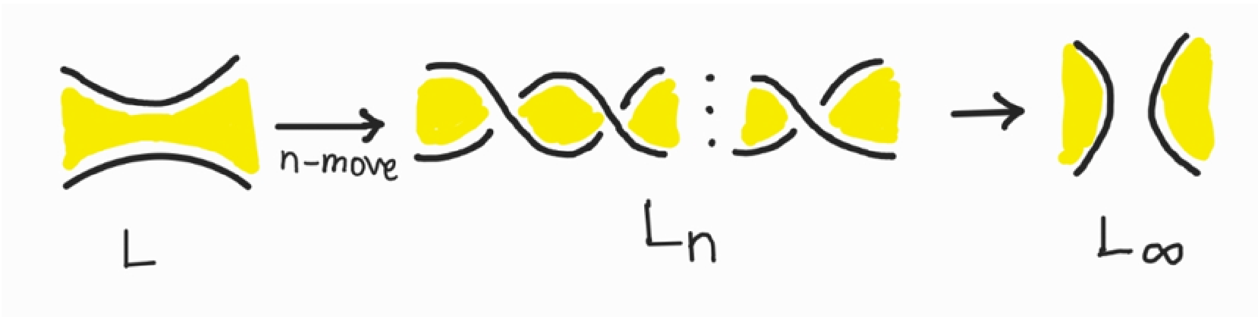


Figure 9: n-move on a link; L_0, L_n and L_∞

For compution the Goeritz matrices of L_0, L_n and L_∞, we will choose the yellow regions as in above figure, and the white region X in $\mathbb{R}^2 - L_\infty$ is divided into two regions X_0 and X_1 in $\mathbb{R}^2 - L$.

Lemma 4.2. $G(L_n) = \begin{pmatrix} G(L_\infty) & \alpha \\ \alpha^T & n+q \end{pmatrix}$

Corollary 4.4. 1. $\text{Det}G(L_n) - \text{Det}G(L_0) = n\text{Det}G(L_\infty)$.

2. $\sigma(G(L_0)) + 2 \geq \sigma(G(L_n)) \geq \sigma(G(L_0)),\ n \geq 0.$

3. $|\sigma(G(L_n)) - \sigma(G(L_\infty))| \leq 1$. Moreover, $\sigma(G(L_\infty)) = \sigma(G(L_n))$ if and only if $r(G(L_n)) = r(G(L_\infty)) + 2$, or $r(G(L_n)) = r(G(L_\infty))$, where $r(A)$ denoted the rank of the matrix A .

Corollary 4.5. 1. Let the orientation of L_0 be such that the strings are parallel. Let we obtain L_n from L_0 by a t_n-move (as shown in figure); then

$$n - 2 \leq \sigma(L_0) - \sigma(L_\infty) \leq n.$$

2. Let the orientation of L_0 be such that the strings are anti-parallel, and let n be an even number $n = 2m$. Let L_{2m} be obtained from L_0 by a $\bar{t}_{2m}$-move (as shown in figure); then

$$-2 \leq \sigma(L_0) - \sigma(L_{2m}) \leq 0.$$

5 Seifert Surfaces

In 1930, P. Frankl and L. Pontrjagin demonstrated ([Prz11]) that every knot is a bound of some oriented surface. The surface is named Seifert surface after H. Seifert, as he found a very simple way to construct such a surface, and later several applications of the surface were developed.

Definition 5.1 (Seifert Surface for a knot)**.** A Seifert surface for a link K in $\mathbb{R}^3$, is a connected, orientable surface embedded in $\mathbb{R}^3$ with boundary K.

Theorem 5.1. Every knot admits a Seifert surface.

Example 5.1. Seifert surfaces for a given knot L are not unique. For example, given below are two non-isotopic Seifert surfaces for the same link:

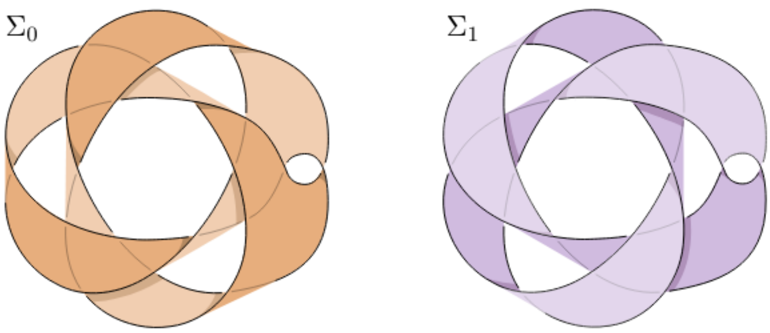


Figure 10: Two non-isotopic Seifert surfaces with genus 1 $\sum_0$ and $\sum_1$ for the same knot L. ([HKM+22])

5.1 Construction of Seifert surface

Let L be an oriented link in $\mathbb{R}^3$, and L_D be a fixed diagram of L. The Seifert surface for the link L is constructed ([Prz11]) as follows:

1. **Step 1.** In the neighborhood of each crossing of L, we modify as shown in the figure below:

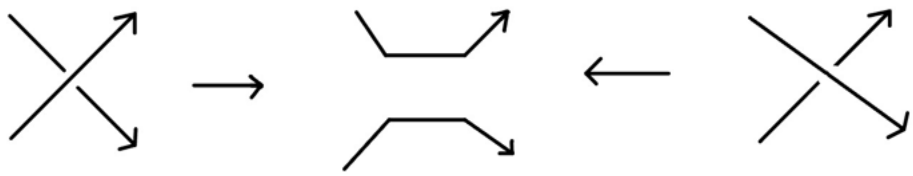


Figure 11: Smoothing the crossing

This modification of link L is called as smoothing of crossings.

2. **Step 2.** We get a collection of disjoint and oriented simple closed curves in the plane after smoothing all the crossings of L_D. These closed curves are denoted by $D_{\vec{s}}$ and called Seifert circles.

