Critical values of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$ over a number field

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by

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This thesis is dedicated to my mom.

She has been my inspiration and motivation throughout this work.

Certificate

Certified that the work incorporated in the thesis entitled "*Critical values of L-functions for* $\mathrm{GL}_3 \times \mathrm{GL}_1$ *over a number field*", submitted by *Gunja Sachdeva* was carried out by the candidate under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date:

Prof. A. Raghuram
Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Abstract

We prove an algebraicity result for all the critical values of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$ over a totally real field, and a CM field separately. These L-functions are attached to a cohomological cuspidal automorphic representation of GL_3 having cohomology with respect to a general coefficient system and an algebraic Hecke character of GL_1. This is derived from the theory of Rankin–Selberg L-functions attached to pairs of automorphic representations on $\mathrm{GL}_3 \times \mathrm{GL}_2$. Our results are a generalization and refinement of the results of Mahnkopf [26] and Geroldinger [14]. The resulting expressions for critical values of the Rankin-Selberg L-functions are compatible with Deligne's conjecture. As an application, we obtain algebraicity results for symmetric square L-functions.

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Statement of Originality

The main results of this thesis which constitute original research are Theorems 1.2 and 1.3. This leads to Corollaries 1.4 and 1.5.
Sections 4.2.2 and 5.4; Propositions 4.6, 4.19 and 4.20; Lemma 5.20 as well as Theorem 5.11 are original subsidiary results that are required to prove the main results. As an application, main theorem helps to prove Theorem 1.6.

Contents

Chapter 1

Introduction

1.1 History and motivation of the problem.

There has been a long history involving special values of automorphic L-functions for $\mathrm{GL}_n \times \mathrm{GL}_m$, where the special values are written as algebraic multiples of complex invariants defined by means of representation theory and cohomological tools. More precisely, given a cuspidal automorphic representation Π on a reductive algebraic group G, there have been attempts to answer the following questions:

- *What are the interesting integers $s = m$ to consider for $L(s, \Pi)$?*
- *What can we say about algebraicity properties of $L(m, \Pi)$?*

This work is related to a conjecture of Deligne on special values of motivic L-functions. The statement of the conjecture is as follows (see Deligne [11, Conj. 2.8]):

Conjecture 1.1 *Let M be a pure motive over $\mathbb{Q}$ with coefficients in a number field $\mathbb{Q}(M)$. It asserts that the critical values at $s = m \in \mathbb{Z}$ of the $L-$function attached to Motive M can be described, upto multiplication by elements in a number field $\mathbb{Q}(M)$, in terms of geometric motivic periods $c^{\pm}(M)$ and certain explicit power of $(2\pi i)$ as follows:*

$$L(m, M) \sim_{\mathbb{Q}(M)} (2\pi i)^{d(m)} c^{(-1)^m}(M).$$

In 1998, Mahnkopf [26] started looking at the problem of proving an algebraicity result for all critical values of Rankin–Selberg L-functions on $\mathrm{GL}_3 \times \mathrm{GL}_1$ over a number field F. In his paper, he proved the algebraicity of the critical values of the L-function attached to a cuspidal automorphic representation of GL_3 over $\mathbb{Q}$ having cohomology with respect to constant coefficients. Later in 2015, his student Geroldinger [14], generalized his work to arbitrary cohomological weights (μ_1, μ_2, μ_3) of GL_3 over $\mathbb{Q}$ and also proved a functional equation for $p-$adic automorphic L-functions. This thesis deals with proving an algebraicity result for the special values of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$ in the following two situations:

1. Over a totally real field having cohomology with general coefficients $\mu = (\mu_1, \mu_2, \mu_3)$;
2. Over a CM field (totally imaginary quadratic field over a totally real field) having cohomology with coefficients $\mu = (\mu^\iota, \mu^{\bar{\iota}})$ where $\mu^\iota = (\mu_1, \mu_2, \mu_3)$ and $\mu^{\bar{\iota}} = (\mu_1^*, \mu_2^*, \mu_3^*)$ such that $\mu_2 = \mu_2^*$ and also μ is a "parallel" weight.

Such results can be proved by giving a cohomological interpretation to an integral representing a critical L-value.

1.2 Statements of the theorems.

Algebraicity results for all critical values of certain Rankin-Selberg L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$ over a number field F derives from the theory of L-functions attached to pairs of automorphic representations on $\mathrm{GL}_3 \times \mathrm{GL}_2$. Once we have L-functions on $\mathrm{GL}_3 \times \mathrm{GL}_2$, we adapt general techniques and methods of Raghuram's paper [29] to prove the main theorems. To describe the theorems in greater detail, we need some notations. Suppose $\mathbb{A}_F$ is the ring of adeles of F. Given a regular algebraic cuspidal automorphic representation

Π of $\mathrm{GL}_3(\mathbb{A}_F)$, one knows from Clozel [9] that there is a pure dominant integral weight μ such that Π has a nontrivial contribution to the cohomology of some locally symmetric space of GL_3 with coefficients coming from the finite-dimensional representation with highest weight μ. We denote this as $\Pi \in \mathrm{Coh}(G_3, \mu)$, for $\mu \in X_0^+(T_3)$, where T_3 is the diagonal torus of $G_3 = \mathrm{GL}_3$. Let $\Pi = \Pi_\infty \otimes \Pi_f$ be the usual decomposition of Π into its archimedean part Π_∞ and its finite part Π_f. One knows that its rationality field $\mathbb{Q}(\Pi)$ is a number field and that Π is defined over this number field. For a given weight μ, the representation $\mathcal{M}_\mu$ is defined over a number field $\mathbb{Q}(\mu)$, and by Clozel [9], it is known that cuspidal cohomology has a $\mathbb{Q}(\mu)$-structure; hence the realization of Π_f as a Hecke-summand in cuspidal cohomology in lowest possible degree has a $\mathbb{Q}(\Pi)$-structure. On the other hand, the Whittaker model $\mathcal{W}(\Pi_f)$ of the finite part of the representation admits a $\mathbb{Q}(\Pi)$-structure. Following Raghuram-Shahidi [33], on comparing these two $\mathbb{Q}(\Pi)$-structures, certain periods $p^{\epsilon_\Pi}(\Pi) \in \mathbb{C}^\times$ were defined and studied; here $\epsilon_\Pi = (\epsilon_v)_{v \in S_r}$ is a collection of signs indexed by the set S_r of real places of F. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, one knows that ${}^\sigma\Pi \in \mathrm{Coh}(G_3, {}^\sigma\mu)$ and one can define periods simultaneously for all ${}^\sigma\Pi$. Henceforth, let $\mu \in X_0^+(T_3)$ stand for a dominant integral pure weight and consider $\Pi \in \mathrm{Coh}(G_3, \mu)$. The statement of the theorems are as follows:

Theorem 1.2 *(**F is totally real**) Let $\Pi \in \mathrm{Coh}(G_3, \mu)$ with $\varepsilon_{\Pi_v} = \mathbb{1}$ for all $v \in S_\infty$ (see Proposition 3.9 for the definition of ε_{Π_v}), and let $\mu \in X_0^+(T_3)$ such that for each $\mu = (\mu_v)_{v \in S_\infty}$, $\mu_v = (n_v, 0, -n_v)$ with n_v a non-negative integer. Put $n = \min\{n_v\}$. Let $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a character of finite order, and define $\mathbb{Q}(\chi) := \mathbb{Q}(\{\text{values of } \chi\})$. Suppose that $m \in \mathbb{Z}$ is critical for $L_f(s, \Pi \otimes \chi)$, the finite part of the standard degree-3 L-function attached to Π and χ. Then*

$$m \in \begin{cases} \{1 - n_{ev}, \dots, -3, -1;\ 2, 4, \dots, n_{ev}\}, & \text{if } \chi \text{ is totally even,} \\ \{1 - n_{od}, \dots, -4, -2, 0;\ 1, 3, \dots, n_{od}\}, & \text{if } \chi \text{ is totally odd,} \end{cases}$$

where $n_{ev} = 2\left[\frac{n+1}{2}\right]$ = *the largest even positive integer less than or equal to* $n+1$*, and* $n_{od} = 2\left[\frac{n}{2}\right]+1$ = *the largest odd positive integer less than or equal to* $n+1$*. (If* χ *is even at one place and odd at another place then there are no critical points.) Fix a quadratic totally odd character* ξ *once and for all (which will be relevant only when* χ *is totally odd). Consider the four cases:*

Case 1a. χ **is totally even and** $m \in \{2, 4, \cdots, n_{\mathbf{ev}}\}$.
Define $\Omega_r^+(\Pi) := p^{\epsilon_\Pi}(\Pi) L_f(-1, \Pi)^{-1}$*. There exists a nonzero complex number* $P_\infty^1(\mu, m)$ *depending only the weight* μ *and the critical point* m *such that*

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P_\infty^1(\mu, m)\, \Omega_r^+(\Pi)\, \mathcal{G}(\chi)^2,$$

where, by $\approx_{\mathbb{Q}(\Pi,\chi)}$*, we mean up to an element of the number field which is the compositum of the rationality fields* $\mathbb{Q}(\Pi)$ *and* $\mathbb{Q}(\chi)$*; and* $\mathcal{G}(\chi)$ *is the Gauß sum of* χ*.*

Case 1b. χ **is totally even and** $m \in \{1 - n_{\mathbf{ev}}, \cdots, -3, -1\}$.
Define $\Omega_l^+(\Pi) := p^{\epsilon_\Pi}(\Pi) L_f(2, \Pi)^{-1}$*. There exists a nonzero complex number* $P_\infty^2(\mu, m)$ *such that*

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P_\infty^2(\mu, m)\, \Omega_l^+(\Pi)\, \mathcal{G}(\chi).$$

Case 2a. χ **is totally odd and** $m \in \{1, 3, \cdots, n_{\mathbf{od}}\}$.
Define $\Omega_r^-(\Pi) := p^{\epsilon_\Pi}(\Pi) L_f(0, \Pi \otimes \xi)^{-1}$*. There exists a nonzero complex number* $P_\infty^3(\mu, m)$ *such that*

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P_\infty^3(\mu, m)\, \Omega_r^-(\Pi)\, \mathcal{G}(\chi)^2 \mathcal{G}(\xi),$$

where $\mathcal{G}(\xi)$ *is the Gauß sum of* ξ*.*

Case 2b. χ **is totally odd and** $m \in \{1 - n_{\mathbf{od}}, \cdots, -4, -2, 0\}$.
Define $\Omega_l^-(\Pi) := p^{\epsilon_\Pi}(\Pi) L_f(1, \Pi \otimes \xi)^{-1}$. *There exists a nonzero complex number* $P_\infty^4(\mu, m)$ *such that*

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P_\infty^4(\mu, m)\, \Omega_l^-(\Pi)\, \mathcal{G}(\chi).$$

Moreover, in each of the cases, the ratio of the L-value on the left hand side divided by all the quantities in the right hand side is equivariant for the action of $\mathrm{Aut}(\mathbb{C})$.

This theorem has appeared in the article [31]. For $F = \mathbb{Q}$, $\mu = 0$ and $m = 1$, the case 2a above is the main rationality result in Mahnkopf [26]; and for $F = \mathbb{Q}$ and general μ, a weak form of the above theorem is implicit in the construction of the p-adic L-functions in Geroldinger [14]. Let's mention *in passing* that if $n = 0$ and χ is totally even, then there are no critical points. Now we come to CM case where the shape of the main theorem is similar to the totally real case but the input data is different and more complicated.

Theorem 1.3 *(F* **is a CM field)** *Let* $\Pi \in \mathrm{Coh}(G_3, \mu)$ *with* $\mu \in X_0^+(T_3)$. *We suppose that* μ *is a parallel weight, that is,* $\mu = (\mu_v)_{v \in S_\infty}$,

$$\mu_v = (n_1,\ 0,\ n_2;\ -n_2,\ 0,\ -n_1)$$

with n_1 *a non-negative integer and* n_2 *a non-positive integer. (See Section 2.1 for the definition of* S_∞*.) Furthermore, let* $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ *be an algebraic Hecke character also of parallel weight such that*

$$\chi_\infty(z_\infty) = \prod_{v \in S_\infty} \left(\frac{z_v}{|z_v|} \right)^{-2t}$$

for some $t \in \mathbb{Z}$. *For integers a and b, let*

$$[a,\ b] := \{m \in \mathbb{Z} \mid a \le m \le b\}.$$

Suppose that $m \in \mathbb{Z}$ *is critical for* $L_f(s, \Pi \otimes \chi)$, *the finite part of the standard degree-3 L-function attached to* Π *and* χ. *Then*

- *for t strictly positive,*

$$m \in \begin{cases} [2+n_1-t,\ t-n_1-1] & \text{if} \quad 0 \le n_1 \le t-2, \\ [t-n_1,\ n_1+1-t] & \text{if} \quad t \le n_1 \le 2t-1, \\ [1-t,\ t] & \text{if} \quad n_1 \ge 2t; \end{cases}$$

if $n_1 = t-1$ then there are no critical points;

- *for t strictly negative,*

$$m \in \begin{cases} [2-n_2+t,\ n_2-1-t] & \text{if} \quad t+2 \le n_2 \le 0, \\ [n_2-t,\ 1+t-n_2] & \text{if} \quad 2t+1 \le n_2 \le t, \\ [1+t,\ -t] & \text{if} \quad n_2 \le 2t; \end{cases}$$

if $n_2 = t+1$, there are no critical points.

(If $t = 0$, that is, χ is finite order character, then there are no critical points.) Furthermore, fix once and for all the unitary algebraic Hecke character ϕ of parallel weight such that $\phi_\infty(z) = \left(\frac{z}{|z|}\right)^2$. Consider the cases:

Case 1. *t is strictly positive, $n_2 \le -2t$, $n_1 \ge 1$ and*

$$m \in \begin{cases} [2+n_1-t,\ t-n_1-1] & \text{if} \quad n_1 \le t-2, \\ [t-n_1,\ n_1+1-t] & \text{if} \quad t \le n_1 \le 2t-1, \\ [1-t,\ t] & \text{if} \quad n_1 \ge 2t. \end{cases}$$

Define $\Omega^+(\Pi) := p(\Pi) L_f(0, \Pi \otimes \phi)^{-1}$. Then there exists nonzero complex numbers $P^+_\infty(\mu, m)$ (depending only the weight μ and the critical point m) and $c^+(\phi\chi^{-1})$ (depending on characters χ and ϕ) such that

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi,\phi)} \ P^+_\infty(\mu, m)\, \Omega^+(\Pi)\, c^+(\phi\chi^{-1})\, \mathcal{G}(\chi)^2\ \mathcal{G}(\phi),$$

where, by $\approx_{\mathbb{Q}(\Pi,\chi,\phi)}$, we mean up to an element of the number field which is the compositum of the rationality fields $\mathbb{Q}(\Pi)$, $\mathbb{Q}(\chi)$ and $\mathbb{Q}(\phi)$; and $\mathcal{G}(\chi)$ (resp. $\mathcal{G}(\phi)$) is the Gauß sum of χ (resp. ϕ).

Case 2. *t is strictly negative, $n_1 \geq -2t$, $n_2 \leq -1$ and*

$$m \in \begin{cases} [2 - n_2 + t,\ n_2 - 1 - t] & \textit{if} \quad t + 2 \leq n_2, \\ [n_2 - t,\ 1 + t - n_2] & \textit{if} \quad 2t + 1 \leq n_2 \leq t, \\ [1 + t,\ -t] & \textit{if} \quad n_2 \leq 2t. \end{cases}$$

Define $\Omega^-(\Pi) := p(\Pi) L_f(1, \Pi \otimes \phi^{-1})^{-1}$. There exists nonzero complex numbers $P_\infty^-(\mu, m)$ and $c^+(\chi\phi)$ such that

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi,\phi)} \ P_\infty^-(\mu, m)\, \Omega^-(\Pi)\, c^+(\chi\phi)\, \mathcal{G}(\chi)\ \mathcal{G}(\phi)^{-2}.$$

Moreover, in each of the cases, the ratio of the L-value on the left hand side divided by all the quantities in the right hand side is equivariant for the action of $\mathrm{Aut}(\mathbb{C})$*.*

This theorem will appear in the forthcoming article [35], in which author will address the general μ situation. For a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ which is regular conjugate self-dual, cohomological, the above theorem is contained in the main rationality results of Jie Lin's thesis [25].