3. **Step 3.** Each curve of $D_{\vec{s}}$ is boundary of a disc in the plane. Note that these discs are not always disjoint. They might be nested as shown the the figure below:

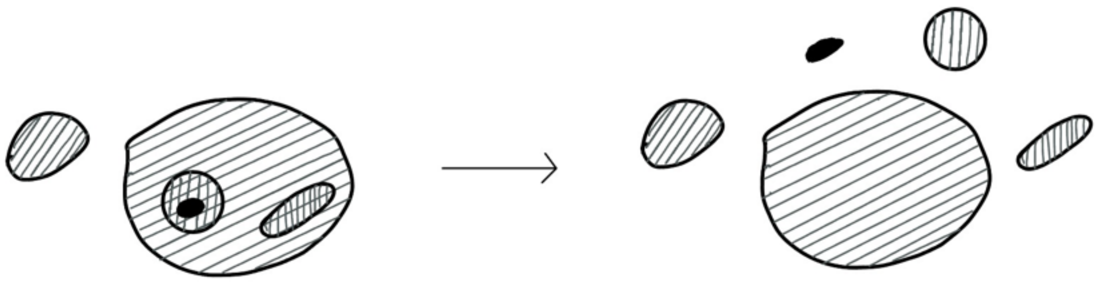


Figure 12: Pushing the circle above the projection

We will observe which of the discs are not disjoint and push them above the projection plane slightly to make them disjoint starting with the innermost disc first, and proceeding outwards as shown in figure 10 above.

4. **Step 4.** Assign signs + and − to discs according to the orientation of their boundaries as:

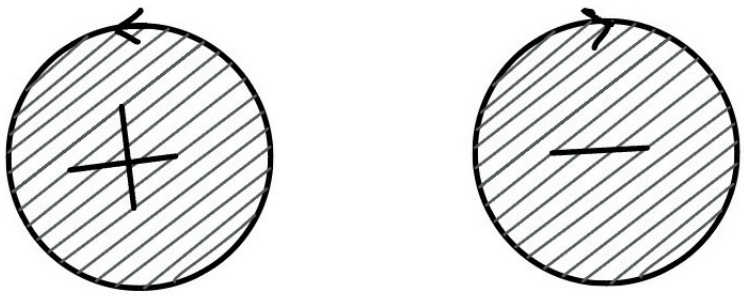


Figure 13: Assigning signs to the discs

5. **Step 5.** We will use half-twisted bands to connect the discs at the original crossings of the diagram L_D as shown in the figure below:

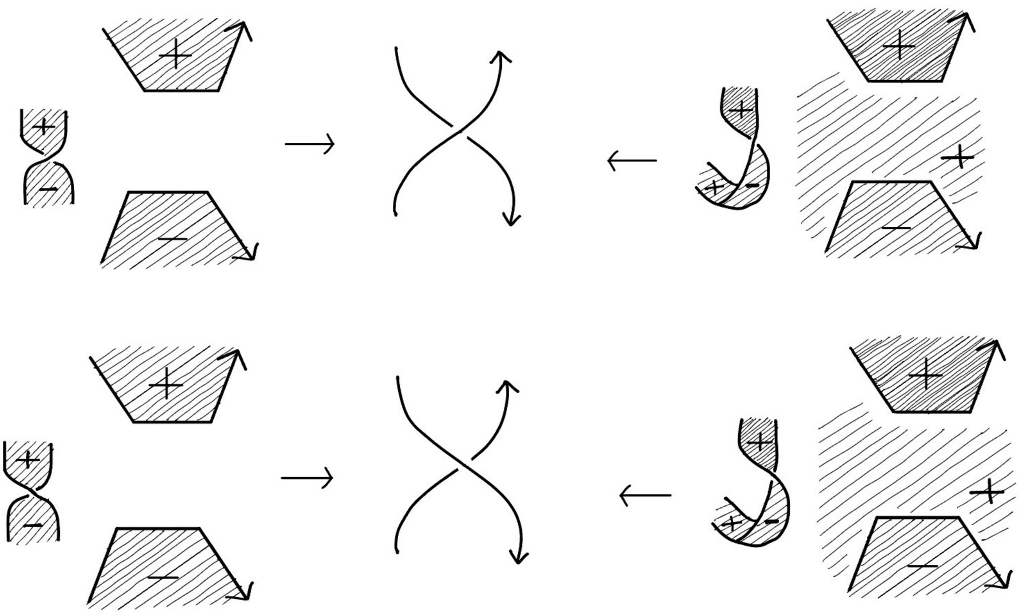


Figure 14: Adding half twists to join the discs

If the projection of the link is connected, then the resulting surface is also connected. That means if L is a knot. If L is a link with more than one component, then we join its components by tubes in such an orientation-preserving manner as shown below:

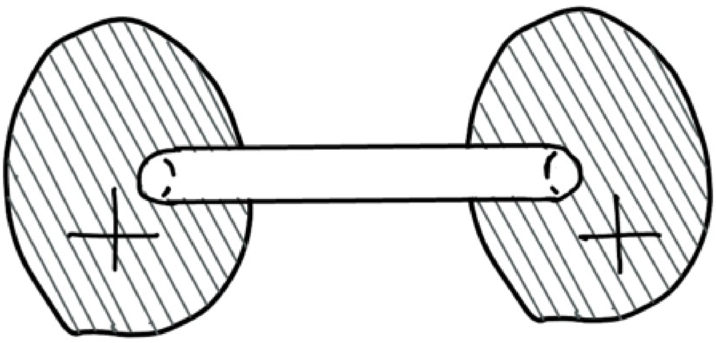
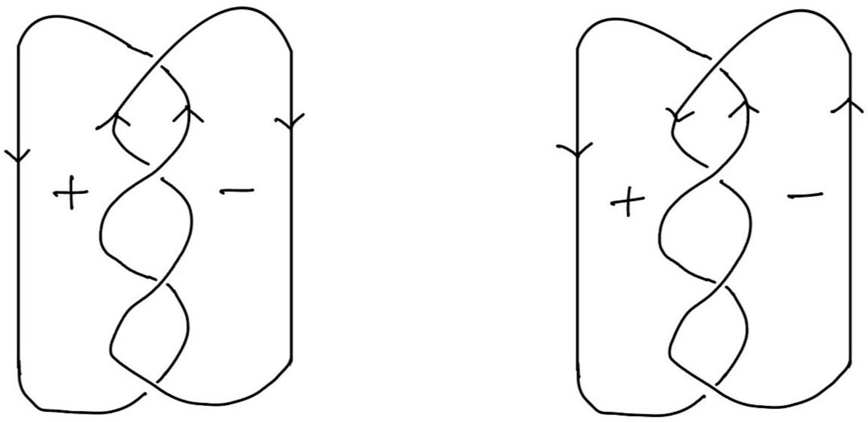


Figure 15: Joining components by tubes for constructing Seifert surface

Since we have joined a "+ side" to another "+ side"(see figure 14, and figure 15), the surface is orientable as well. This connected, orientable manifold will thus have L as its boundary.

Remark 5.1. [Prz11] If the link K has multiple components then the constructed Seifert surface depends on the orientation of the components of K.
For example, we will have a look at the link $T_{(2,4)}$ with two different orientations.



(a) (b)

Figure 16: The link $T_{(2,4)}$ with two different orientations

Let's call the link in figure 14(a) as T_1 and the link in figure 14(b) as T_2. The Seifert surface of T_1 and T_2 have genus 1 and 0 respectively.([Ada94])

Corollary 5.1. ([Przll]) Let K be a knot with n_c number of crossings of the projection of K, and n_s be the number of Seifert circles. Then the resultant surface from the construction mentioned above, is unknotted. It means its complement in $\mathbb{R}^3$ is a handlebody. The handlebody has genus equal to $n_c + 1 - n_s$ and Euler characteristic equal to $n_s - n_c$.

Corollary 5.2. ([Przll])([BV10]) Let L be a link having n components and D be its diagram. Moreover, let c be the number of connected components in D, n_c be the number of crossings in D and n_s be the number of Seifert circles. The genus of the resulting Seifert surface S_L is given by:

$$genus(S_L) = c - \frac{n_s + n - n_c}{2}.$$

Definition 5.2. The genus of the minimal genus Seifert surface of knot L is said to be the genus of knot L.

Corollary 5.3. ([Przll]) A knot in $\mathbb{R}^3$ is trivial if and only if it has genus equal to 0.

Proposition 5.1. ([Przll]) For any given link K in $\mathbb{R}^3$, the first homology of the exterior of K is freely generated by meridians of components of K. In particular,

$$H_1(\mathbb{R}^3 - K) = Z^{\text{com}(K)}.$$

Theorem 5.2. ([Prz11]) Let S be a Seifert surface of a link K, then $H_1(\mathbb{R}^3 - S)$ and $H_1(F)$ are isomorphic. Also, there exist a non-singular bilinear form ψ defined as

$$\psi : H_1(\mathbb{R}^3 - K) \times H_1(S) \to \mathbb{Z}$$

given by $\psi(\alpha, \beta) = l(\alpha, \beta)$, where $l(\alpha, \beta)$ is defined as the intersection number of α and a 2-chain with β as its boundary.

6 Connected sum of links

([Prz11]) Let L_1 and L_2 be oriented knots in $\mathbb{R}^3$. Let $x_1 \in L_1$ and $x_2 \in L_2$, N_1 be the regular neighbourhood of L_1 in the pair $(\mathbb{R}^3, L_1)$, and N_2 be the regular neighbourhood of L_2 in the pair $(\mathbb{R}^3, L_2)$. Consider an orientation-reversing homeomorphism $\Phi \colon \partial N_1 \to \partial N_2$ mapping end of $L_1 \cap (\mathbb{R}^3 - \text{int}(N_1))$ to beggining of $L_2 \cap (\mathbb{R}^3 - \text{int}(N_2)$ and vice versa. Now consider a pair $((\mathbb{R}^3 - \text{int}(N_1) \cup_\Phi \mathbb{R}^3) - \text{int}(N_2)), (L_1 - \text{int}(N_1) \cup_\Phi L_2) - \text{int}(N_2)))$. We see that $(\mathbb{R}^3 - \text{int}(N_1) \cup_\Phi (\mathbb{R}^3 - \text{int}(N_2))$ is a 3-dimensional sphere, and hence $L = (L_1 - \text{int}(N_1) \cup_\Phi (L_2 - \text{int}(N_2)$ is an oriented knot. The knot L obtained from L_1 and L_2 by above way is called as the connected sum of knots L_1 and L_2, and denoted by $L = L_1 \# L_2$.

Lemma 6.1. In the category of oriented knots in $\mathbb{R}^3$, the connected sum of knots is a well-defined, associative, and commutative operation up to ambient isotopy.

Definition 6.1. A prime knot is a knot that can not be presented as the connected sum of any non-trivial knot.

Theorem 6.1. ([Prz11])([Sch53]) Genus of knots in $\mathbb{R}^3$ is additive, i.e.

$$\text{genus}(L_1 \# L_2) = \text{genus}(L_1) + \text{genus}(L_2).$$

Corollary 6.1. ([Prz11]) Every knot in $\mathbb{R}^3$ can be decomposed into a finite connected sum of prime knots.

Corollary 6.2. ([Prz11]) The trefoil knot is prime.

Proof. We know from the definition of trefoil knot that it is not a trivial knot, and hence its genus will not be zero (by corollary 3.1.2). Also, using Seifert's construction, the Seifert surface of trefoil is double twisted mobius strip, with genus, $g = 1 - \frac{2+1-3}{2} = 1 - 0 = 1$. Thus, trefoil knot has genus 1.
Let trefoil knot T is not prime, which means it is a connected sum of atleast two prime components K_1 and K_2. This implies that $1 = g(T) = g(K_1) + g(K_2)$. Thus, either one of K_1 or K_2 must have genus 0, and hence is unknot. This shows that Trefoil knot can not be decomposed further into non-trivial knots, as hence is prime. □

7 Linking number; Seifert forms and matrices

In this section we will introduce the linking number for any pair A and B of disjoint knots. We will give the topological definition and then further show it's agreement with the diagrammatic definition we stated before.

Definition 7.1. Linking number([Prz11]) The linking number $\text{lk}(A, B)$ is an integer such that $[A] = \text{lk}(A, B)[m]$, where $[A]$ andd $[m]$ are homology classes of the oriented curve A and the meridian m of the oriented knot B, respectively.