The proof of theorems, following [26], is based on an integral representation for the value $L_f(m, \Pi \times \chi)$, which we derive from the Rankin–Selberg theory of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_2$, by taking Π on GL_3 and an induced representation $\Sigma(\chi_1, \chi_2)$ on GL_2. Furthermore, assume that the representations are such that $s = 1/2$ is critical for the Rankin-Selberg L-function attached to $\Pi \times \Sigma(\chi_1, \chi_2)$. We note that

$$L(s, \Pi \times \Sigma(\chi_1, \chi_2)) = L(s + 1/2, \Pi \otimes \chi_1) L(s - 1/2, \Pi \otimes \chi_2).$$

Using results from [26] and [30], we can arrange for the data d_1, d_2, χ_1^0 and χ_2^0 in totally real case (Proposition 4.19) and for the data $\sigma_1, \sigma_2, \chi_1^1$ and χ_2^1 in CM case (Proposition 4.20) so as to afford an interpretation of the critical L-value $L(\frac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2))$ as a Poincaré pairing between the pull-back to GL_2 of a cuspidal cohomology class $\vartheta^\circ_{\Pi,\epsilon_\Pi}$ for Π and an Eisenstein cohomology class

ϑ°_Σ for $\Sigma(\chi_1, \chi_2)$ (see Theorem 5.11). Now we freeze one of the characters χ_1, χ_2, and let the other vary, to capture all the critical values $L(m, \Pi \otimes \chi)$. In Section 5.4, for the each of the cases above in both the theorems, we express $L(m, \Pi \otimes \chi)$ in terms of certain periods and the Poincaré pairing of $\vartheta^\circ_{\Pi, \epsilon_\Pi}$ and ϑ°_Σ, from which we deduce the required algebraicity result in Sections 6.2 and 6.3.

Let's now briefly address the compatibility of algebraicity results with motivic periods and motivic L-functions. Let M be a pure motive over $\mathbb{Q}$ with coefficients in a number field $\mathbb{Q}(M)$. Suppose M is critical, then a celebrated conjecture of Deligne [11, Conjecture 2.8] relates the critical values of its L-function $L(s, M)$ to certain periods that arise out of a comparison of the Betti and de Rham realizations of the motive. One expects a cohomological cuspidal automorphic representation Π to correspond to a motive $M(\Pi)$; one of the properties of this correspondence is that the standard L-function $L(s, \Pi)$ is the motivic L-function $L(s, M(\Pi))$ up to a shift in the s-variable; see Clozel [9, Section 4]. With the current state of technology, it seems impossible to compare our periods $p^\epsilon(\Pi)$ with Deligne's periods $c^\pm(M(\Pi))$. Be that as it may, one can still claim that Theorems 1.2 and 1.3 are compatible with Deligne's conjecture by considering the behavior of L-values under twisting by characters. Blasius [2] and Panchishkin [28] have independently studied the behavior of $c^\pm(M(\Pi))$ upon twisting the motive $M(\Pi)$ by a Dirichlet character (more generally by Artin motives). Using Deligne's conjecture, they predict the behavior of critical values of motivic L-functions upon twisting by algebraic Hecke characters. This takes the following form in case of Theorem 1.2 which we state only when the twisting character is a totally even finite order Dirichlet character:

Corollary 1.4 *(F is totally real) Let* $\Pi \in \mathrm{Coh}(G_3, \mu)$ *and* $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ *be of finite order and which is totally even. If the critical point* m *is to the* *right of the center of symmetry then*

$$L_f(m, \Pi \otimes \chi) \;\approx\; L_f(m, \Pi)\mathcal{G}(\chi)^2,$$

but if the critical point m is to the left of the center of symmetry then we have

$$L_f(m, \Pi \otimes \chi) \;\approx\; L_f(m, \Pi)\mathcal{G}(\chi).$$

In both the cases the ratio is $\mathrm{Aut}(\mathbb{C})$*-equivariant.*

From the above relation between critical values for twisted L-functions with the corresponding values of the untwisted L-functions we may claim that our result is compatible with Deligne's conjecture. See also [32, Section 7] where such relations for twisted critical values are conjectured for symmetric power L-functions of a modular form.

Analogously it takes the following form in case of Theorem 1.3 where the twisting character is a unitary algebraic Hecke character, which is enough to state for a particular sub-case of each case:

Corollary 1.5 *(F is CM field) Let $\Pi \in \mathrm{Coh}(G_3, \mu)$ and $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a unitary algebraic Hecke character, defined as in Theorem 1.3. Also fix a unitary Hecke character ϕ as stated in Theorem 1.3. If $t \geq 1;\ n_1 \geq 1;\ n_2 \leq -2t$ and the critical point m satisfies $1 - t \leq m \leq t$ then*

$$\frac{L_f(m, \Pi \otimes \chi)}{\mathcal{G}(\chi)^2} \;\approx\; \frac{L_f(m, \Pi \otimes \phi)}{\mathcal{G}(\phi)^2} \cdot c^+(\phi\chi^{-1}),$$

but if $t \leq -1;\ n_1 \geq -2t;\ n_2 \leq -1$ and the critical point m is such that $1 + t \leq m \leq -t$ then we have

$$\frac{L_f(m, \Pi \otimes \chi)}{\mathcal{G}(\chi)} \;\approx\; \frac{L_f(m, \Pi \otimes \phi^{-1})}{\mathcal{G}(\phi)^{-1}} \cdot c^+(\chi\phi).$$

In both the cases the ratio is $\mathrm{Aut}(\mathbb{C})$*-equivariant.*

The above corollary suggests a factorization of the periods of $\chi\phi$ in terms of the periods of χ and of ϕ, possibly giving a symmetric form to the above equations.

The proof of both the corollaries follows by taking ratio of L-values: $L_f(m, \Pi \otimes \chi)$ and $L_f(m, \Pi \otimes \eta)$ where η is the trivial character when F is totally real or η is a fixed unitary algebraic Hecke character when F is CM field.

Finally, as an application let's discuss the case of symmetric square L-functions for GL_2. For a totally real case Theorem 1.2 applies to the symmetric square L-function $L(s, \mathrm{Sym}^2\varphi, \chi)$ attached to a holomorphic cuspidal Hilbert modular form φ, twisted by a finite order Dirichilet character χ. See Section 6.4. Furthermore, for a CM field case we wish to apply Theorem 1.3 to obtain a rationality result for all the critical values of the symmetric–square L-function $L(s, \mathrm{Sym}^2(\pi), \chi)$ attached to cohomological cuspidal automorphic representation π, twisted by a unitary Hecke character χ. This leads us to the following theorem:

Theorem 1.6 *Let $\pi \in \mathrm{Coh}(G_2, \mu)$ with $\mu \in X_0^+(T_2)$, a 'parallel' dominant integral weight such that for each $v \in S_\infty$, $\mu_v = (a, -a; a, -a)$ for some $a \geq 1$. Let χ be a unitary algebraic Hecke character of a CM field such that $\chi_\infty(z) = (z/|z|)^{-2t}$ for some $t > 0$. Assume that $a \geq t$. Suppose a character ϕ is same as defined in Theorem 1.3. Then the critical set consists of integers $m \in [1-t,\ t]$ and furthermore,*

$$L_f(m, \mathrm{Sym}^2(\pi) \otimes \chi)$$
$$\approx_{\mathbb{Q}(\pi,\chi,\phi)} \; P_\infty^+(\mathrm{Sym}^2(\mu)), m)\, \Omega^+(\mathrm{Sym}^2(\pi))\, c^+(\phi\chi^{-1})\, \mathcal{G}(\chi)^2\; \mathcal{G}(\phi).$$

The proof of the above theorem is given in Section 6.4.

In Chapter 2, we give a dictionary of terminologies which will be needed later to develop the theory. The reader may quickly skim through this chapter to acquaint himself/herself with the notations and cohomological groups we deal with.

In Chapter 3, we begin with a cuspidal automorphic representation on GL_3 and an induced representation on GL_2 and study their cohomological nature. In Section 3.2 we see the general form of an algebraic Hecke character, which later helps in finding the critical values of L-function and defining the induced representation. Furthermore, Sections 3.3 and 3.4 deal with cuspidal cohomology on GL_3 and Eisentein cohomology on GL_2 respectively.

In Chapter 4, we study the analytic interpretation of L-function on $\mathrm{GL}_3 \times \mathrm{GL}_1$. In Section 4.1, we attach an L-function to a pair of representations on $\mathrm{GL}_3 \times \mathrm{GL}_2$, using Rakin–Selberg integrals. In Section 4.2 we calculate the critical set for L-functions on $\mathrm{GL}_3 \times \mathrm{GL}_1$ in terms of weights associated to representations. Furthermore, we arrange everything for the compatibility of weight systems in Section 4.3.

In Chapter 5, we study the cohomological interpretation of Rankin-Selberg integral, using tools available in chapter 3 and then prove the main identity which relates the L-value with the global pairing of cohomology classes. Finally in Chapter 6, we give the Galois equivariant version of both the main theorems followed by an application to the symmetric square L-functions, by thinking of the L-function on $\mathrm{GL}_3 \times \mathrm{GL}_1$ as the standard L-function of the symmetric-square–which is a cohomological cuspidal representation of G_3–twisted by an algebraic Hecke character χ.

Chapter 2

Preliminaries

2.1 Notations and Definitions

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the set of natural numbers, integers, rational numbers, and real numbers, respectively.
- $\mathbb{C}$ denotes the field of complex numbers; for $z \in \mathbb{C}$, $\Re(z)$ will denote its real part, $|z|$ its absolute value and $\bar{z}$ its complex conjugate.
- For integers a and b, define $[a,\ b] := \{m \in \mathbb{Z} \mid a \leq m \leq b\}$.
- $\mathbb{1}$ stands for trivial character.
- **The base field.** Let F be a number field of degree $d_F = [F : \mathbb{Q}]$ with ring of integers $\mathcal{O} = \mathcal{O}_F$. For any place v we write F_v for the topological completion of F at v. Let S_∞ be the set of archimedean places of F. Let $S_\infty := S_r \cup S_c$, where S_r (resp., S_c) is the set of real (resp., complex) places. Let $\varepsilon_F = \mathrm{Hom}(F, \mathbb{C})$ be the set of all embeddings of F as a field into $\mathbb{C}$. There is a canonical surjective map $\varepsilon_F \longrightarrow S_\infty$, which is a bijection on the real embeddings and real places, and identifies a pair of complex conjugate embeddings $\{\iota_v, \bar{\iota}_v\}$ with the complex place v. For each $v \in S_r$, we fix an isomorphism $F_v \cong \mathbb{R}$ which is canonical. Similarly for $v \in S_c$, we fix $F_v \cong \mathbb{C}$ given by (say) ι_v; this choice is not canonical. Let $r_1 = |S_r|$ = number of real places and $r_2 = |S_c|$ =

number of complex places; hence $d_F = r_1 + 2r_2$.
In particular, if we separate the case of totally real and totally imaginary number fields (or CM fields) then:

1. **F is totally real.** In this case $S_\infty = S_r$ and hence $d_F = r_1$.

2. **F is CM field.** A number field F is a **CM** field if it is a totally imaginary quadratic extension F/F_0 where the base field F_0 is totally real. Put $[F_0 : \mathbb{Q}] = d_0$. Then $d_F = [F : \mathbb{Q}] = 2d_0$. Furthermore, in this case $S_\infty = S_c$ and hence $d_F = 2r_2$. This implies $r_2 = d_0$.

Moreover, if $v \notin S_\infty$, and $\mathfrak{p}$ denotes the prime ideal of $\mathcal{O}$ corresponding to v, then we let $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$, and $\mathcal{O}_\mathfrak{p}$ the ring of integers of $F_\mathfrak{p}$. Sometimes, F_v is used for $F_\mathfrak{p}$ and similarly $\mathcal{O}_v$ for $\mathcal{O}_\mathfrak{p}$. The unique maximal ideal of $\mathcal{O}_\mathfrak{p}$ is $\mathfrak{p}\mathcal{O}_\mathfrak{p}$ and is generated by a uniformizer $\varpi_\mathfrak{p}$. Let $\mathfrak{D}_F$ denote the absolute different of F, that is,

$$\mathfrak{D}_F^{-1} = \{x \in F : T_{F/\mathbb{Q}}(x\mathcal{O}) \subset \mathbb{Z}\}.$$

For any prime ideal $\mathfrak{p}$ of F define $r_\mathfrak{p} \geq 0$ by: $\mathfrak{D}_F = \prod_\mathfrak{p} \mathfrak{p}^{r_\mathfrak{p}}$. Let $\mathbb{A}_F$ stand for its adèle ring, with $\mathbb{A}_{F,f}$ and $\mathbb{A}_F^\times$ the ring of finite adèles and group of idèles, respectively. For brevity, $\mathbb{A}_\mathbb{Q}$ will be denoted by $\mathbb{A}$, and similarly, $\mathbb{A}^\times$ for $\mathbb{A}_\mathbb{Q}^\times$.

- We let $\| \, \|_F$: $\mathbb{A}_F^\times \longrightarrow \mathbb{R}_{>0}$ be the adèlic norm of F defined by

$$\| x \| = \prod_{v-\text{finite ramified}} |x_v|_v \prod_{v \in S_\infty} |x_v|_v^{[F_v:\mathbb{R}]}.$$

- **Lie groups.** The algebraic group GL_n/F will be denoted as $\underline{G}_n$, and we put $G_n = R_{F/\mathbb{Q}}(\underline{G}_n)$. An F-group will be denoted by an underline and the corresponding $\mathbb{Q}$-group via Weil restriction of scalars will be denoted without the underline; hence for any $\mathbb{Q}$-algebra A the group of A-points of G_n is $G_n(A) = \underline{G}_n(A \otimes_{\mathbb{Q}} F)$. Let $\underline{B}_n = \underline{T}_n \underline{U}_n$ stand for the standard Borel subgroup of $\underline{G}_n$ of all upper triangular matrices, where $\underline{U}_n$ is the unipotent radical of $\underline{B}_n$, and $\underline{T}_n$ the diagonal torus. The center of $\underline{G}_n$ will be denoted by $\underline{Z}_n$. These groups define the corresponding $\mathbb{Q}$-groups $G_n \supset B_n = T_n U_n \supset Z_n$. Observe that Z_n is not $\mathbb{Q}$-split, and we let S_n be the maximal $\mathbb{Q}$-split torus in Z_n; we have $S_n \cong \mathbb{G}_m$ over $\mathbb{Q}$.