Lemma 7.1. Let K be a knot with Seifert surface F, and $F \subset \mathbb{R}^3 - intV_K$, such that its orientation determines the orientation of ∂F compatible with that of the longitude. Then $\text{lk}(A, K)$ is equal as the algebraic intersection number of A and F.

Lemma 7.2. Let $D_{A\cup B}$ diagram $A \cup B$ of a link such that A and B are disjoint oriented knots. The orientation of $S^3 = R^3 \cup \infty$ is assumed to be induced by the orientation of the plane which contains the diagram of $A \cup B$ and the third axis directed upwards.

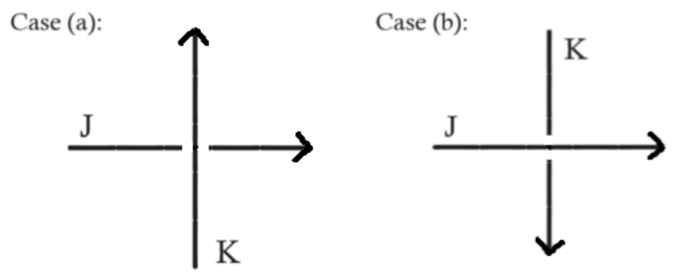


Figure 17: Assigning signs to crossings

Assign the signs "$+1$" and "-1" to the crossings as given in the figure below. Then, the linking number $\text{lk}(A, B)$ is equal to the sum of all such numbers assigned to crossings.

Remark 7.1.

1. $\text{lk}(A, K) = \text{lk}(K, A) = -\text{lk}(-K, A)$, where $-K$ denoted the knot K with reversed orientation.

2. Let γ and ω are disjoint 1-cycles in S^3 then we define $\text{lk}(\gamma, \omega)$ as the intersection number of γ with a 2-chain in S^3 having ω as a boundary. Moreover $\text{lk}(\gamma, \omega)$ does not depend on the 2-chain in S^3 having ω as a boundary.

3. $\mathrm{lk}(\gamma, \omega) = \mathrm{lk}(\omega, \gamma)$, and $\mathrm{lk}(\gamma, n\omega) = n \cdot \mathrm{lk}(\gamma, \omega)$.
   Let ω' be a cycle disjoint from γ then $\mathrm{lk}(\gamma, \omega + \omega') = \mathrm{lk}(\gamma, \omega) + \mathrm{lk}(\gamma, \omega')$

4. Let the cycles ω and ω' be homologous in the complement of γ, then $\mathrm{lk}(\gamma, \omega) = \mathrm{lk}(\gamma, \omega')$.

After defining the linking number and studying its properties, for a given knot or a link we can define a Seifert form. The Seifert surface of a knot or a link, S is a two-sided surface in S^3. We select a regular neighbourhood of S in S^3. Let that neighbourhood be $S \times [-1, 1]$. Let γ be a 1-cycle in int$S$, we can consider the cycle γ^+ (resp. γ^-) in $S \times 1$ obtained by pushing γ to $S \times 1$ (resp. $S \times -1$ obtained by pushing γ to $S \times -1$).

Definition 7.2. Seifert form of a knot([Prz11]) The function $f : H_1(\mathrm{int}S) \times H_1(\mathrm{int}S) \to \mathbb{Z}$ such that $f(\gamma, \omega) = \mathrm{lk}(\gamma^+, \omega)$ is called Seifert form of the knot K. The Seifert form of an oriented link L is defined using an oriented Seifert surface S of L.

Lemma 7.3. The function f is a well defined bilinear form on the $\mathbb{Z}$-module $H_1(\mathrm{int}S)$.

Definition 7.3. ([PBI+23]) Seifert matrix $M = \{a_{i,j}\}$ in a basis $x_1, x_2, \ldots, x_{2g+com(L)-1}$ of $H_1(S)$ is defined as the matrix of f in this basis, that is

$$a_{i,j} = \mathrm{lk}(x_i^+, x_j).$$

Then for $\gamma, \omega \in H_1(S)$, we have $f(\gamma, \omega) = \gamma^T M \omega$. We will write the coefficients of a vector as column matrix.

Example 7.1. If we compute the Seifert surface S of the right-handed trefoil knot, and then compute the Seifert matrix V of S in the basis $[\gamma], [\omega]$ is given as

$$V = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

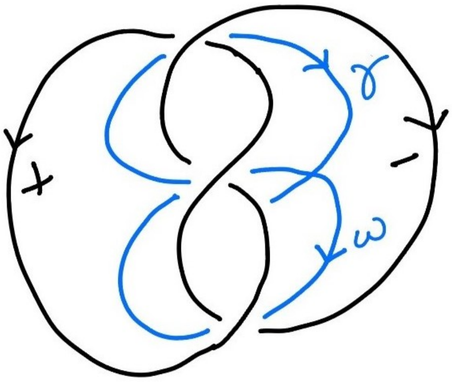


Figure 18: a right-handed trefoil

Example 7.2. Let P_{k_1,k_2,k_3} denote pretzel link of type (k_1, k_2, k_3). For the Seifert surface of the pretzel knot $P_{2m_1+1,2m_2+1,2m_3+1}$, the Seifert matrix computed in basis $[\gamma], [\omega]$ is equal to $\begin{pmatrix} -m_1 - m_2 & m_2 \\ m_2 + 1 & -m_1 - m_2 \end{pmatrix}$

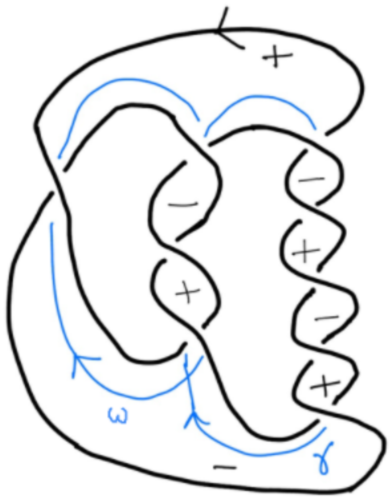
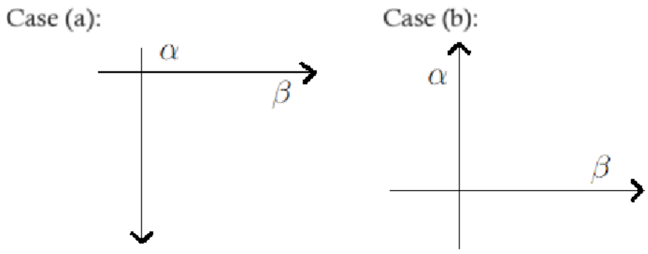