Note that the field F at infinity is

$$F_\infty := F \otimes \mathbb{R} \simeq \prod_{\iota \in \varepsilon_F} F_\iota \simeq \prod_{v \in S_r} \mathbb{R} \times \prod_{v \in S_c} \mathbb{C}.$$

Then the group at infinity is

$$G_{n,\infty} := G_n(\mathbb{R}) = \prod_{v \in S_\infty} \mathrm{GL}_n(F_v) \cong \prod_{v \in S_r} \mathrm{GL}_n(\mathbb{R}) \times \prod_{v \in S_c} \mathrm{GL}_n(\mathbb{C}).$$

We have the center $Z_n(\mathbb{R}) = \prod_{v \in S_r} \mathbb{R}^\times \times \prod_{v \in S_c} \mathbb{C}^\times$, where each copy of $\mathbb{R}^\times$ (resp., $\mathbb{C}^\times$) consists of nonzero scalar matrices in the corresponding copy of $\mathrm{GL}_n(\mathbb{R})$ (resp., $\mathrm{GL}_n(\mathbb{C})$). The subgroup $S_n(\mathbb{R})$ of $Z_n(\mathbb{R})$ denotes the split component of center consisting of $\mathbb{R}^\times$ diagonally embedded in $\prod_{v \in S_r} \mathbb{R}^\times \times \prod_{v \in S_c} \mathbb{C}^\times$. Furthermore, suppose $C_{n,\infty} := \prod_{v \in S_r} \mathrm{O}(n) \times \prod_{v \in S_c} \mathrm{U}(n)$ be the maximal compact subgroup of $G_n(\mathbb{R})$. Put

$$\begin{aligned} K_{n,\infty} &= S_n(\mathbb{R}) C_{n,\infty} \\ &\cong \mathbb{R}^\times \left(\prod_{v \in S_r} \mathrm{O}(n) \times \prod_{v \in S_c} \mathrm{U}(n) \right) \\ &\cong \mathbb{R}_+^\times \left(\prod_{v \in S_r} \mathrm{O}(n) \times \prod_{v \in S_c} \mathrm{U}(n) \right) \\ &= S_n(\mathbb{R})^0 C_{n,\infty}, \end{aligned}$$

where $S_n(\mathbb{R})^0$ denotes the topological connected component of the identity of the split component $S_n(\mathbb{R})$. Let $K_{n,\infty}^0$ be the topological connected component of $K_{n,\infty}$. Hence

$$K_{n,\infty}^0 = S_n(\mathbb{R})^0 C_{n,\infty}^0 \cong \mathbb{R}_+^\times \left(\prod_{v \in S_r} \mathrm{SO}(n) \times \prod_{v \in S_c} \mathrm{U}(n) \right).$$

For any topological group $\mathfrak{G}$, we will let $\pi_0(\mathfrak{G}) := \mathfrak{G}/\mathfrak{G}^0$ stand for the group of connected components. We will identify

$$\pi_0(G_{n,\infty}) = \pi_0(K_{n,\infty}) \cong \prod_{v \in S_\infty} \{\pm 1\} = \prod_{v \in S_r} \{\pm\} \times \prod_{v \in S_c} \{+\}.$$

Furthermore, we identify $\pi_0(G_n(\mathbb{R}))$ inside $G_n(\mathbb{R})$ via the $\delta_n' s$ where the matrix $\delta_n = \mathrm{diag}(-1, 1, \ldots, 1)$ represents the nontrivial element in $\mathrm{O}(n)/\mathrm{SO}(n)$. The character group of $\pi_0(K_{n,\infty})$ is denoted by $\widehat{\pi_0(K_{n,\infty})}$.

- $\mathcal{M}_\mu$ denotes an irreducible finite dimensional complex representation of $G_{n,\infty}$ with highest weight μ.

- Fix a global measure dg on $G_n(\mathbb{A})$, which is a product of local measures dg_v. The local measures are normalized as follows: For a finite place v, if $\mathcal{O}_v$ is the ring of integers of F_v, then we assume that $\mathrm{Vol}(G_n(\mathcal{O}_v)) = 1$, and at infinity assume that $\mathrm{Vol}(C_{n,v}^0) = 1$.

- **Lie algebras.** For a real Lie group G, we denote its Lie algebra by $\mathfrak{g}^0$ and the complexified Lie algebra by $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{g}^0 \otimes_{\mathbb{R}} \mathbb{C}$. Thus, for example, if G is the Lie group $\mathrm{GL}_n(\mathbb{R})$ then $\mathfrak{g}^0 = \mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Let $\mathfrak{g}_{n,\infty}$ and $\mathfrak{k}_{n,\infty}$ denoting the complexified Lie algebras of $G_{n,\infty}$ and $K_{n,\infty}$, respectively.

- Let $\iota : \mathrm{GL}_{n-1} \longrightarrow \mathrm{GL}_n$ be the map $g \mapsto \left(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right)$. Then ι induces a map at the level of local and global groups and between appropriate symmetric spaces of G_{n-1} and G_n, all of which will also be denoted by ι again; we hope that this will cause no confusion. The pullback (of a subset, a function, a differential form, or a cohomology class) via ι will be denoted by ι^*.

- We fix, once and for all, a non-trivial, continuous, additive character $\psi : F \setminus \mathbb{A}_F \longrightarrow \mathbb{C}^\times$. We assume that $\psi_v : F_v^+ \longrightarrow \mathbb{C}^\times$ is unramified for all finite places v. That is, if $\mathfrak{D}_F = \prod_{\mathfrak{p}} \mathfrak{p}^{r_{\mathfrak{p}}}$, the product running over all prime ideals $\mathfrak{p} \subset \mathcal{O}$, then the conductor of the local character ψ_v is $\mathcal{O}_v$, i.e., ψ_v is trivial on $\mathcal{O}_v$ and non-trivial on $\mathfrak{p}_v^{-1}\mathcal{O}_v$.

- **Gauß sums of Adèlic characters.** For a Dirichlet character χ modulo an integer N, following Shimura [39], we define its Gauß sum $\mathfrak{g}(\chi)$ as the Gauß sum of its associated primitive character, say χ_0 of conductor c, where $\mathfrak{g}(\chi_0) = \sum_{a=0}^{c-1} \chi_0(a)\, e^{2\pi i a/c}$. For a Hecke character ξ of F, by which we mean a continuous homomorphism $\xi : F^\times \setminus \mathbb{A}_F^\times \longrightarrow \mathbb{C}^*$, following Weil [44, Chapter VII, Section 7], we define the Gauß sum of ξ as follows: We let $\mathfrak{c}$ stand for the conductor ideal of ξ_f. Let $y = (y_v)_{v\neq\infty} \in \mathbb{A}_f^\times$ be such that $\mathrm{ord}_v(y_v) = -\mathrm{ord}_v(\mathfrak{c})$. The Gauß sum of ξ is defined as
$$\mathcal{G}(\xi_f, \psi_f, y) = \prod_{v\neq\infty} \mathcal{G}(\xi_v, \psi_v, y_v),$$
where the local Gauß sum $\mathcal{G}(\xi_v, \psi_v, y_v)$ is defined as
$$\mathcal{G}(\xi_v, \psi_v, y_v) = \int_{\mathcal{O}_v^\times} \xi_v(u_v)^{-1}\psi_v(y_v u_v)du_v.$$
For almost all v, where everything in sight is unramified, we have $\mathcal{G}(\xi_v, \psi_v, y_v) = 1$, and for all v we have $\mathcal{G}(\xi_v, \psi_v, y_v) \neq 0$. Note that, unlike Weil, we do not normalize the Gauß sum to make it have absolute value one and we do not have any factor at infinity. Suppressing the dependence on ψ and y, we denote $\mathcal{G}(\xi_f, \psi_f, y)$ simply by $\mathcal{G}(\xi_f)$ or even $\mathcal{G}(\xi)$.

- **Locally symmetric spaces.** (See [16, Section 1.1].) Let K_f be an open-compact subgroup of $G_n(\mathbb{A}_f)$. Let us write $K_f = \prod_p K_p$ where each K_p is an open compact subgroup of $G_n(\mathbb{Q}_p)$ and for almost all p we have $K_p = \prod_{v|p} \mathrm{GL}_n(\mathcal{O}_v)$. Define the double-coset space
$$S_n(K_f) = G_n(\mathbb{Q}) \setminus G_n(\mathbb{A})/K_{n,\infty}^0 K_f = \mathrm{GL}_n(F) \setminus \mathrm{GL}_n(\mathbb{A}_F)/K_{n,\infty}^0 K_f.$$
For brevity, let $K = K_{n,\infty}^0 K_f$, and define
$$X = G_n(\mathbb{A})/K = G_n(\mathbb{R})/K_{n,\infty}^0 \times G_n(\mathbb{A}_f)K_f,$$

i.e., X is the product of the symmetric space $G_n(\mathbb{R})/K^0_{n,\infty}$ with a totally disconnected space; any connected component of X is of the form $X_g = G_n(\mathbb{R})^0(g_\infty; g_f)K_f/K$ where $g = (g_\infty; g_f) \in G_n(\mathbb{A})$ with $g_\infty \in \pi_0(G_n(\mathbb{R})) \subset G_n(\mathbb{R})$. The stabilizer of X_g inside $G_n(\mathbb{Q})$ is $\Gamma_g := \{\gamma \in G_n(\mathbb{Q}) : \gamma \in G_n(\mathbb{R})^0 \cap g_f K_f g_f^{-1}\}$. Any connected component of $S_n(K_f)$ is of the form $\Gamma_g \setminus X_g \cong \Gamma_g \setminus G_n(\mathbb{R})^0/K^0_{n,\infty}$. However, Γ_g does not act freely on X_g since $S_{n,\infty} \subset K_{n,\infty}$. Indeed, the stabilizer of every point in X_g contains a congruence subgroup Δ of $S_n(\mathcal{O}_F)$; this Δ is independent of the point in X_g, but the congruence conditions on Δ depend on K_f. The group $\bar{\Gamma}_g = \Gamma_g/\Delta$ acts freely on X_g and the quotient $\bar{\Gamma}_g \setminus X_g$ is a locally symmetric space. We will abuse terminology and sometimes refer to $S_n(K_f)$ as a locally symmetric space of G_n with level structure K_f.

Similarly, define

$$\tilde{S}_n(K_f) := G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/C^0_{n,\infty}K_f \ = \ \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/C^0_{n,\infty}K_f,$$

where $C^0_{n,\infty}$ is the connected component of the identity of the maximal compact subgroup $C_{n,\infty}$ of $G_n(\mathbb{R})$. We get a canonical fibration ϕ given by:

$$\begin{array}{ccc} \tilde{S}_n(K_f) & = & G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/C^0_{n,\infty}K_f \\ \big\downarrow & & \big\downarrow \phi \\ S_n(K_f) & = & G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/K^0_{n,\infty}K_f. \end{array}$$

- **Automorphic representations.** An irreducible representation of $G_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}_F)$ is said to be *automorphic*, following Borel–Jacquet [4], if it is isomorphic to an irreducible subquotient of the representation of $G_n(\mathbb{A})$ on its space of automorphic forms. We say an automorphic representation is *cuspidal* if it is a subrepresentation of the representation of $G_n(\mathbb{A})$ on the space of cusp forms $\mathcal{A}_{\mathrm{cusp}}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) = \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F))$. Let V_π be the subspace of cusp forms realizing a cuspidal automorphic representation π. For an automorphic representation π of $G_n(\mathbb{A})$, we have $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is a representation of $G_{n,\infty}$ and $\pi_f = \otimes_{v \notin S_\infty} \pi_v$ is a representation of $G_n(\mathbb{A}_f)$. The central character of π will be denoted ω_π.

- **Rationality field of π.** Given π, suppose V is the representation space of π_f , any $\sigma \in \mathrm{Aut}(\mathbb{C})$ defines a representation π_f^σ on $V \otimes_{\mathbb{C}} \mathbb{C}_{\sigma^{-1}}$ where $G_n(\mathbb{A}_f)$ acts on the first factor. Let $S(\pi_f)$ be the subgroup of $\mathrm{Aut}(\mathbb{C})$ consisting of all σ such that $\pi_f^\sigma \simeq \pi_f$. Define the rationality field $\mathbb{Q}(\pi_f)$ of π_f as the subfield of $\mathbb{C}$ fixed by $S(\pi_f)$; we denote this as $\mathbb{Q}(\pi) \equiv \mathbb{Q}(\pi_f) = \mathbb{C}^{S(\pi_f)}$. (See [32] for details.)

- The finite part of a global L-function attached to a representation π is denoted by $L_f(s, \pi)$ and for any place v the local L-factor at v is denoted by $L(s, \pi_v)$.

2.2 Various Cohomologies

- **Relative Lie algebra cohomology.** (See Borel-Wallach [5] for details.) If V is a $\mathfrak{g}-$module, and $q \in \mathbb{N}$, then

$$C^q = C^q(\mathfrak{g}; V) = \mathrm{Hom}_F(\wedge^q \mathfrak{g}, V),$$

and $d : C^q \longrightarrow C^{q+1}$ is defined as

$$\begin{aligned} df(x_0, \dots, x_q) &= \sum_j (-1)^j x_j . f(x_0, \dots, \hat{x_j}, \dots, x_q) \\ &+ \sum_{j<k} (-1)^{j+k} f([x_j, x_k], x_0, \dots, \hat{x_j}, \dots, \hat{x_k}, \dots, x_q). \end{aligned}$$

Here a hat over an argument means that it is omitted. An elementary calculation shows that d intertwines the action of $\mathfrak{g}$, and that $d^2 = 0$. Furthermore, to $x \in \mathfrak{g}$ there is associated an endomorphism θ_x of C^q and a linear map $i_x : C^q \longrightarrow C^{q-1}$ defined by

$$(\theta_x f)(x_1, \dots, x_q) = \sum_j f(x_1, \dots, [x_j, x], \dots, x_q) + x \cdot f(x_1, \dots, x_q),$$

$$(i_x f)(x_1, \ldots, x_{q-1}) = f(x, x_1, \ldots, x_{q-1}).$$

Let $C^q(\mathfrak{g}, \mathfrak{k}, V)$ be the subspace of $C^q(\mathfrak{g}, V)$ consisting of the elements annihilated by the maps i_x and θ_x for all $x \in \mathfrak{k}$. Then $C^q(\mathfrak{g}, \mathfrak{k}, V)$ is stable under the map d and we have

$$C^q(\mathfrak{g}, \mathfrak{k}; V) = \mathrm{Hom}_{\mathfrak{k}}(\wedge^q(\mathfrak{g}/\mathfrak{k}), V),$$

where the action of $\mathfrak{k}$ on $\wedge^q(\mathfrak{g}/\mathfrak{k})$ is induced by the adjoint representation.

The cohomology groups of complex $C^q(\mathfrak{g}, \mathfrak{k}; V)$ are the *relative lie algebra cohomology groups* $H^q(\mathfrak{g}, K; V)$ of $\mathfrak{g}$ mod $\mathfrak{k}$, with coefficients in V, where K is the connected subgroup such that $\mathrm{Lie}(K) = \mathfrak{k}$. We are interested in the cohomology groups: $H^\bullet(\mathfrak{g}_{n,\infty}, K^0_{n,\infty}; V)$.

Observe that if K_0 is a normal subgroup of K, since K acts on the space $\mathrm{Hom}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), V)$, this implies K/K_0 acts $\mathrm{Hom}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), V)^{K_0} = \mathrm{Hom}_{K_0}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), V)$. Here K^0 is a topological connected component of K

- **Sheaf cohomology.** (Reference: see Harder-Raghuram [17]) Given a dominant-integral weight $\mu \in X^+(T_n)$ and the associated representation $\mathcal{M}_{\mu,E}$, where E is an extension of $\mathbb{Q}(\mu)$, we get a sheaf $\widetilde{\mathcal{M}}_{\mu,E}$ of $E-$vector spaces on symmetric space $S_{G_n}(K_f)$ as follows: Let $\pi : G_n(\mathbb{A})/K^0_{n,\infty}K_f \to S_n(K_f)$ be the canonical projection. For any open subset U of $S_n(K_f)$ define the sections over U by:

$$\widetilde{\mathcal{M}}_\mu(U) := \{s : \pi^{-1}(U) \to \mathcal{M}_{\mu,E} \mid s \text{ is locally constant, and} \\ s(\gamma u) = \rho_\mu(\gamma)s(u); \forall\, \gamma \in G_n(\mathbb{Q}), u \in \pi^{-1}(U)\},$$

where ρ_μ is the finite dimensional representation of $G_n(\mathbb{R})$ with highest weight μ. This defines a sheaf of complex vector spaces on $S_n(K_f)$. Note that even if $\mathcal{M}_{\mu,E} \neq 0$ it is possible that the sheaf $\widetilde{\mathcal{M}}_{\mu,E} = 0$.