Figure 19: $P_{1,3,5}$

Definition 7.4. Let S be an oriented surface. For two homology classes $\alpha, \beta \in H_1(S)$ represented by transversal cycles,the their algebraic intersection number $\tau(\alpha, \beta)$ is defines as the sum of the signed intersection points where the sign is defined in the following if α meets β transversally at a point q, then the sign of the intersection at point q is $+1$ if the intersection is as shown in the case(a), and -1 if the intersection is as shown in case (b) as shown in figure below:



Remark 7.2. $\tau : H_1(S) \times H_1(S) \to \mathbb{Z}$ is bilinear and $\tau(\alpha, \beta) = -\tau(\beta, \alpha)$

Remark 7.3. Let T of matrix of τ, then

$$\det(T) = \begin{cases} 1 & \text{if } \partial S \text{ is either } S^1 \text{ or } \phi. \\ 0 & \text{otherwise} \end{cases}$$

Remark 7.4. Let S be a Seifert surface of a link then $\tau(\alpha, \beta) = f(\alpha, \beta) - f(\beta, \alpha,)$.

Corollary 7.1. ([Rol03]) Let M be the Seifert matrix of a knot K in S^3. Then M satisfies the following equation:

$$\det(M - M^T) = 1.$$

Definition 7.5. Two matrices are said to be S-equivalent if by using the following modifications for a finite number of times on one, other can be obtained:

1. $A \Leftrightarrow BAB^T$, where B is a matrix with integer entries and $\det B = \pm 1$.

2. $$A \Leftrightarrow \begin{pmatrix} A & \zeta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A \Leftrightarrow \begin{pmatrix} A & 0 & 0 \\ \eta & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

   where ζ is a column and η is a row.

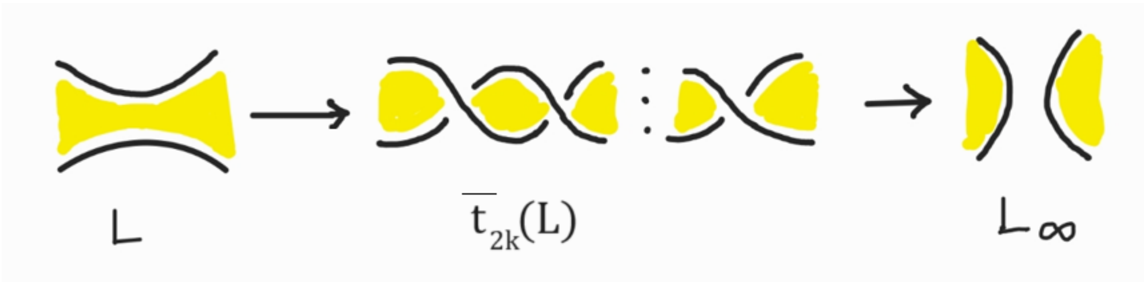
Theorem 7.1. Let F_1 and F_2 be the Seifert surfaces of two isotopic links L_1, and L_2 respectively. If M_1 and M_2 are their Seifert matrices computed in some basis $\mathcal{B}_1$, and respectively $\mathcal{B}_2$, then M_1 is S-equivalent to M_2.

8 Alexander polynomial from Seifert form, signatures of links

Lemma 8.1. Let K be an oriented link, and M be a Seifert matrix. and define the potential function $\Omega_K(x) = \det(xA - x^{-1}M^T)$. Then $\Omega_K(x)$ does not depend on the choice of a Seifert surface and its Seifert matrix. As a particular example, if T is the trivial knot then $\Omega_T x = 1$.

Theorem 8.1 (Kauffman)**.** $\Omega_K(x) = \Delta_K(t) = \nabla_K(z)$, where $x = -\sqrt{t}$, and $z = x^{-1} - x = \sqrt{t} - \frac{1}{\sqrt{t}}$.

Definition 8.1. The $\bar{t}_{2k}$-move on the link L, which means introducing k number of full-twists given anti-parallel oriented arcs in L, is the elementary operation on an oriented diagram L resulting in $\bar{t}_{2k}(L)$ as illustrated in figure below:



We see that $\bar{t}_2$-move changes the crossing from positive crossing to negative crossing. Choose Seifert surfaces $S(L)$, $S(\bar{t}_{2k}(L))$, and $S(L_\infty)$. Choose a basis for $H_1(S(L_\infty))$, and add one standard element, e_∞ to obtain a basis for $H_1(S(L))$, and $e_{\bar{t}_{2k}(L)}$ to get a basis of $H_1(S(\bar{t}_{2k}(L))$. Let M_{L_∞} be the Seifert matrix of L_∞ in the chosen basis.
Thus we have the following lemma:

Lemma 8.2.

$$M_L = \begin{pmatrix} M_{L_\infty} & \zeta \\ \rho & p \end{pmatrix},$$

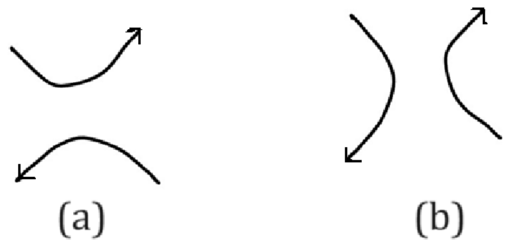
$$M_{\bar{t}_{2k}(L)} = \begin{pmatrix} M_{L_\infty} & \zeta \\ \rho & p+k \end{pmatrix},$$

where ζ is the column given by linking numbers of $e^+_{\bar{t}_{2k}(L)}$ with basis elements of $H_1(S(L_\infty))$, ρ is the row given by linking numbers of basis elements of $H_1(S(L_\infty))$ with $e^-_{\bar{t}_{2k}(L)}$, and $p = \operatorname{lk}(e^+_{\bar{t}_{2k}(L)}, e^-_{\bar{t}_{2k}(L)})$

Corollary 8.1. ([Prz11])

1. If L_1 and L_2 be two oriented links which are $\bar{t}_{2k}$ equivalent, then the Seifert matrices of L_1 and L_2 are S-sequivalent modulo k.