(See Harder [16, 1.1.3].) Indeed, $\widetilde{\mathcal{M}}_{\mu,E} = 0$ unless the central character of ρ_μ has the infinity type of an algebraic Hecke character of F. We are interested in the sheaf cohomology groups

$$H^\bullet(S_n(K_f), \widetilde{\mathcal{M}}_{\mu,E}).$$

It is convenient to pass to the limit over all open-compact subgroups K_f and let $H^\bullet(S_n, \widetilde{\mathcal{M}}_{\mu,E}) := \varinjlim_{K_f} H^\bullet(S_n(K_f), \widetilde{\mathcal{M}}_{\mu,E})$. There is an action of $\pi_0(G_{n,\infty}) \times G_n(\mathbb{A}_f)$ on $H^\bullet(S_n, \widetilde{\mathcal{M}}_{\mu,E})$, and the cohomology of $S_n(K_f)$ is obtained by taking invariants under K_f, i.e.,

$$H^\bullet(S_n(K_f), \widetilde{\mathcal{M}}_{\mu,E}) = H^\bullet(S_n, \widetilde{\mathcal{M}}_{\mu,E})^{K_f}.$$

Working at a transcendental level, i.e., taking $E = \mathbb{C}$, we can compute the above sheaf cohomology via the de Rham complex, and then reinterpreting the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

$$H^\bullet(S_n, \widetilde{\mathcal{M}}_\mu) \;\simeq\; H^\bullet(\mathfrak{g}_{n,\infty}, K^0_{n,\infty}; C^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) \otimes \mathcal{M}_\mu).$$

With level structure K_f it takes the form:

$$H^\bullet(S_n(K_f), \widetilde{\mathcal{M}}_\mu) \;\simeq\; H^\bullet(\mathfrak{g}_{n,\infty}, K^0_{n,\infty}; C^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))^{K_f} \otimes \mathcal{M}_\mu).$$

We will also consider the cohomology groups $H^\bullet(\tilde{S}_n(K_f), \widetilde{\mathcal{M}}_\mu)$.

- **Cuspidal cohomology.** The inclusion

$$C^\infty_{\rm cusp}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})) \hookrightarrow C^\infty(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))$$

of the space of smooth cusp forms in the space of all smooth functions induces, via results of Borel [3], an injection in cohomology; this defines cuspidal cohomology:

$$H^\bullet_{\rm cusp}(S_n(K_f), \widetilde{\mathcal{M}}_\mu) \;\simeq\; H^\bullet(\mathfrak{g}_n, K^0_{n,\infty}; C^\infty_{\rm cusp}(G_n(\mathbb{Q})\backslash G_n(\mathbb{A}))^{K_f} \otimes \mathcal{M}_\mu).$$

Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition of $\pi_0(G_{n,\infty}) \times G_n(\mathbb{A}_f)$-modules:

$$H^{\bullet}_{\text{cusp}}(S_n, \widetilde{\mathcal{M}}_\mu) \;=\; \bigoplus_{\Pi} H^{\bullet}(\mathfrak{g}_n, K^0_{n,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu) \otimes \Pi_f.$$

We say that Π contributes to the cuspidal cohomology of G_n with coefficients in $\mathcal{M}_\mu$ if Π has a nonzero contribution to the above decomposition. Equivalently, if Π is a cuspidal automorphic representation whose representation at infinity Π_∞ after twisting by $\mathcal{M}_\mu$ has nontrivial relative Lie algebra cohomology, i.e., $H^{\bullet}(\mathfrak{g}_n, K^0_{n,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu) \neq 0$ for some $\bullet$. In this situation, we write $\Pi \in \text{Coh}(G_n, \mu)$. It is well known (see [9]) that only pure weights support cuspidal cohomology.

Chapter 3

Representation Theory and Cohomology

3.1 Finite dimensional representations

Consider $T_{n,\infty} = \prod_{v \in S_\infty} T_n(F_v)$. Let $X^*(T_n) = X^*(T_{n,\infty})$ be the group of all algebraic characters of $T_{n,\infty}$, and let $X^+(T_n) = X^+(T_{n,\infty})$ be the subset of $X^*(T_{n,\infty})$ which are dominant integral with respect to Borel subgroup B_n. A weight $\mu \in X^+(T_{n,\infty})$ is described as follows: $\mu = (\mu_v)_{v \in S_\infty}$, where

- For $v \in S_r$, we have $\mu_v = (\mu_1^v, \ldots, \mu_n^v)$, $\mu_i^v \in \mathbb{Z}$, $\mu_1^v \geq \ldots \geq \mu_n^v$, and the character μ_v sends $t = \mathrm{diag}(t_1, \ldots, t_n) \in T_n(F_v)$ to $\prod_i t_i^{\mu_i^v}$.

- If $v \in S_c$ then μ_v is the pair $(\mu^{\iota v}, \mu^{\bar{\iota} v})$, with $\mu^{\iota v} = (\mu_1^{\iota v}, \cdots, \mu_n^{\iota v}), \mu_i^{\iota v} \in \mathbb{Z}, \mu_1^{\iota v} \geq \cdots \geq \mu_n^{\iota v}$; likewise $\mu^{\bar{\iota} v} = (\mu_1^{\bar{\iota} v}, \cdots, \mu_n^{\bar{\iota} v})$ and $\mu_1^{\bar{\iota} v} \geq \cdots \geq \mu_n^{\bar{\iota} v}$; the character μ_v is given by sending

$$t = \mathrm{diag}(z_1, \cdots, z_n) \in T_n(F_v) \text{ to } \prod_{i=1}^{n} z_i^{\mu^{\iota v}} \bar{z}_i^{\mu^{\bar{\iota} v}},$$

  where $\bar{z}_i$ is the complex conjugate of z_i.

Furthermore, if there is an integer $\mathsf{w}(\mu)$ such that

(1) For $v \in S_r$ and $1 \leq i \leq n$ we have $\mu_i^v + \mu_{n-i+1}^v = \mathsf{w}(\mu)$;

(2) For $v \in S_c$ and $1 \leq i \leq n$ we have $\mu_i^{\bar{\iota} v} + \mu_{n-i+1}^{\iota v} = \mathsf{w}(\mu)$,

then we call such a weight μ a *pure weight* and call $\mathsf{w}(\mu)$ the *purity weight* of μ. We denote the set of dominant integral pure weights as $X_0^+(T_n) = X_0^+(T_{n,\infty})$. Furthermore, take an integer b and integers $a_1 \geq a_2 \geq \cdots \geq a_n$ such that

$$a_j + a_{n-j+1} = b;$$

now for each $v \in S_\infty$ put $\mu_v = (a_1, \cdots, a_n)$; then μ is pure with $\mathsf{w}(\mu) = b$. Such a weight is called a *parallel weight*.

For $\mu \in X^+(T_{n,\infty})$, we define $(\rho_\mu, \mathcal{M}_{\mu,\mathbb{C}})$ an irreducible finite dimensional complex representation of $G_{n,\infty}$ with highest weight μ as follows: Since $G_{n,\infty} = \prod_{v \in S_r} \mathrm{GL}_n(\mathbb{R}) \times \prod_{v \in S_c} \mathrm{GL}_n(\mathbb{C})$, it is clear that

$$(\rho_\mu, \mathcal{M}_\mu) = (\otimes_v \rho_{\mu_v}, \otimes_v \mathcal{M}_{\mu_v})$$

such that for $v \in S_r$, $(\otimes_v \rho_{\mu_v}, \otimes_v \mathcal{M}_{\mu_v})$ being the irreducible finite dimensional representation of $\mathrm{GL}_n(\mathbb{R})$ of highest weight μ_v, and $v \in S_c$, $(\otimes_v \rho_{\mu_v}, \otimes_v \mathcal{M}_{\mu_v})$ is the complex representation of the real algebraic group $G(F_v) = \mathrm{GL}_n(\mathbb{C})$ defined as $\rho_{\mu_v}(g) = \rho_{\mu^{\iota_v}}(g) \otimes \rho_{\mu^{\bar{\iota}_v}}(\bar{g})$; here $\rho_{\mu^{\iota_v}}$ (resp., $\rho_{\mu^{\bar{\iota}_v}}$) is the irreducible representation $\mathcal{M}_{\mu^{\iota_v}}$ (resp., $\mathcal{M}_{\mu^{\bar{\iota}_v}}$) of the complex group $\mathrm{GL}_n(\mathbb{C})$ with highest weight μ^{ι_v} (resp., $\mu^{\bar{\iota}_v}$).

3.2 Algebraic Hecke Characters

(See Weil [43] for more details.) Recall $\mathbb{A}_F$ is the adèle ring of F, and $\mathbb{I}_F = \mathbb{A}_F^\times$ is the group of idèles of F. Let E be the group of all units ε in F and $C_F = \mathbb{I}_F / F^\times$ denotes the idèle class group of F.

Definition. A Hecke character is a continuous homomorphism

$$\chi : \mathbb{I}_F / F^\times \longrightarrow \mathbb{C}^\times.$$

Recall the norm map $\parallel \parallel$ of an idèle $\alpha \in \mathbb{I}_F$ which is defined as

$$\parallel \alpha \parallel = \prod_v |\alpha_v|_v,$$

where all the valuations are normalized valuations. Then $\mathbb{I}_F/F^\times \longrightarrow \mathbb{R}_+^\times$ is a surjective homomorphism with kernel $=: {}^0\mathbb{I}_F$. Clearly $F^\times \subset {}^0\mathbb{I}_F$. We have the following exact sequence

$$0 \longrightarrow {}^0\mathbb{I}_F/F^\times \longrightarrow \mathbb{I}_F/F^\times \longrightarrow \mathbb{R}_+^\times \longrightarrow 0.$$

This sequence splits. For example, two of the splittings are given by mapping $t \in \mathbb{R}_+^\times$ into the idéle which is t at a particular real infinite place and 1 elsewhere, or by mapping t to $t^{1/2}$ at a particular complex infinite place and 1 elsewhere. This splitting gives

$$\mathbb{I}_F/F^\times \simeq {}^0\mathbb{I}_F/F^\times \times \mathbb{R}_+^\times.$$

It is a fundamental fact that ${}^0\mathbb{I}_F/F^\times$ is compact (Neukirch [27, Theorem VI. 1.6]). A continuous homomorphism of ${}^0\mathbb{I}_F/F^\times$ into $\mathbb{C}^\times$ has compact image and so lands in S^1. Furthermore, any homomorphism $\mathbb{R}_+^\times \longrightarrow \mathbb{C}^\times$ is of the form $x \longmapsto |x|^w$ for a complex number $w = \sigma + i\varphi$. Putting these remarks together, any Hecke character χ can be uniquely factored as

$$\chi = \chi^\circ \otimes \| \, \|^\sigma \tag{3.1}$$

where $\chi^\circ : \mathbb{I}_F/F^\times \longrightarrow S^1$ is a unitary Hecke character and $\sigma \in \mathbb{R}$.

The character at infinity of a Hecke character:

• *Characters of* $\mathbb{R}^\times$. Any continuous homomorphism $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$ is of the form

$$\chi(x) = \operatorname{sgn}(x)^{-f}|x|^w = \left(\frac{x}{|x|}\right)^{-f} |x|^w,$$

with $f \in \{0, 1\}$ and $w \in \mathbb{C}$. Such a character χ is unitary if and only if $w = i\varphi \in i\mathbb{R}$.

• *Characters of* $\mathbb{C}^\times$. First let us note that for $z = x + iy \in \mathbb{C}$, its normalized real absolute value is defined as $|z| := |z|_\mathbb{R} := \sqrt{x^2 + y^2}$ and its normalized complex absolute value is $|z|_{\mathbb{C}} := |z|_{\mathbb{R}}^2 = x^2 + y^2$. Any continuous homomorphism $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$ is of the form

$$\chi(z) = \left(\frac{z}{|z|}\right)^{-f} |z|_{\mathbb{C}}^{w},$$

with $f \in \mathbb{Z}$ and $w \in \mathbb{C}$. Such a character χ is unitary if and only if $w = i\varphi \in i\mathbb{R}$.

• *Description of* χ_∞. For a Hecke character χ, let $\chi_\infty = \chi|_{F_\infty^\times}$ where $F_\infty^\times \hookrightarrow \mathbb{I}_F$. We want to describe this character explicitly. We will use the following notations: λ is any infinite place; $\lambda \in S_\infty$, ρ is any real place; $\rho \in S_r$ and $F_\rho \simeq \mathbb{R}$ canonically, and ι is any complex place; $\iota \in S_c$ and $F_\iota \simeq \mathbb{C}$ non-canonically. Further $|x_\infty|_\infty = \prod\limits_{\lambda} |x_\lambda|_\lambda$ for $x_\infty \in F_\infty$ is the product of normalized valuations. Keep in mind that a Hecke character factorizes as $\chi = \chi^\circ \otimes \| \|^\sigma$; see Equation (3.1). We can write the character at infinity χ_∞ on $x_\infty \in F_\infty^\times$ as

$$\chi_\infty(x_\infty) = \left(\prod_{\lambda \in S_\infty} \left(\tfrac{x_\lambda}{|x_\lambda|}\right)^{-f_\lambda} |x_\lambda|_\lambda^{i\varphi_\lambda} \right) |x_\infty|_\infty^{\sigma},$$

where $f_\rho \in \{0,\, 1\}$, $f_\iota \in \mathbb{Z}$, $\varphi_\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}$.
Using above description of χ at infinity one can classify all finite order characters (say χ^0): A character χ^0 of C_F is of **finite order** if and only if it is 1 on the connected component of identity in $\mathbb{I}_F$, that is, if and only if $f_\iota = 0$ for all complex places ι, $\varphi_\lambda = 0$ for all infinite places λ, and $\sigma = 0$. This implies, for example, a finite order character on $\mathbb{R}^\times$ will be either $\mathbb{1}$ or sgn and a finite order character on $\mathbb{C}^\times$ is always $\mathbb{1}$.

After classifying finite order characters, we want to classify Hecke characters χ (given by Equation (3.1)): Following Weil, given f_λ and σ, a necessary and sufficient condition for the existence of a character χ is that there should be an integer m such that

$$\prod_{\lambda \in S_\infty} (\varepsilon_\lambda / |\varepsilon_\lambda|)^{m f_\lambda} = 1 \quad \Longrightarrow \quad \prod_{\lambda \in S_\infty} (\varepsilon_\lambda / \bar{\varepsilon_\lambda})^{m f_\lambda} = 1, \tag{3.2}$$

for all $\varepsilon \in E$. The last equation is obtained by replacing m by $2m$.
Clearly, if F is a totally real number field, then Equation (3.2) always holds. Further if F is a CM field, that is, totally imaginary quadratic extension of a totally real number field F_0, then, by Dirichlet's theorem ε^m is totally real for every $\varepsilon \in E$ because the group E_0 of the units in F_0 is of finite index in E and take m to be that index. Hence Equation (3.2) holds on E for that value of m and for arbitrary values of f_ι. It implies then for a CM field, character χ exists.

Note: The above argument may not work in the case of totally imaginary fields, i.e., for all $\varepsilon \in E$, ε^m need not be totally real. For a totally imaginary field F, it may happen that there does not exist any totally real field say F_r between F and $\mathbb{Q}$ such that necessary condition Equation (3.2) holds. For example take $F = \mathbb{Q}(\sqrt{2} + i\sqrt{3})$. Its easy to check that F is totally imaginary number field of degree 4, which is not a CM field as $\mathbb{Q}(i\sqrt{6})$ is the only proper subfield of F. Hence we work with CM fields only.

We say that χ is of **type(A)** if all the φ_λ are 0 and σ is rational. In order for a field to have non-trivial characters of type(A), it is necessary and sufficient that it should contain a totally imaginary quadratic extension F_1 of its maximal totally real subfield F_0, for example, F is CM field with $F = F_1$. Otherwise F will have trivial characters.

A *trivial algebraic Hecke character* is a trivial character of type(A), which is the form $\chi^0 \parallel \parallel^m$, with χ^0 of finite order and m an integer. If F is a totally real number field, then F will have only trivial algebraic Hecke characters.

Hence an algebraic Hecke character of a totally real field looks like

$$\chi = \chi^0 \parallel \parallel^m,$$

with $m \in \mathbb{Z}$ and χ^0 a finite order Dirichlet character.

A *non-trivial algebraic Hecke character* is a non-trivial character of type(A) for which 2σ is an integer and $f_\lambda \equiv 2\sigma \pmod 2$ for all λ. Hence an algebraic Hecke character of a CM field looks like

$$\chi = \chi^1 \parallel \parallel^\sigma, \tag{3.3}$$

with a unitary character χ^1 whose infinity part is

$$\chi^1_\infty(x_\infty) = \prod_\lambda (x_\lambda/|x_\lambda|)^{-f_\lambda}$$

such that $f_\lambda \equiv 2\sigma$ (mod 2) for some $f_\lambda \in \mathbb{Z}$ and $\sigma \in \frac{1}{2}\mathbb{Z}$. According to Weil, such characters are called characters of **type($\mathbf{A}_0$)**.