2. Let (a), and (b) be defined as in the below figure.



The potential function satisfies:

$$\Omega_{\bar{t}_{2k}(L)} - \Omega_{(a)} = k(x - x^{-1})\Omega_{(b)}.$$

In particular, when $k = -1$, then $\Omega_{L_+} - \Omega_{L_-} = (x^{-1} - x)\Omega_{L_0}(x)$.

3. $\Omega_{\bar{t}_{2k}(L)} = \det(xA_{\bar{t}_{2k}(L)} - x^{-1}A^T_{\bar{t}_{2k}(L)}) = \det\begin{pmatrix} A_{L_\infty} & x\zeta - x^{-1}\rho^T \\ x\rho - x^{-1}\zeta^T & (x - x^{-1})(p + k)\end{pmatrix}$, and $\Omega_{(a)} = \det\begin{pmatrix} A_{L_\infty} & x\zeta - x^{-1}\rho^T \\ x\rho - x^{-1}\zeta^T & (x - x^{-1})p\end{pmatrix}$

Example 8.1. For the pretzel link $K = P_{2m_1+1,2m_2+1,\ldots,2m_k+1}$ the Alexander-Conway polynomial can be calculated using the above corollary by applying the formula from Corollary 7.1 (2), for a given column of the pretzel link.
For $z = x^{-1} - x$, we get,
$\Omega_K(x) = \nabla_K(z) = \sum_{i=0}^{k-1} a_{k,i}z^i\nabla_{T_{2,k-i}(z)}$

$$= z^{k-1}(\binom{k-1}{0}+a_{k,1}\binom{k-2}{0}+a_{k,2}\binom{k-3}{0}+\ldots)+z^{k-3}(\binom{k-2}{1}+a_{k,1}\binom{k-3}{1}+a_{k,2}\binom{k-4}{1}+\ldots)\cdot$$

$$= \sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\Big(\sum_{j=0}^{k-1-2i}\binom{k-1-i-j}{i}a_{k,i}\Big)z^{k-1-2i}$$

where $a_{k,i}$ is an elementary symmetric polynomial in the variables $m_1, m_2, \ldots m_k$ of degree i, i.e. $\prod_{j=1}^{k}(z+m_j) = \sum_{i=0}^{k} a_{k,i}z^{k-i}$ and the Alexander-Conway polynomial of the torus link $T_{2,k-i}$ is given by $\nabla_{T_{2,k-i}}(z) = \nabla_{P_{1,1,\ldots,1(z)}}$.
This leads us to the observation that the pretzel knot $P_{5,7,-3}$ (refer figure 17) has trivial Alexander-Conway polynomial..

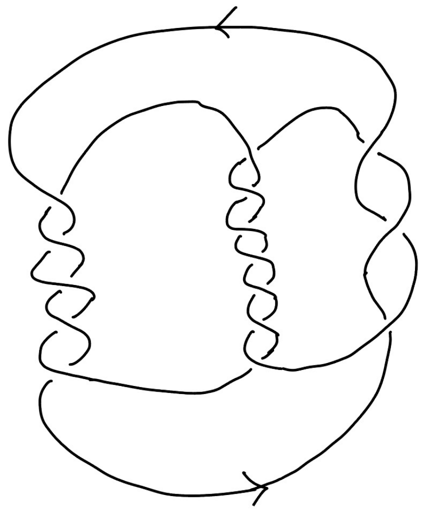


Figure 20: $P_{5,7,-3}$, i.e. the link $P_{2m_1+1,2m_2+1,2m_3+1}$ with $m_1 = 2, m_2 = 3, m_3 = -2$

8.1 Tristram-Levine signature

Definition 8.2. Let K be a link with Seifert matrix V_K. Let $\zeta(\neq 1)$ be any complex number. Consider, for each ζ, a Hermitian matrix $H_K(\zeta) = (1-\bar{\zeta})V_K + (1-\zeta)V_K^T$. The Tristam-Levine signature of K is defined as the signature of the matrix $H_K(\zeta)$. We denote the signature by $\sigma_K(\zeta)$ if the parameter ζ is considered. If we consider $\rho = 1-\zeta$ as a parameter, then we denote the signature by $\sigma_\rho(K)$. Moreover, using this convention $\sigma(K) = \sigma_1(K) = \sigma_K(0) = \sigma_K(-1)$, and $\sigma_K(1) = 0$.

Corollary 8.2. Let L be a link, and $\bar{t}_{2m}$-move be the move as mentioned in definition 7.1 with $k = m$,

1. For every $\bar{t}_{2m}$-move and $Re(1-\zeta) \geq 0$, we have

$$0 \leq \sigma_{\bar{t}_{2m}(L)}(\zeta) - \sigma_L(\zeta) \leq 2.$$

   In particular, $-2 \leq \sigma_{L_+}(\zeta) - \sigma_{L_-}(\zeta) \leq 0$.

2. Moreover, for any m and ζ,

$$0 \leq |\sigma_{L_\infty}(\zeta) - \sigma_{\bar{t}_{2m}(L)}| \leq 1.$$

   In particular, $0 \leq |\sigma_{L_+}(\zeta) - \sigma_{L_0}(\zeta)| \leq 1$

Corollary 8.3. Let $\bar{K}$ be the mirror image of link K, then $V_{\bar{K}} = -V_K, H_K(\zeta) = -H_{\bar{K}}(\zeta), \sigma_K(\zeta) = \sigma_{\bar{K}}(\zeta)$, and $\sigma_\rho(K) = -\sigma_\rho(\bar{K})$. Thus when K is amphichiral, $\sigma_K(\zeta) = 0$.

8.2 The relation between $\Omega_K(x)$ and $\sigma_\rho(K)$

Lemma 8.3. Let the potential function at $i\rho$ be non-zero. Then

$$i^{\sigma_\rho(K)} = \frac{\Omega_K(i\rho)}{|\Omega_K(i\rho)|} = \frac{\Delta_K(t_0)}{|\Delta_K(t_0)|} = \frac{\nabla_K(-i(\rho+\bar{\rho}))}{|\nabla_K(-i(\rho+\bar{\rho}))|}$$

, where $\Delta_K(t_0)$ is the Alexander-Conway polynomial and $t_0 = -\rho^2$ that is $\sqrt{t_0} = -i\rho$.

Corollary 8.4. The classical signature $\sigma(K) = \sigma_1(K)$ satisfies the following relation:

$$i^{\sigma(K)} = i^{\sigma(V_K+V_K^T)} = \frac{\Omega_K(i)}{|\Omega_K(i)|} = \frac{\mathrm{Det}_K}{|\mathrm{Det}_K|} = \frac{\Delta_K(-1)}{|\Delta_K(-1)|} = \frac{\nabla_K(-2i)}{|\nabla_K(-2i)|}$$

Here, we take $\sqrt{t} = -i$, and thus $\Delta_K(-1) = \Delta_K(1)$. We also assume that $\mathrm{Det}_K \neq 0$.

Example 8.2. The smallest non-amphichiral knot whose Jones, Kauffman, and Homflypt polynomials are symmetric is the knot 9_{42}.

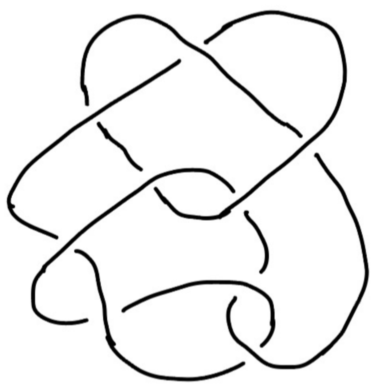


Figure 21: The knot 9_{42}

Theorem 8.2. For a given knot L,

$$\sigma_L \equiv |\mathrm{Det}_L| - 1 \mod 4$$

Corollary 8.5. For a given knot L,

$$\mathrm{Det}_L = (-1)^{\frac{|\mathrm{Det}_L|-1}{2}} |\mathrm{Det}_L|.$$

Corollary 8.6. Let K be a link with Alexander polynomial $\Delta_K(t)$. Let $\Delta_K(t)$ be non-zero on the unit circle then for any ρ such that $|\rho| = 1$, then $\sigma_\rho(L) = 0$.

Proposition 8.1. Let L be a knot with $\text{Det}_L = 1$, then

$$\sigma(L) \equiv 0 \mod 8$$

9 Finding the signature of alternating and Quasi-alternating links

Let $L = L_0$, and L_∞ be as defined in definition 7.1.

Theorem 9.1. Given that $\text{Det}_0 \neq 0$ and $\text{Det}_\infty \neq 0$, following conditions are equivalent:

1. $|\text{Det}_{L_+}| = |\text{Det}_{L_0}| = |\text{Det}_{L_\infty}|$
2. $\sigma(L_+) = \sigma(L_0) - 1$ and $\sigma(L_+) = \sigma(L_\infty) - 1/2(\omega(L_0) - \omega(L_\infty))$.

The similar relations hold for negative crossings too.

Definition 9.1. Let $\overrightarrow{G}$ be an oriented link diagram. We construct a signed graph $\Gamma(\overrightarrow{G})$ whose vertices are in correspondence with Seifert circles of $\overrightarrow{G}$, and edges are in correspondence with the crossings of $\overrightarrow{G}$. The signs are given to the edges with respect to the sign of the crossing. This graph is called as Seifert graph of $\overrightarrow{G}$ and denoted by $\Gamma(\overrightarrow{G})$.

Definition 9.2. Let G be a link diagram (not necessarily oriented). A function s from the set of crossings of G to the set $\{\pm 1\}$ is said to be the Kauffman state of G. A marker is assigned to each crossing of G according to the convention given in the figure 22:

We denote the system of circles obtained by smoothing the crossings of the diagram G by G_s according to the markers of the state s. The number of such circles in G_s is denoted by $|G_s|$.

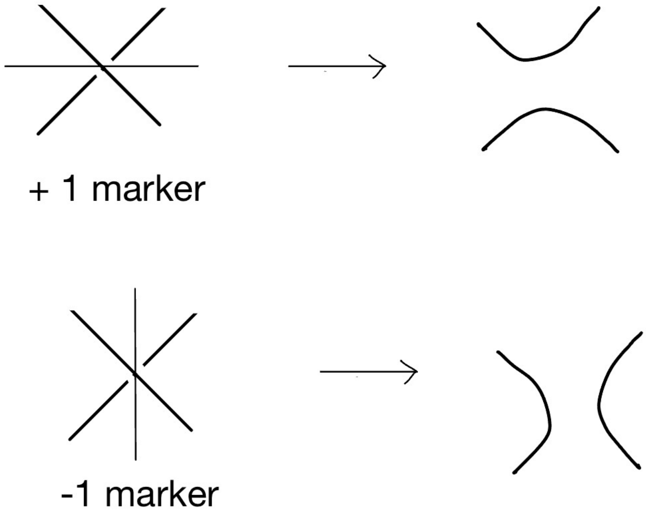


Figure 22: convention for markers and their respective smoothings

Definition 9.3. ([BV10]) Let K be a link and G be its link diagram with Kauffman state s.

1. Construct a graph $A_s(G)$, from G as follows. Vertices of $A_s(G)$ correspond to circles of G_s, and the edges correspond to crossings of G. In case of the Tait graph, $A_s(G)$ is a signed graph where the edge $e(q)$ is given the sign $s(q)$, where $s(q)$ is the sign of marker at the crossing q.

2. If the graph $A_s(G)$ has no loops, then the diagram G is s-adequate.

3. We construct a surface $F_s(G)$ embedded in $\mathbb{R}^3$ and $\partial F_s(G) = G$, for each Kauffman state s of the diagram G.