Note. In this thesis, unless otherwise mention, for CM case we deal with algebraic Hecke characters of type(A_0) with *parallel* weight. Hence under this assumption, an algebraic Hecke character χ of a CM field looks like:

$$\chi = \chi^1 \parallel \parallel^\sigma,$$

with a unitary character χ^1 whose infinity part is

$$\chi^1_\infty(x_\infty) = \prod_\lambda (x_\lambda/|x_\lambda|)^{-f} = \left(\frac{x_\infty}{|x_\infty|}\right)^{-f}$$

such that $f \equiv 2\sigma$ (mod 2) for some $f \in \mathbb{Z}$ and $\sigma \in \frac{1}{2}\mathbb{Z}$.

Fact 3.4 *If a character χ of the idèle class group $C_\mathbb{F}$ of the field F is of type(A_0), then the coefficients of the Hecke L-series associated with χ lie in a finite algebraic extension of $\mathbb{Q}$; we denote this field as $\mathbb{Q}(\chi)$ and it is described below.*

More facts:

- Assume that F has at least one real place. Define the *signature* ε_χ of an algebraic Hecke character χ as follows: By the purity constraint, with purity weight w, the character $\chi^0 := \chi \| \|^{-\mathsf{w}(\chi)}$ is a character of finite order. For $v \in S_r$, define

$$\varepsilon_{\chi_v} = (-1)^{\mathsf{w}} \chi_v^0(-1).$$

Now put $\varepsilon_\chi = (\varepsilon_{\chi_v})_{v \in S_r}$. The signature is an r_1-tuple of signs indexed by real embeddings of F. This implies when F is a CM field, no signature appears because a finite order character of $\mathbb{C}^\times$ is always trivial.

- For each finite place v and any smooth character $\omega_v : F_v^\times \to \mathbb{C}^\times$, define the rationality field $\mathbb{Q}(\omega_v)$ of ω_v as the field obtained by adjoining the values of ω_v to $\mathbb{Q}$. For an algebraic Hecke character χ, we define its rationality field $\mathbb{Q}(\chi)$ as the compositum of the fields $\mathbb{Q}(\chi_v)$ for all finite places v that are unramified for χ. Then $\mathbb{Q}(\chi)$ is a number field, and the field $\mathbb{Q}(\chi)$ need not contain the field F.

Critical values of an algebraic Hecke character.

1. *χ is a finite order Dirichlet character.* Let χ be a Dirichlet character mod m with $\chi(-1) = (-1)^p$, where $p \in \{0, 1\}$. The following results hold for the special values of L-function attached to χ (see Neukirch [27, Chapter 7] for details):

   - For any integer $k \geq 1$, one has
   $$L(1-k, \chi) = -\frac{B_{k,\chi}}{k}, \tag{3.5}$$
   where $B_{k,\chi}$ is the generalized Bernoulli number.

   - For $k \equiv p \pmod 2$, $k \geq 1$, one has
   $$L(k, \chi) = (-1)^{1+\frac{k-p}{2}} \frac{\mathcal{G}(\chi)}{2i^p} \left(\frac{2\pi}{m}\right)^k \frac{B_{k,\bar{\chi}}}{k}. \tag{3.6}$$

2. χ *is a unitary Hecke character of a CM field.* Let χ be an algebraic Hecke character of $F^\times$ such that $\chi_\infty(z_\infty) = \left(\frac{z_\infty}{|z_\infty|}\right)^{-f}$ for $f \in \mathbb{Z}$ and for all $z_\infty \in \mathbb{C}^{r_2}$. Using Section 3.2, it is clear that $f \equiv 0 \pmod 2$, that is, f is an even integer. Put $f = 2a$; $a \in \mathbb{Z}$. Rewriting the character at infinity we get:

$$\begin{aligned}\chi_\infty(z_\infty) &= \prod_{v \in S_\infty} \left(\frac{z_v}{|z_v|}\right)^{-2a} \\ &= \prod_{v \in S_\infty} z_v^{-2a}|z_v|^{2a} \\ &= \prod_{v \in S_\infty} z_v^{-2a+a}\bar{z}_v{}^{a} = r_2 \cdot (z_v^{-a}\bar{z}_v{}^{a}).\end{aligned}$$

Let $L(s, \chi)$ be the Hecke L-function attached to character χ. Then the infinite part of L-value is

$$L_\infty(s, \chi) = \prod_{v \in S_c} L(s, \chi_v) = r_2 \cdot (2(2\pi)\Gamma(s + |a|)).$$

Similarly the Hecke L-factor associated with $\chi^\vee$ is

$$L_\infty(1 - s, \chi^\vee) = \prod_{v \in S_c} L(1 - s, \chi_v^\vee) = \prod_{v \in S_c} 2(2\pi)\Gamma(1 - s + |a|).$$

An integer m is critical if for all $v \in S_c$ the Gamma factors $\Gamma(s + |a|)$ and $\Gamma(1 - s + |a|)$ have no poles, that is,

$$m + |a| \geq 1; \quad \text{and} \quad 1 - m + |a| \geq 1.$$

On simplifying we get:

$$m \in \mathbb{Z} \text{ is critical if and only if } \{1 - |a| \leq m \leq |a|\}. \tag{3.7}$$

Fact 3.8 *Blasius [2] proved that Deligne's conjecture [11, Conjecture 2.8] (which relates critical values of a motivic L-function $L(s, M)$ to certain motivic periods) holds for a motive $M(\chi)$ (of pure weight w)*

attached to an algebraic Hecke character χ over CM fields. It is stated as follows: If s is critical for the L-function attached to an algebraic Hecke character χ over a CM field F, then

$$\frac{L_f(s,\chi)}{c^+(\chi \parallel \cdot \parallel^s)} \in \mathbb{Q}(\chi) \otimes_{\mathbb{Q}} \mathbb{C},$$

where $c^+(\chi \parallel \cdot \parallel^s) = (2\pi i)^{\frac{s[F\,:\,\mathbb{Q}]}{2}} c^{(-1)^s}(\chi)$. Here $c^{\pm}$ are the motivic periods introduced by Deligne and $\mathbb{Q}(\chi)$ is a number field as in Fact 3.4. (See also [37].) Moreover $M(\chi)$ has no $\left(\frac{w}{2},\frac{w}{2}\right)-$classes unless F is totally real and hence using ([18], for example) $c^+(\chi) \sim_{\mathbb{Q}(\chi)} c^-(\chi)$.

3.3 Cuspidal Cohomological representations of GL_3

Let $\mu \in X_0^+(T_{3,\infty})$ be a pure weight, and let $\Pi \in \mathrm{Coh}(G_3,\mu)$. The purpose of this section is to first write down explicitly the representation Π_∞ in terms of μ. Since $\Pi_\infty = \prod_{v\in S_\infty} \Pi_v$, if we have local representations at hand then the problem of describing Π_∞ is a purely local one. This helps in computing the set of critical points of Rankin-Selberg L-functions. Secondly, we study the connection of representation Π_∞ with cohomology groups $H^\bullet(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu)$, giving the possible degrees in which one has non-trivial cuspidal cohomology. This permits us to give a cohomological interpretation to Rankin-Selberg L-functions. We begin by taking up real and complex places separately. Before that we make the following observation:

• The group $K_{3,\infty}/K^0_{3,\infty} = (\mathbb{Z}/2\mathbb{Z})^{r_1}$ acts on $H^\bullet(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu)$. We consider certain isotypic components for this action. Consider an r_1 tuple of signs indexed by the set S_r of real places in S_∞. Let

$$\epsilon = (\epsilon_v)_{v\in S_r} \in \{\mathbb{1}, \mathrm{sgn}\}^{r_1} = (K_{3,\infty}/K^0_{3,\infty})^\wedge.$$

When F is totally real field, Π uniquely determines ϵ, and when F is a CM field then no sign appears as F has no real place. For later use let $H^{\bullet}(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu)(\epsilon_\Pi)$ be the corresponding isotypic component.

3.3.1 Cohomological representations of $\mathrm{GL}_3(\mathbb{R})$

We review some well known details that will be relevant later on. (See [29] for more details and further references.) For any integer $\ell \geq 1$, let D_ℓ stand for the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with lowest non-negative $\mathrm{SO}(2)$-type given by the character $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto \exp^{-i(\ell+1)\theta}$, and central character $a \mapsto \mathrm{sgn}(a)^{\ell+1}$.

Suppose $\mu \in X^+_0(T_3)$ is a pure dominant integral weight written as $\mu = (\mu_v)_{v \in S_\infty}$ with $\mu_v = (\mu^v_1, \mu^v_2, \mu^v_3)$ and let $\mathsf{w} = \mu^v_1 + \mu^v_3 = 2\mu^v_2$ be the purity weight of μ. Note that w is an even integer. Let's write it as $\mathsf{w} =: 2\mathsf{w}^\circ = 2\mu^v_2$. Suppose $\Pi \in \mathrm{Coh}(G_3, \mu)$, then it is clear that

$$\Pi \otimes \| \ \|^{\mathsf{w}^\circ} \in \mathrm{Coh}(G_3, \mu - \mathsf{w}^\circ)$$

because

$$H^{\bullet}(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \| \ \|^{\mathsf{w}^\circ} \otimes \mathcal{M}_\mu \otimes (\det)^{-\mathsf{w}^\circ}) = H^{\bullet}(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu).$$

The purity weight of $\mu - \mathsf{w}^\circ$ is 0, and furthermore $\Pi \otimes | \ |^{\mathsf{w}^\circ}$ is a unitary cuspidal representation. As far as L-functions (and their special values) are concerned, we have not lost any information since $L(s, \Pi \otimes \| \ \|^{\mathsf{w}^\circ}) = L(s + \mathsf{w}^\circ, \Pi)$. We will henceforth assume:

1. μ is a pure dominant integral weight with purity weight 0; so $\mu = (\mu_v)_{v \in S_\infty}$ with $\mu_v = (n_v, 0, -n_v)$ for a non-negative integer n_v.

2. $\Pi \in \mathrm{Coh}(G_3, \mu)$, i.e., Π is a unitary cuspidal automorphic representation of GL_3/F that has nontrivial cohomology with respect to $\mathcal{M}_\mu$.

The following well-known proposition records some basic information about the relative Lie algebra cohomology groups in this context.

Proposition 3.9 *Let $\mu \in X_0^+(T_3)$ be a pure dominant integral weight with purity weight 0; we write $\mu = (\mu_v)_{v \in S_\infty}$ with $\mu_v = (n_v, 0, -n_v)$ for an integer $n_v \geq 0$. Put $\ell_v = 2n_v + 2$. Suppose $\Pi \in \mathrm{Coh}(G_3, \mu)$. Then for every $v \in S_\infty$ we have*

$$\Pi_v \; = \; \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})} \left(D_{\ell_v} \otimes \varepsilon_{\Pi_v}\right),$$

where, $P_{(2,1)}$ is the standard parabolic subgroup of $\mathrm{GL}_3(\mathbb{R})$ with Levi quotient $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$, and ε_{Π_v} is a quadratic character of $\mathbb{R}^\times$. In terms of the central character, we have $\varepsilon_{\Pi_v}(-1) = -\omega_{\Pi_v}(-1)$. (We also write $\varepsilon_{\Pi_v} = \mathrm{sgn}^{e_{\Pi_v}}$ with $e_{\Pi_v} \in \{0, 1\}$.)
Define $b_3^F = 2d_F = 2[F : \mathbb{Q}]$. The smallest degree $\bullet$ for which

$$H^\bullet(\mathfrak{g}_{3,\infty}, K_{3,\infty}^0; \Pi_\infty \otimes \mathcal{M}_\mu) \neq 0 \text{ is } \bullet = b_3^F = 2d_F,$$

and in this degree, the cohomology group is one-dimensional. Further as a $K_{3,\infty}/K_{3,\infty}^0$-module we denote the cohomology group by ϵ_Π, which is a d_F-tuple of signs: $(\mathrm{sgn}^{1+e_{\Pi_v}})_{v \in S_r}$.

3.3.2 Cohomological representations of $\mathrm{GL}_3(\mathbb{C})$

(See [29] for more details and further references.) Suppose $\Pi \in \mathrm{Coh}(G_3, \mu)$ and $\mu \in X_0^+(T_3)$ is a pure dominant integral weight written as $\mu = (\mu_v)_{v \in S_\infty}$. We shall drop v from the notations for this section. Then, for each complex place v, μ is a pair $(\mu^\iota, \mu^{\bar\iota})$, where ι is a complex embedding that has been non-canonically chosen and fixed, and $\bar\iota$ is the conjugate embedding; and we have 3-tuples $\mu^\iota = (\mu_1, \mu_2, \mu_3)$ and $\mu^{\bar\iota} = (\mu_1^*, \mu_2^*, \mu_3^*)$ of integers. Let $\mathsf{w} = \mu_1^* + \mu_3 = \mu_2^* + \mu_2 = \mu_3^* + \mu_1$ be the purity weight of μ. Then

$$\mu^\iota = (\mu_1, \mu_2, \mu_3), \text{ and } \mu^{\bar\iota} = (\mathsf{w} - \mu_3, \mathsf{w} - \mu_2, \mathsf{w} - \mu_1).$$

Reduction of μ: Let χ be an algebraic Hecke character such that $\chi_\infty(z) = z^m \bar{z}^n$ for some $m, n \in \mathbb{Z}$. Before we proceed, we need to prove the existence of character χ. We have the following lemma:

Lemma 3.10 *For a fixed $v \in S_c$ and given integers m and n. There exist an algebraic Hecke character χ such that $\chi_\infty(z) = z^m \bar{z}^n$.*

Proof. Take $f := n - m$ and $\sigma := \frac{m+n}{2}$. From Section 3.2, given $f \in \mathbb{Z}$ and $\sigma \in \frac{1}{2}\mathbb{Z}$, there exists a character of type(A_0) belonging to f which has following shape:

$$\chi = \chi^1 \parallel . \parallel^\sigma,$$

such that $f \equiv 2\sigma \pmod 2$ and $\chi^1 : F^\times \setminus \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ with $\chi^1_\infty(z) = \left(\frac{z}{|z|}\right)^{-f}$. Then for each $v \in S_c$, we have

$$\begin{aligned}\chi_v(z) &= (z/|z|)^{-f}|z|^{2\sigma}\\ &= z^{-f}|z|^{f+2\sigma}\\ &= z^{-f} z^{\frac{f+2\sigma}{2}} \bar{z}^{\frac{f+2\sigma}{2}}\\ &= z^{-f/2+\sigma}\bar{z}^{f/2+\sigma}.\end{aligned}$$

Clearly $m = -\frac{f}{2} + \sigma$, and $n = \frac{f}{2} + \sigma$. Then $\chi_\infty(z) = z^m \bar{z}^n$. Hence χ is the required character. □

Now using [29, Proposition 2.14],

$$\Pi_\infty = \mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}_3(\mathbb{C})}(z^{a_1}\bar{z}^{b_1} \otimes z^{a_2}\bar{z}^{b_2} \otimes z^{a_3}\bar{z}^{b_3}),$$

where $(a_1, a_2, a_3) = (\mu_1 + 1, \mu_2, \mu_3 - 1)$; and $(b_1, b_2, b_3) = (\mu_3^* - 1, \mu_2^*, \mu_1^* + 1)$. Then

$$L_\infty(s, \Pi_\infty \otimes \chi_\infty) \approx \prod_{i=1}^{3} \Gamma\left(s + \frac{a_i + b_i + m + n}{2} + \frac{|a_i - b_i + m - n|}{2}\right).$$

If $m = n$,

$$\begin{aligned}L_\infty(s, \Pi_\infty \otimes \chi_\infty) &\approx \prod_{i=1}^{3} \Gamma\left(s + \frac{a_i + b_i + 2m}{2} + \frac{|a_i - b_i|}{2}\right)\\ &\approx \prod_{i=1}^{3} \Gamma\left(s + m + \frac{a_i + b_i}{2} + \frac{|a_i - b_i|}{2}\right)\\ &= L_\infty(s + m, \Pi_\infty).\end{aligned} \tag{3.11}$$

Thus for the game to work we need to impose the following hypothesis: Consider $\Pi \in \mathrm{Coh}(G_3, \mu)$ with μ in $X_0^+(T_{3,\infty})$ such that for each $v \in S_c$, $\mu_2 = \mu_2^*$. It is clear then that the representation Π under a Tate twist,

$$\Pi \otimes \| \cdot \|^{\mu_2} \in \mathrm{Coh}(G_3, \mu - \mu_2)$$

because

$$H^\bullet(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; V_{\Pi_\infty} \otimes \| \cdot \|_\infty^{\mu_2} \otimes \mathcal{M}_\mu \otimes \det{}^{-\mu_2} \otimes \bar{\det}{}^{-\mu_2})$$
$$= H^\bullet(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; V_{\Pi_\infty} \otimes \mathcal{M}_\mu).$$

The purity weight of $\mu - \mu_2$ is 0, and furthermore $\Pi \otimes \| \cdot \|^{\mu_2}$ is a unitary cuspidal representation. As far as L-functions (and their special values) are concerned, we have not lost any information since

$$L(s, \Pi \otimes \| \ \|^{\mu_2}) \ = \ L(s + \mu_2, \Pi).$$

Twisted Representation. Consider ${}^{\mathrm{t}}\Pi = \Pi \otimes \| \ \|^{\mu_2}$ be a Tate twist of Π along with the condition that for each complex place v assume $\mu_2 = \mu_2^*$. Put ${}^{\mathrm{t}}\mu = \mu - \mu_2$. This gives us ${}^{\mathrm{t}}\Pi \in \mathrm{Coh}(G_3, {}^{\mathrm{t}}\mu)$, where ${}^{\mathrm{t}}\mu_v = (\mu^\iota, \mu^{\bar{\iota}})$, such that

$$\mu^\iota = (\mu_1 - \mu_2,\ \mu_2 - \mu_2,\ \mu_3 - \mu_2),\ \text{and}\ \mu^{\bar{\iota}} = (\mu_1^* - \mu_2^*,\ \mu_2^* - \mu_2^*,\ \mu_3^* - \mu_2^*).$$

Then for each $v \in S_c$ we have

$${}^{\mathrm{t}}\mu = (n_1,\ 0,\ n_2;\ -n_2,\ 0,\ -n_1), \tag{3.12}$$

with $\mu_1 - \mu_2 = n_1 \in \mathbb{Z}^+$ and $\mu_3 - \mu_2 = n_2 \in \mathbb{Z}^-$, where $\mathbb{Z}^+ = \{n \in \mathbb{Z}\ :\ n \geq 0\}$ and $\mathbb{Z}^- = \{n \in \mathbb{Z}\ :\ n \leq 0\}$.
Henceforth we will assume:

1. μ is a pure dominant integral weight with purity weight 0; so $\mu = (\mu_v)_{v \in S_\infty}$ with $\mu_v = (\mu^{\iota_v}; \mu^{\bar{\iota}_v}) = (n_{1v}, 0, n_{2v};\ -n_{2v}, 0, -n_{1v})$ for a non-negative integer n_{1v} and a non-positive integer n_{2v}. Furthermore, assume μ to be a parallel weight; so there exists

$$\mu_0 = (\mu_0^\iota; \mu_0^{\bar{\iota}}) = (n_1, 0, n_2; -n_2, 0, -n_1)$$

such that for each $v \in S_\infty$, $\mu_v = \mu_0$.

2. $\Pi \in \mathrm{Coh}(G_3, \mu)$, i.e., Π is a unitary cuspidal automorphic representation of GL_3/F that has nontrivial cohomology with respect to $\mathcal{M}_\mu$,

The following well-known proposition records some basic information about the relative Lie algebra cohomology groups in this context.

Proposition 3.13 *Let $\mu \in X_0^+(T_3)$ be a pure dominant integral 'parallel' weight with purity weight 0; we write $\mu = (\mu_v)_{v \in S_\infty}$ with*

$$\mu_v = (n_1,\ 0,\ n_2; -n_2,\ 0,\ -n_1)$$

for integers $n_1 \geq 0 \geq n_2$. Define the cuspidal parameters as:

$$a = (a_1, a_2, a_3) := (n_1 + 1,\ 0,\ n_2 - 1)$$

$$b = (b_1, b_2, b_3) := (-n_1 - 1,\ 0,\ -n_2 + 1).$$

Suppose $\Pi \in \mathrm{Coh}(G_3, \mu)$. Then for every $v \in S_c$ and corresponding to parameters a, b we have

$$\Pi_v := \mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}_3(\mathbb{C})}(z^{a_1}\bar{z}^{b_1} \otimes z^{a_2}\bar{z}^{b_2} \otimes z^{a_3}\bar{z}^{b_3}),$$

where for integers a, b, $z^a\bar{z}^b$ is the character of $\mathbb{C}^\times$ which sends z to $z^a\bar{z}^b$. It is well known that Π_v is irreducible, essential tempered, and generic.
Define $b_3^F = 3d_0$. The smallest degree $\bullet$ for which

$$H^\bullet(\mathfrak{g}_{3,\infty}, K_{3,\infty}^0; \Pi_\infty \otimes \mathcal{M}_\mu) \neq 0 \text{ is } \bullet = b_3^F = 3d_0,$$

and in this degree, the cohomology group is one-dimensional, and since $K_{3,\infty} = K_{3,\infty}^0$ is connected, $\pi_0(K_{3,\infty})$ is trivial, and hence acts trivially on this cohomology group.

3.4 Eisentein Cohomology for GL_2

For $i = 1, 2$, let $\chi_i : F^\times \setminus \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ be algebraic Hecke characters. Then according to Section 3.2, χ_i's have the following shape:

• If F is a totally real field, then

$$\chi_i = \| \, . \, \|^{d_i} \chi_i^0,$$

where χ_1^0, and χ_2^0 are of finite order and $d_1, d_2 \in \mathbb{Z}$. Clearly for every archimedean place v, χ_i^0 is a quadratic character of $\mathbb{R}^\times$. Thus we have $\chi_{iv}^0 = (\mathrm{sgn})^{e_{iv}}$, for $e_{iv} \in \{0, 1\}$.

• If F is a CM field, then

$$\chi_i = \| \, . \, \|^{\sigma_i} \chi_i^1,$$

for some $\sigma_i \in \frac{1}{2}\mathbb{Z}$ and χ_i^1 are the characters of parallel weight such that at each infinite place $v \in S_c$,

$$\chi_{i,v}^1(z) = (z/|z|)^{-f_i},$$

with $f_i \in \mathbb{Z}$ and $f_i \equiv 2\sigma_i \pmod 2$.

Define the globally induced representation

$$\Sigma(\chi_1, \chi_2) := \mathrm{Ind}_{B_2(\mathbb{A})}^{G_2(\mathbb{A})}(\chi_1 \parallel \, \parallel^{1/2}, \chi_2 \parallel \, \parallel^{-1/2}),$$

which decomposes into a restricted tensor product $\Sigma(\chi_1, \chi_2) = \otimes_v' \Sigma(\chi_{1,v}, \chi_{2,v})$, where $\Sigma(\chi_{1,v}, \chi_{2,v})$ denotes the normalized parabolically induced representation $\mathrm{Ind}_{B_2(F_v)}^{\mathrm{GL}_2(F_v)}(\chi_{1,v}| \, |_v^{1/2}, \chi_{2,v}| \, |_v^{-1/2})$ of $\mathrm{GL}_2(F_v)$. Let

$$\Sigma_f(\chi_1, \chi_2) := \otimes_{v \nmid \infty} \Sigma(\chi_{1,v}, \chi_{2,v})$$

and

$$\Sigma_\infty(\chi_1, \chi_2) := \otimes_{v | \infty} \Sigma(\chi_{1,v}, \chi_{2,v})$$

denote the finite and infinite part of $\Sigma(\chi_1, \chi_2)$, respectively. For simplicity of notations take $\Sigma(\chi_{1,v}, \chi_{2,v}) =: V_{\chi_v}$.

3.4.1 Choice of λ for non-zero cohomology

Let $\mathcal{M}_\lambda$ be a finite dimensional representation of $G_{2,\infty}$, with highest integral dominant weight $\lambda = (\lambda_v)_{v\in S_\infty} \in X_0^+(T_2)$ which decomposes as $\mathcal{M}_\lambda = \otimes_{v|\infty}\mathcal{M}_{\lambda_v}$ where $\mathcal{M}_{\lambda_v}$ is the finite-dimensional irreducible representation of $\mathrm{GL}_2(F_v)$ with highest weight λ_v. In this subsection for each fix $v \in S_\infty$, we want to find λ_v in terms of parameters appear in the characters χ_1 and χ_2 such that

$$H^\bullet(\mathfrak{g}_{2,v}, K^0_{2,v}; V_{\chi_v} \otimes \mathcal{M}_{\lambda_v}) \neq 0,$$

for some $\bullet \in \mathbb{Z}$. We handle real and complex case separately:

Case 1. *If v is a real place* (The basic reference here is Harder [16]): If we write $\lambda_v = (\lambda_{v,1}, \lambda_{v,2})$, with integers $\lambda_{v,j}$ and $\lambda_{v,1} \geq \lambda_{v,2}$, then $\mathcal{M}_{\lambda_v} = \mathrm{Sym}^{\lambda_{v,1}-\lambda_{v,2}}(\mathbb{C}^2) \otimes \det^{\lambda_{v,2}}$. Hence, the dimension of $\mathcal{M}_{\lambda_v}$ is $\lambda_{v,1} - \lambda_{v,2} + 1$, and its central character is $t \mapsto t^{\lambda_{v,1}+\lambda_{v,2}}$. We want to find $(\lambda_{v,1}, \lambda_{v,2})$ in terms of $d_1, d_2, \epsilon_{1v}, \epsilon_{2v}$ such that

$$H^\bullet(\mathfrak{g}_{2,v}, K^0_{2,v}; V_{\chi_v} \otimes \mathcal{M}_{\lambda_v}) \neq 0.$$

This is well-known, and we follow [16], however we need to transcribe his notation in terms of our notation. Take $d \in \mathbb{N}$ and $\nu \in \mathbb{Z}$, and define the finite-dimensional representation $M(d,\nu)$ acting on the space of homogeneous polynomials of degree d in two variables X, Y given by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\right)(X,\ Y) = P(aX + cY,\ bX + dY)(ad - bc)^\nu.$$

The dimension of $M(d,\ \nu)$ is $d{+}1$, and it's central character is $t \mapsto t^{d+2\nu}$. For $m \in \mathbb{Z}$, let ξ_m denote the character of $\mathbb{R}^\times$ that sends t to t^m. Furthermore, for a pair of characters $\chi_i(\mathrm{i} = 1, 2)$ of $\mathbb{R}^\times$, let

$${}^{\mathrm{a}}\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$$

denote the algebraic (un-normalized) parabolic induction of the character $\chi_1 \otimes \chi_2$ of the Borel subgroup to $\mathrm{GL}_2(\mathbb{R})$. Then we have (see [16, page 69])

$$H^q(\mathfrak{gl}_2(\mathbb{R}),\ \mathrm{SO}(2);\ {}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{-\nu-d}\otimes\xi_{-\nu})\otimes M(d,\ \nu)) = \begin{cases} \mathbb{C} & \text{if } q=\{0,\ 1\} \\ 0 & \text{otherwise.}\end{cases}$$

and similarly we have

$$H^q(\mathfrak{gl}_2(\mathbb{R}),\ \mathrm{SO}(2);\ {}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{1-\nu}\otimes\xi_{-d-\nu-1})\otimes M(d,\ \nu))$$
$$= \begin{cases} \mathbb{C} & \text{if } q=\{1,\ 2\} \\ 0 & \text{otherwise.}\end{cases}$$

Since the central characters of ${}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{-\nu-d}\otimes\xi_{-\nu})\otimes M(d,\ \nu)$ and ${}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{1-\nu}\otimes\xi_{-d-\nu-1})\otimes M(d,\ \nu)$ are trivial (a necessary condition for non-vanishing of cohomology dictated by Wigner's Lemma), we also have $H^q(\mathfrak{gl}_2(\mathbb{R}),\ \mathrm{SO}(2);-) = H^q(\mathfrak{gl}_2(\mathbb{R}),\ \mathrm{SO}(2)\mathbb{R}^\times;-)$ for the modules above. Comparing $M(d,\ \nu)$ with $\mathcal{M}_{\lambda_v}$ we get:

$$d = \lambda_{v,1}-\lambda_{v,2},\ \nu = \lambda_{v,2}.$$

Keeping later application in mind, we would like to arrange for V_{χ_v} to be ${}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{1-\nu}\otimes\xi_{-d-\nu-1}) = {}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{1-\lambda_{v,2}}\otimes\xi_{-1-\lambda_{v,1}})$ to get nontrivial cohomology for $q=1,\ 2$. Using definitions, we get

$$V_{\chi_v} := \mathrm{Ind}_{B_2(F_v)}^{\mathrm{GL}_2(F_v)}(\chi_{1,v}|\ |_v^{1/2},\ \chi_{2,v}|\ |_v^{-1/2}) = {}^{\mathrm{a}}\mathrm{Ind}_B^G(\chi_{1,v}|\ |_v,\ \chi_{2,v}|\ |_v^{-1})\ .$$

Therefore, we would like to have:

$$|t_1|^{d_1+1}|t_2|^{d_2-1}(\mathrm{sgn}(t_1))^{\epsilon_{1v}}(\mathrm{sgn}(t_2))^{\epsilon_{2v}} = t_1^{1-\lambda_{v,2}}t_2^{-1-\lambda_{v,1}},$$

which may be written as

$$|t_1|^{d_1+1}|t_2|^{d_2-1}(\mathrm{sgn}(t_1))^{\epsilon_{1v}}(\mathrm{sgn}(t_2))^{\epsilon_{2v}}$$
$$= |t_1|^{1-\lambda_{v,2}}(\mathrm{sgn}(t_1))^{1-\lambda_{v,2}}|t_2|^{-1-\lambda_{v,1}}(\mathrm{sgn}(t_2))^{-1-\lambda_{v,1}}.$$

On comparing both sides we get:

$$\lambda_{v,1} = -d_2,\ \lambda_{v,1} \equiv \epsilon_{2v} + 1 \pmod 2$$

$$\lambda_{v,2} = -d_1,\ \lambda_{v,2} \equiv \epsilon_{1v} + 1 \pmod 2\ . \tag{3.14}$$

Case 2. If v is a complex place (See Harder [16] for reference):
If we write $\lambda_v = (\lambda_v^\iota, \lambda_v^{\bar\iota})$ with $\lambda_v^\iota = (\lambda_{v,1}, \lambda_{v,2})$, and $\lambda_v^{\bar\iota} = (\lambda_{v,1}^*, \lambda_{v,2}^*)$ where $\lambda_{v,j}$, and $\lambda_{v,j}^*$ are integers such that $\lambda_{v,1} \geq \lambda_{v,2}$ and $\lambda_{v,1}^* \geq \lambda_{v,2}^*$. Then

$$\begin{aligned}\mathcal{M}_{\lambda_v} &= \mathcal{M}^\iota_{\lambda_v} \otimes \mathcal{M}^{\bar\iota}_{\lambda_v} \\ &= \mathrm{Sym}^{\lambda_{v,1}-\lambda_{v,2}}(\mathbb{C}^2) \otimes \det{}^{\lambda_{v,2}} \otimes \overline{\mathrm{Sym}}^{\lambda^*_{v,1}-\lambda^*_{v,2}}(\mathbb{C}^2) \otimes \overline{\det}^{\lambda^*_{v,2}}.\end{aligned}$$

We want to find $(\lambda_{v,1}, \lambda_{v,2})$ and $(\lambda^*_{v,1}, \lambda^*_{v,2})$ in terms of $f_1, f_2, \sigma_1, \sigma_2$ such that

$$H^\bullet(\mathfrak{g}_{2,v}, K^0_{2,v}; V_{\chi_v} \otimes \mathcal{M}_{\lambda_v}) \neq 0.$$