    - Each of the circles in G_s bound a disc in the projection plane. We push the discs slightly above the plane to make them disjoint.
    - Now connect the discs together at the original crossings of G by half twisted bands (ignoring the orientation). The obtained 2-manifold had G as the boundary.

There is another surface associated to graph $A_s(D)$, called as Turaev surface, $M(s)$, for which the positive state (s_+) or the negative state (s_-) of an alternating diagram is a planar surface. The construction of $M(s)$ for a given state s of G is as shown in the figure:

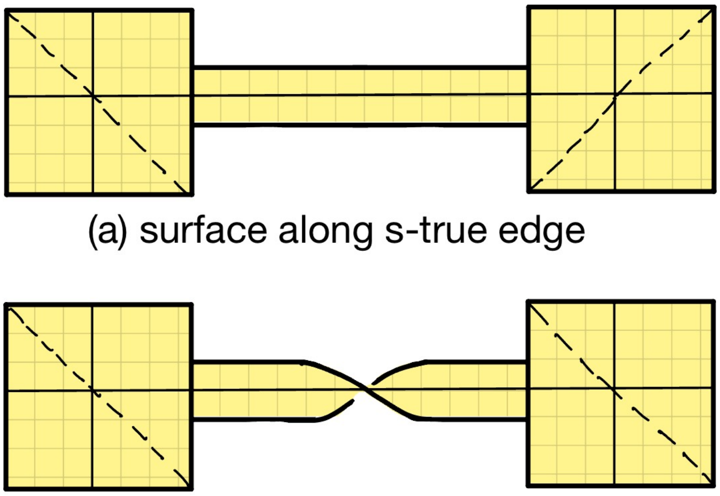


Figure 23: Construction of Turaev surface

$M(s)$ is obtained from a regular neighbourhood of projection of links by adding half-twists to modify the neighbourhoods of s-wrong edges. Clearly, $M(s)$ depends on s and the link projection but not on the over-under information of the diagram.

Definition 9.4. The minimal genus of Turaev surface over diagrams G of a link K with $s_+(D)$ states is called as the Turaev genus of the link K.

Remark 9.1. Turaev genus of an alternating link is 0.

10 Working with Quasi-alternating links

Definition 10.1. ([Ter15]) A family of links, $\mathcal{F}$ is said to be the family of quasi-alternating links if it is the smallest family such that:

1. $\mathcal{F}$ contains the trivial knot.

2. If K is a link with crossing such that
   (1) both K_0 and K_∞ are in $\mathcal{F}$.
   (2) $|\text{Det}_K| = |\text{Det}_{K_0}| + |\text{Det}_{K_\infty}|$, then K is in $\mathcal{F}$.

Example 10.1.

1. A split link can never be quasi-alternating as it has a determinant equal to 0.

2. A no-split alternating link is quasi-alternating.

3. 9_{46} is not quasi-alternating.

4. The knot 13_{n1659} shown in below figures which shows us 2 different diagrams of 13_{n1659} is quasi-alternating with 13 crossings.

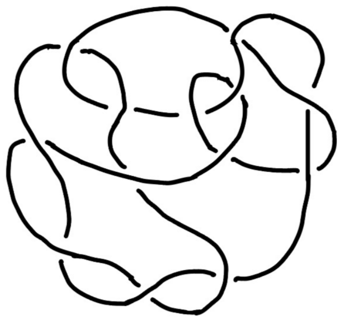


Figure 24: (a)

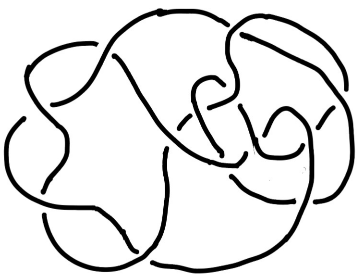


Figure 25: (b)

The characterisation that if a given Pretzel link is quasi-alternating or not is done by the following theorem.

Theorem 10.1. The Pretzel link $P_{1,\ldots,1,a_1,\ldots,a_k,-b_1,\ldots,-b_n}$ with $e + k + n \geq 3$, and $a_i \geq 2, b_i \geq 3$ is quasi-alternating if and only if one of the conditions below holds:

1. $e \geq n$,

2. $e = k - 1 \geq 0$,

3. $e = 0, k = 1$, and $a_1 > min(b_1, \ldots, b_n)$,

4. $e = 0, n = 1$, and $b_1 > min(a_1, \ldots, a_n)$

Definition 10.2. Quasi-alternating computational tree index, QACTI(L) We introduce QACTI(L) to have a measure of complexity of quasi-alternating links. We define it inductively as:
For the trivial knot K_0, QACTI(K_0)=0. QACTI(K) is the minimum over all quasi-alternating crossings q (of any diagram) of K of maxQACTI(K_0^q), QACTI(K_∞^q)+1.

Corollary 10.1. Let K be a quasi-alternating link, then:

1. $|\text{Det}(K)| - 1 \geq QACTI(\text{K}) \geq \log_2(|\text{Det}(K)|)$.

2. For all orientations of K, QACTI(K) $\geq |\sigma(\bar{K})|$.

3. Let q be a quasi-alternating crossing, then QACTI(K) $\leq$ QACTI(K_0^q) $+ 1$, and QACTI(K) $\leq$ QACTI(K_∞^q) $+ 1$

Conclusion

In this project, we have constructed a pathway which starts from the Tait Graph and Goeritz matrix to define what is the family of quasi-alternating links, and study some of its important properties. we first constructed a graph H corresponding to a given link. For this graph H, we defined a Goeritz matrix whose determinant is an invariant of knots upto ambient isotopy. Also, the signature, and nullity of Goeritz matrix has a direct relation to the signature and nullity of the link (both of which are invariants of links). The constructed graph H helps us construct Seifert surface, and we subsequently calculate the Seifert matrix for the surface, which provides us with the tools of calculating the Alexander polynomial, and the Conway polynomial. We also calculated the Tristam-Levine signature, which is a link invariant. Then we defined Quasi-alternating links and then studied some of the properties of the quasi alternating links. Then we defined QACTI which is a measure of the depth or complexity of quasi-alternating links.

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