Again we need to transcribe the notation of [16] in terms of our notation. Take $d \in \mathbb{N}$ and $\nu \in \mathbb{Z}$, and define the finite-dimensional representation $M(d,\nu) = M(d_1,\nu_1) \otimes M(d_2,\nu_2)$ where each $M(d_i,\nu_i)$ is the finite dimensional representation of $\mathrm{GL}_2(\mathbb{R})$ acting on the space of homogeneous polynomials of degree d_i in two variables X, Y given by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\right)(X,\ Y) = P(aX + cY,\ bX + dY)(ad - bc)^{\nu_i}$$

The dimension of $M(d_i,\nu_i)$ is $d_i + 1$, and its central character is $t \mapsto t^{d_i+2\nu_i}$. For $m, n \in \mathbb{Z}$, let $\xi_{(m,n)}$ denote the character of $\mathbb{C}^\times$ that sends z to $z^m \bar{z}^n$. Furthermore, for a pair of characters χ_i (i $= 1, 2$) of $\mathbb{C}^\times$, let ${}^{\mathrm{a}}\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ denote the algebraic (un-normalized) parabolic induction of the character $\chi_1 \otimes \chi_2$ of the Borel subgroup $B(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})$. Then we have (see [16, page 74])

$$H^q(\mathfrak{gl}_2(\mathbb{C}),\ \mathrm{U}(2)\mathbb{C}^\times;\ {}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{(1-\nu_1,-d_2-\nu_2)} \otimes \xi_{(-d_1-1-\nu_1,-\nu_2)}) \otimes M(d,\ \nu)) = \begin{cases} \mathbb{C} & \text{if } q \in \{0,\ 1\}, \\ 0 & \text{otherwise,} \end{cases}$$

if $d_1 = d_2$. (The cases that will be relevant to us later on, based on the choices in Proposition 4.20, indeed satisfy the condition $d_1 = d_2$.) Comparing $M(d,\ \nu)$ with $\mathcal{M}_{\lambda_v}$ we get:

$$d_1 = \lambda_{v,1} - \lambda_{v,2},\ \nu_1 = \lambda_{v,2},$$

and

$$d_2 = \lambda^*_{v,1} - \lambda^*_{v,2},\ \nu_2 = \lambda^*_{v,2}. \tag{3.15}$$

Keeping this in mind, we would like to arrange for V_{χ_v} to be ${}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{(1-\nu_1,-d_2-\nu_2)} \otimes \xi_{(-d_1-1-\nu_1,-v_2)})$

$$= {}^{\mathrm{a}}\mathrm{Ind}_B^G(\xi_{(1-\lambda_{v,2},-\lambda^*_{v,1})} \otimes \xi_{(-\lambda_{v,1}-1,-\lambda^*_{v,2})})$$

to get nontrivial cohomology for $q = 1,\ 2$. Using definitions, we get

$$V_{\chi_v} := \mathrm{Ind}_{B_2(F_v)}^{\mathrm{GL}_2(F_v)}(\chi_{1,v}|\ |^{1/2}_{v,\mathbb{C}},\ \chi_{2,v}|\ |^{-1/2}_{v,\mathbb{C}}) = {}^{\mathrm{a}}\mathrm{Ind}_{B(F_v)}^{\mathrm{GL}_2(F_v)}(\chi_{1,v}|\ |_{v,\mathbb{C}},\ \chi_{2,v}|\ |^{-1}_{v,\mathbb{C}})\,.$$

Therefore, we would like to have:

$$\left(\frac{t_1}{|t_1|}\right)^{-f_1}|t_1|^{\sigma_1+2}\left(\frac{t_2}{|t_2|}\right)^{-f_2}|t_2|^{\sigma_2-2} = t_1^{1-\lambda_{v,2}}\bar{t}_1^{-\lambda^*_{v,1}}t_2^{-\lambda_{v,1}-1}\bar{t}_2^{-\lambda^*_{v,2}},$$

which may be written as

$$t_1^{\frac{-f_1+2\sigma_1}{2}+1}\bar{t}_1^{\frac{f_1+2\sigma_1}{2}+1}t_2^{\frac{-f_2+2\sigma_2}{2}-1}\bar{t}_2^{\frac{f_2+2\sigma_2}{2}-1} = t_1^{1-\lambda_{v,2}}\bar{t}_1^{-\lambda^*_{v,1}}t_2^{-\lambda_{v,1}-1}\bar{t}_2^{-\lambda^*_{v,2}}.$$

On comparing both sides we get:

$$\lambda_{v,1} = \left(\frac{f_2}{2} - \sigma_2\right),\ \lambda_{v,2} = \left(\frac{f_1}{2} - \sigma_1\right);$$

and

$$\lambda^*_{v,1} = -\left(\frac{f_1}{2} + \sigma_1\ + 1\right),\ \lambda^*_{v,2} = \left(1 - \frac{f_2}{2} - \sigma_2\right). \tag{3.16}$$

Summarizing both cases, we have the following lemma:

Lemma 3.17 *For $i = 1, 2$, let χ_i be algebraic Hecke characters of F. Let $\lambda \in X^+(T_{2,\infty})$ be a dominant integral parallel weight written as $\lambda = (\lambda_v)_{v \in S_\infty}$. Suppose that λ satisfies Equation (3.14) if F is totally real, and satisfies Equation (3.16) if F is CM field. Then*

$$H^q(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1, \chi_2) \otimes \mathcal{M}_\lambda) \neq 0 \Longleftrightarrow q = 1 \text{ or } 2.$$

This lemma leads to the following proposition:

Proposition 3.18 *Assume the following hypothesis:*

- **For totally real field:** *Let $\chi_i = \|\ .\ \|^{d_i}\ \chi_i^\circ$ be algebraic Hecke characters of F with $d_i \in \mathbb{Z}$ and χ_i° finite-order character. Suppose that $d_1 \geq d_2$, and for $v \in S_\infty$ suppose also that $\chi_{iv}^\circ = (\mathrm{sgn})^{e_{iv}}$ for $e_{iv} \in \{0, 1\}$ such that $e_{iv} \not\equiv d_i \pmod 2$. Let $\lambda \in X_0^+(T_2)$ be the dominant integral 'parallel' weight determined by d_1, d_2 as: $\lambda = (\lambda_v)_{v \in S_\infty}$, where each $\lambda_v = (-d_2, -d_1)$. Define $b_2^F = d_F$.*
- **For CM field:** *Let $\chi_i = \|\ .\ \|^{\sigma_i}\ \chi_i^1$ be algebraic Hecke characters of F with $\sigma_i \in \frac{1}{2}\mathbb{Z}$ and χ_i^1 are the characters such that $\chi_{i\infty}^1(z_\infty) = (z_\infty/|z_\infty|)^{-f_i}$ for some $f_i \in \mathbb{Z}$ with the property $f_i \equiv 2\sigma_i \pmod 2$. Further let $\lambda \in X^+(T_2)$ be the dominant integral 'parallel' weight determined by $f_1, f_2, \sigma_1, \sigma_2$ as: $\lambda = (\lambda_v)_{v \in S_\infty}$, where each $\lambda_v = (\lambda_v^\iota, \lambda_v^{\bar\iota})$; with*

$$\lambda_v^\iota = \left(\tfrac{f_2}{2} - \sigma_2, \tfrac{f_1}{2} - \sigma_1\right) \text{ and } \lambda_v^{\bar\iota} = \left(-\left(\tfrac{f_1}{2} + \sigma_1 + 1\right), 1 - \tfrac{f_2}{2} - \sigma_2\right).$$

Define $b_2^F = d_0$.
Then

$$H^\bullet(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1, \chi_2) \otimes \mathcal{M}_\lambda) \neq 0 \iff b_2^F \leq \bullet \leq 2b_2^F.$$

Furthermore, in the extremal degrees of b_2^F and $2b_2^F$, the cohomology group is one-dimensional.

Proof. The proof follows from Künneth formula for relative Lie algebra cohomology (and the details recalled above)

$$H^\bullet(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1,\chi_2)\otimes\mathcal{M}_\lambda)(\epsilon_\Pi)$$
$$= \bigoplus_{\Sigma a_v=\bullet} (\bigotimes_{v\in S_r} H^{a_v}(\mathfrak{g}_{2,v}, K^0_{2,v}; \Sigma(\chi_{1v},\chi_{2v})\otimes\mathcal{M}_{\lambda_v})(\epsilon_v)$$
$$\bigotimes_{v\in S_c} H^{a_v}(\mathfrak{g}_{2,v}, K^0_{2,v}; \Sigma(\chi_{1v},\chi_{2v})\otimes\mathcal{M}_{\lambda_v})(1\!\!1)).$$

□

3.4.2 Action of $K_{2,\infty}/K^0_{2,\infty}$ on group cohomology

We need to determine the action of $\pi_0(K_{2,\infty})$ on Lie algebra cohomology groups $H^\bullet(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1,\chi_2)\otimes\mathcal{M}_\lambda)$.

Case 1. *F is totally real field.* Let's recall some standard convention: For a representation Π of GL_n (in any suitable local or global context) and for a real number t, we denote $\Pi\otimes|.|^t$ by $\Pi(t)$. Also, we will abbreviate the normalized parabolically induced representation $\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{R})}_{B_2(\mathbb{R})}(\xi_{m_1}\otimes\xi_{m_2})$ simply as $\xi_{m_1}\times\xi_{m_2}$. Observe that, for $v\in S_\infty$ we have

$$\begin{aligned}\Sigma_v(\chi_1,\chi_2) &= \chi_{1v}(1/2)\times\chi_{2v}(-1/2)\\ &= |\cdot|^{d_1}\,\mathrm{sgn}^{e_{1v}}(1/2)\times|\cdot|^{d_2}\,\mathrm{sgn}^{e_{2v}}(-1/2)\\ &= \xi_{d_1}(\mathrm{sgn})(1/2)\times\xi_{d_2}(\mathrm{sgn})(-1/2)\\ &= \xi_{-\lambda_{v,2}}(\mathrm{sgn})(1/2)\times\xi_{-\lambda_{v,1}}(\mathrm{sgn})(-1/2).\end{aligned}$$

Now, we need to determine the action of $\pi_0(K_{2,\infty})$ on the cohomology group $H^1(\mathfrak{g}_{2,\infty}, K^0_{2,\infty};\ \Sigma_\infty(\chi_1,\chi_2)\otimes\mathcal{M}_\lambda)$. To this end, take two integers a and b with $a\geq b$. Consider the following exact sequence of $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2)\mathbb{R}^\times_+)$-modules:

$$0\longrightarrow D_{a-b+1}\left(\frac{a+b}{2}\right)\longrightarrow\xi_a(1/2)\times\xi_b(-1/2)\longrightarrow\mathcal{M}_{a,b}\longrightarrow 0.$$

Twisting by the sgn character, while noting that twisting commutes with induction, we get

$$0 \longrightarrow D_{a-b+1}\left(\frac{a+b}{2}\right) \longrightarrow \xi_a(\mathrm{sgn})(1/2) \times \xi_b(\mathrm{sgn})(-1/2) \\ \longrightarrow \mathcal{M}_{a,b} \otimes \mathrm{sgn} \longrightarrow 0.$$

Note that the discrete series representation D_l is invariant under twisting by sgn character. For brevity, let $\nu := (a,\ b)$; $\nu^\vee = (-b,\ -a)$;

$$V_\nu := \xi_a(\mathrm{sgn})(1/2) \times \xi_b(\mathrm{sgn})(-1/2),$$
$$D_{\nu^\vee} = D_{a-b+1}\left(\frac{a+b}{2}\right),$$

and

$$\mathcal{M}_\nu^- := \mathcal{M}_\nu \otimes \mathrm{sgn}\,.$$

The above exact sequence may then be written as

$$0 \longrightarrow D_{\nu^\vee} \xrightarrow{\ i\ } V_\nu \longrightarrow \mathcal{M}_\nu^- \longrightarrow 0\,.$$

Tensor this sequence by $\mathcal{M}_{\nu^\vee} = \mathcal{M}_\nu^\vee$, and apply $H^\bullet(\mathfrak{gl}_2,\ \mathrm{SO}(2)\mathbb{R}_+^\times; -)$ to get the following long exact sequence:

$$0 \to H^0(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to H^0(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \to H^0(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) \to$$
$$\to H^1(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \to H^1(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) \to$$
$$\to H^2(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to H^2(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \to H^2(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) \to$$
$$H^3(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to \cdots$$

Now, we make precise all the above cohomology groups as O(2)/SO(2)-modules. Recall $\mathbb{1}$ stand for the trivial character, and let sgn the sign-character of O(2)/SO(2) . For the finite-dimensional modules $\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}$, first of all, since $H^0 = \mathrm{Hom}$, we easily see that

$$H^0(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) = \mathrm{sgn}\,.$$

Next, it follows from [42, Propsition I.4], that

$$H^2(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) = \mathbb{1},$$

and furthermore, one may see that $H^q(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) = 0$ for $q \notin \{0, 2\}$. For the discrete series representation, it is well-known that

$H^1(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) = \mathbb{1} \oplus \mathrm{sgn}$, and $H^q(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) = 0$, if $q \neq 1$.

Also, since V_ν doesn't contain a finite-dimensional sub-representation we deduce $H^0(V_\nu \otimes \mathcal{M}_{\nu^\vee}) = 0$, whence, $H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee})$ sits in the short exact sequence:

$$0 \to H^0(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) \to H^1(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee}) \to H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \to 0.$$

Hence, as an $\mathrm{O}(2)/\mathrm{SO}(2)$-module we get

$$H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee}) = \mathbb{1}.$$

Furthermore, if $[D_{\nu^\vee}]^+$ denotes an eigenvector in $H^1(D_{\nu^\vee} \otimes \mathcal{M}_{\nu^\vee})$ for the trivial action of $\mathrm{O}(2)$, then we may take it's image under $i^\bullet$ (the map induced by the inclusion i in cohomology) as a generator $[V_\nu]$ for $H^1(V_\nu \otimes \mathcal{M}_{\nu^\vee})$, i.e.,

$$i^\bullet[D_{\nu^\vee}]^+ = [V_\nu]. \tag{3.19}$$

To complete the picture, since the dimension of the symmetric space is 2, we have $H^q = 0$ for all $q \geq 3$, and that

$$H^2(V_\nu \otimes \mathcal{M}_{\nu^\vee}) \cong H^2(\mathcal{M}_\nu^- \otimes \mathcal{M}_{\nu^\vee}) = \mathbb{1}.$$

What we especially will want later is summarized in the following lemma:

Lemma 3.20 *For integers $a \geq b$, we have as an* $\mathrm{O}(2)/\mathrm{SO}(2)$*-module:*

$$H^1(\mathfrak{gl}_2(\mathbb{R}),\ \mathrm{SO}(2)\mathbb{R}_+^\times;\ (\xi_{-b}(\mathrm{sgn})(1/2) \times \xi_{-a}(\mathrm{sgn})(-1/2)) \otimes \mathcal{M}_{(a,b)}) = \mathbb{C}\mathbb{1}.$$

Case 2. *F is a CM field.* Since $K_{3,v} = \mathrm{U}(2)\mathbb{C}^\times$, it is connected. The group $\pi_0(K_{2,v})$ is trivial and hence it vacuously acts trivially on the cohomology group $H^1(\mathfrak{gl}_2(\mathbb{C}), \mathrm{U}(2)\mathbb{C}^\times; \Sigma(\chi_{1v}, \chi_{2v}) \otimes \mathcal{M}_{\lambda_v})$.

On combining both the cases we return to the global situation and use the above local details to get the following:

Proposition 3.21 *Under the hypothesis of Proposition 3.18 we have: The group $\pi_0(K_{2,\infty})$ acts trivially on $H^{b_2^F}(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1, \chi_2) \otimes \mathcal{M}_\lambda)$.*

Proof. Similar to Proposition 3.18, the proof follows from Künneth formula for relative Lie algebra cohomology and Lemma 3.20. □

3.4.3 Eisenstein cohomology classes corresponding to $\Sigma(\chi_1, \chi_2)$

This works exactly as in Mahnkopf [26, Section 1.1] with the additional book-keeping of having to work over a totally real field or a CM field and a general coefficient system offering no additional complications; so, we merely record the details for later use. To begin, fix a generator $[\Sigma(\chi_1, \chi_2)_\infty] = [\Sigma_\infty]$ of the one-dimensional

$$H^{b_2^F}(\mathfrak{g}_{2,\infty}, K^0_{2,\infty}; \Sigma_\infty(\chi_1, \chi_2) \otimes \mathcal{M}_\lambda) \;=\; \mathbb{C}[\Sigma_\infty].$$

Tensoring by $[\Sigma_\infty]$ and following it up by Eisenstein summation gives us a map:

$$\mathcal{F}_\Sigma \;:\; \Sigma(\chi_1, \chi_2)_f^{R_f} \;\longrightarrow\; H^{b_2^F}(S_2(R_f), \mathcal{M}_\lambda). \tag{3.22}$$

where R_f is any open-compact subgroup for which the R_f-invariants in $\Sigma(\chi_1, \chi_2)_f$, denoted as $\Sigma(\chi_1, \chi_2)_f^{R_f}$, is nonzero. (This is the map denoted as 'Eis' on [26, page 96].) Furthermore, the map $\mathcal{F}_\Sigma$ is $\mathrm{Aut}(\mathbb{C})$-equivariant.

Chapter 4

Analytic Theory of L-functions

4.1 Rankin-Selberg L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_2$

4.1.1 Definition of Whittaker Model

Let (Π, V_Π) be a smooth cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_F)$. Let $\phi \in V_\Pi$ be a smooth cusp form. For each continuous additive character ψ, we define a $\psi-$Fourier coefficient, or $\psi-$Whittaker function of ϕ by

$$w_\phi(g) \equiv w_{\phi,\psi}(g) := \int_{N_3(F)\backslash N_3(\mathbb{A}_F)} \phi(ng)\psi^{-1}(n)dn,$$

which satisfies $w_\phi(ng) = \psi(n)w_\phi(g)$, for all $n \in N_3(\mathbb{A}_F)$.
Define $\mathcal{W}(\Pi, \psi) = \{w_\phi | \phi \in V_\Pi\}$. The group $G_3(\mathbb{A})$ acts on this space by right translation and the map $\phi \longmapsto w_\phi$ intertwines the $G_3(\mathbb{A})$–action. The space $\mathcal{W}(\Pi, \psi)$ is called the *Whittaker model* of Π. We will be working with Whittaker models and without any ado we will freely use standard results. (See, for example, Bump [6, Chapters 3, 4] for reference.)

Theorem 4.1 (Global Whittaker Models). *If* $\Pi = \otimes' \Pi_v$ *is a cuspidal automorphic representation of* $\mathrm{GL}_3(\mathbb{A}_F)$. *Then*

(1) there exists a unique Whittaker model $\mathcal{W}(\Pi, \psi)$ *for* Π *with respect to a non-trivial additive character* ψ.

(2) **(Local Whittaker Models.)** *For any place v of F, representation Π_v of $\mathrm{GL}_3(F_v)$ has a unique space $\mathcal{W}(\Pi_v, \psi_v)$ of smooth functions such that for all $w \in \mathcal{W}(\Pi_v, \psi_v)$,*

1. $w(ng) = \psi_v(n)w(g), \quad \forall n \in N_3(F_v), g \in \mathrm{GL}_3(F_v)$,

2. $w(gk') = w(g)$ *for all* $k' \in K'$ *for some* $K' \subset \mathrm{GL}_3(\mathcal{O}_v)$.

(3) the global space decomposes as a restricted tensor product of local Whittaker models. That is, if $\phi = \otimes_v \phi_v$, where ϕ_v is a spherical vector for almost all v, then $w_\phi = \prod w_{\phi_v}$ with $w_{\phi_v}(k_v) = 1$ for almost all v, where $k_v = \mathrm{GL}_3(\mathcal{O}_v)$.

4.1.2 The global integral

We will apply the Rankin–Selberg theory of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_2$ to the pair (Π, Σ), where Π is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ and $\Sigma = \Sigma(\chi_1, \chi_2)$ the induced representation defined above. (See, for example, [10, Lecture 5].) Take a cusp form $\phi_\Pi \in V_\Pi$, and recall that a cusp form is a rapidly decreasing function. Let $\varphi_{\chi_1,\chi_2} \in \Sigma(\chi_1, \chi_2)$, and note that φ_{χ_1,χ_2} is a function on $B_2(\mathbb{Q})\backslash G_2(\mathbb{A})$. To ensure $G_2(\mathbb{Q})$-invariance we do an Eisenstein summation:

$$E(\varphi_{\chi_1,\chi_2}, g, s) \ := \sum_{\gamma \in B_2(F)\backslash \mathrm{GL}_2(F)} |\alpha|^s \varphi_{\chi_1,\chi_2}(\gamma g).$$

It's well known ([16, page 80]) that $E(\varphi_{\chi_1,\chi_2}, g, s)$ converges for $\Re(s) \gg 0$, and has an analytic continuation to an entire function of s if $\chi_1 \neq \chi_2$. (In all the cases that will be relevant to us later on, based on the choices in Propositions 4.19 and 4.20, we will indeed have $\chi_1 \neq \chi_2$.) Put

$$E(\varphi_{\chi_1,\chi_2})(g) \ := \ E(\varphi_{\chi_1,\chi_2}, g, 0).$$

Consider the global period integral:

$$I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) \ := \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \phi_\Pi(\iota(g)) E(\varphi_{\chi_1,\chi_2})(g) |\mathrm{det} g|^{s-1/2} dg. \quad (4.2)$$

This integral converges for all $s \in \mathbb{C}$ since a cusp form has rapid decay whereas an Eisenstein series slowly increases.

To see the Eulerian nature of the above period integral, we pass to the Whitakker models of the representations. Fix a nontrivial additive character $\psi : \mathbb{A}_F/F \to \mathbb{C}^\times$, and suppose that

$$w_\Pi \in \mathcal{W}(\Pi, \psi), \text{ and } w_E \in \mathcal{W}(\Sigma(\chi_1, \chi_2), \bar{\psi})$$

are global Whittaker vectors corresponding to ϕ_Π and $E(\varphi_{\chi_1,\chi_2})$, respectively. Then

$$\begin{aligned}
I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) &\overset{(\forall s \in \mathbb{C})}{=} \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \phi_\Pi(\iota(g)) E(\varphi_{\chi_1,\chi_2})(g) |\mathrm{det} g|^{s-\frac{1}{2}} dg, \\
&\overset{(\Re(s)\gg 0)}{=} \int_{N_2(\mathbb{A})\backslash G_2(\mathbb{A})} w_\Pi(\iota(g)) \cdot \\
&\qquad \left(\int_{N_2(F)\backslash N_2(\mathbb{A})} E(\varphi_{\chi_1,\chi_2})(ng)\psi(n) dn \right) |\mathrm{det} g|^{s-\frac{1}{2}} dg, \\
&\overset{(\Re(s)\gg 0)}{=} \int_{N_2(\mathbb{A})\backslash G_2(\mathbb{A})} w_\Pi(\iota(g)) w_E(g) |\mathrm{det} g|^{s-\frac{1}{2}} dg.
\end{aligned}$$

Now suppose ϕ_Π and φ_{χ_1,χ_2} are chosen so that w_Π and w_E are pure tensors, written as restricted tensors $w_\Pi = \otimes' w_{\Pi_v}$ and $w_E = \otimes' w_{E_v}$, then we have

$$\begin{aligned}
&\int_{N_2(\mathbb{A})\backslash G_2(\mathbb{A})} w_\Pi(\iota(g)) w_E(g) |\mathrm{det} g|^{s-\frac{1}{2}} dg \\
&\qquad = \prod_v \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w_{\Pi_v}(\iota(g_v)) w_{E_v}(g_v) |\mathrm{det} g_v|_v^{s-\frac{1}{2}} dg_v \\
&\qquad =: \textstyle\prod_v \Psi(s, w_{\Pi_v}, w_{E_v}).
\end{aligned}$$

We need to compute the local integrals $\Psi(s, w_{\Pi_v}, w_{E_v})$, especially at ramified places.

4.1.3 Choice of local Whittaker vectors for induced representations of GL_2

For $i = 1, 2$, let χ_i be algebraic Hecke characters of $F^\times \setminus \mathbb{A}_F^\times$ as defined in Section 3.2. Fix a place v of F. Let F_v be the completion of F at v, with the ring of integers $\mathcal{O}_v$ and $\mathfrak{p}_v$ is its maximal ideal. Let $q_v = \#\mathcal{O}_v/\mathfrak{p}_v$ be the cardinality of the residue field, and ϖ_v denote a fixed generator of $\mathfrak{p}_v$. The normalized valuation **val** on F_v has the property that $\mathbf{val}(\varpi_v) = 1$. For the normalized absolute value we have $|\varpi_v| = q_v^{-1}$. Recall $\Sigma_v := \Sigma(\chi_{1v}, \chi_{2v}) := \mathrm{Ind}_{B(F_v)}^{\mathrm{GL}_2(F_v)}(\chi_{1v}|\ |_v^{1/2}, \chi_{2v}|\ |_v^{-1/2})$ is the induced representation of $\mathrm{GL}_2(F_v)$ on the space

$$V(\chi_{1v}, \chi_{2v}) \ := \ \{f : \mathrm{GL}_2(F_v) \to \mathbb{C} \ \mid \ f\left(\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right)g\right) = |ab^{-1}|_v \chi_{1v}(a)\chi_{2v}(b)f(g)\}.$$

The action of $\mathrm{GL}_2(F_v)$ on $V(\chi_{1v}, \chi_{2v})$ is by right translations. Since it is known that $B(F_v)N_2^-(F_v)$ is dense in $\mathrm{GL}_2(F_v)$, any $f \in V(\chi_{1v}, \chi_{2v})$ is completely determined by its values on elements of the form $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$. So we get a model for $\Sigma(\chi_{1v}, \chi_{2v})$ obtained by restricting functions in $V(\chi_{1v}, \chi_{2v})$ to $N_2^-(F_v)$. We denote the space of functions on $N_2^-(F_v) \simeq F_v$ by $V(\chi_{1v}, \chi_{2v})^-$. We recall some well-known facts about 'new vectors' in induced representations (see [7] and [36, Proposition 2.1.2]):

Proposition 4.3 *Suppose the conductor of χ_{iv} is Cond(χ_{iv}) $= \mathfrak{f}_{\chi_{iv}} = \mathfrak{p}_v^{n_i}$, say. Then the conductor of $\Sigma(\chi_{1v}, \chi_{2v}) = \mathfrak{f}_{\Sigma_v} = \mathfrak{f}_{\chi_1}\mathfrak{f}_{\chi_2} = \mathfrak{p}_v^n$, with $n = n_1 + n_2$. For $m \geq 0$ we define*

$$K_{01}(\mathfrak{p}^m) \ := \ \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_v) : c \equiv 0, d \equiv 1(\mathfrak{p}^m)\},$$

with the understanding that $K_{01}(\mathfrak{p}^0) = \mathrm{GL}_2(\mathcal{O}_v)$. Then, the space of $K_{01}(\mathfrak{p}^n)$-invariant vectors in $\Sigma(\chi_{1v}, \chi_{2v})$ is one-dimensional, say $\mathbb{C}f_v^{\mathrm{new}}$. Moreover, this 'new-vector' as a function on $N^-(F_v)$ may be taken to be of the following shape:

- *If χ_{1v} and χ_{2v} are ramified, then*
$$f_v^{\mathrm{new}}\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) = \begin{cases} \chi_{1v}(x)^{-1}|x|^{-1/2}, & \textit{if } \mathbf{val}(x) = n_2, \\ 0, & \textit{if } \mathbf{val}(x) \neq n_2. \end{cases}$$

- *If* χ_{1v} *is unramified and* χ_{2v} *is ramified, then* $f_v^{\rm new}\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right) = \begin{cases}\chi_{1v}(\varpi_v)^{-n_2}|\varpi_v|^{-n_2/2}, & \textit{if } \mathbf{val}(x) \geq n_2,\\ 0, & \textit{if } \mathbf{val}(x) < n_2.\end{cases}$

- *If* χ_{1v} *is ramified and* χ_{2v} *is unramified, then* $f_v^{\rm new}\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right) = \begin{cases}\chi_{1v}(x)^{-1}\chi_{2v}(x)|x|^{-2}, & \textit{if } \mathbf{val}(x) \leq 0,\\ 0, & \textit{if } \mathbf{val}(x) > 0.\end{cases}$

- *If* χ_{1v} *and* χ_{2v} *are unramified, then we take* $f_v^{\rm new} = f_v^{\rm sp}$*, the spherical vector, i.e., the vector fixed by* $\mathrm{GL}_2(\mathcal{O}_v)$*, normalized such that* $f_v^{\rm sp}(k_v) = 1$ *for* $k_v \in \mathrm{GL}_2(\mathcal{O}_v)$*.*

Now we consider the new-vector $f_v^{\rm new}$ in the local Whittaker model. For the global additive character ψ, we will furthermore assume that the local ψ_v is unramified, i.e., the largest fractional ideal on which ψ_v is trivial is $\mathcal{O}_v$. For $f_v \in V(\chi_{1v}, \chi_{2v})$, the corresponding ψ_v^{-1}-Whittaker function is given by the integral:

$$w_{f_v}(g) \equiv w_{f_v,\psi_v^{-1}}(g) \;=\; \int_{N(F_v)} \psi_v(n) f_v(\mathrm{w}_\circ^{-1} ng)dn,$$

where $\mathrm{w}_\circ = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. The map $f_v \mapsto w_{f_v}$ identifies the local induced representation $\Sigma(\chi_{1v}, \chi_{2v})$ with its Whittaker model $\mathcal{W}(\Sigma(\chi_{1v}, \chi_{2v}), \psi_v^{-1})$. We have the following lemma for new vectors stated in terms of the Whittaker models:

Lemma 4.4 *The space of* $K_{01}(\mathfrak{p}^n)-$*invariant vectors in* $\mathcal{W}(\Sigma(\chi_{1v}, \chi_{2v}), \psi_v^{-1})$ *is one-dimensional, and we may take as generator* $w_{\Sigma_v}^{\rm new} := w_{f_v^{\rm new}}$*. Furthermore, there exists* $t^* = diag(t, 1)$ *such that* $w_{\Sigma_v}^{\rm new}(t^*) \neq 0$*, for some* $t \in F_v^\times$*.*

Proof. First part of the lemma follows from [36, Proposition 2.12] and second part follows from Kirillov theory (see, for example [12, Section 3]). □

We would like to take a convenient t^* and compute the value $w_{\Sigma_v}^{\rm new}(t^*)$. Towards this, to begin, suppose v is a finite unramified place, i.e., Σ_v admits a K_v-fixed vector which is unique up to scalars; take $w^\circ_{\Sigma_v}$ as the unique K_v-fixed vector such that $w^\circ_{\Sigma_v}(1) = 1$. On the other hand, $w_{f_v^{\mathrm{sp}}}$ is also a K_v-fixed vector, and so there exists a $C_v \in \mathbb{C}^\times$ such that $w_{f_v^{\mathrm{sp}}} = C_v w^\circ_{\Sigma_v}$; then $C_v = w_{f_v^{\mathrm{sp}}}(1)$. We have the well-known proposition ([8, Theorem 5.4], [38, Chapter 5, page 352]):

Proposition 4.5 *Suppose χ_{1v} and χ_{2v} are unramified characters, and f_v^{sp} is the spherical vector in the induced representation Σ_v, then*

$$C_v = w_{f_v^\circ}(1) = L(2, \chi_{1v}\chi_{2v}^{-1})^{-1}.$$

Let's note that in the usual Casselman-Shalika formula, one sees the value at $s = 1$ of a local L-function, but recall in our case that for $\Sigma(\chi_1, \chi_2)$ the inducing representation is $\chi_1(1/2) \times \chi_2(-1/2)$ which accounts for the L-value at $s = 2$, since $L(1, \chi_{1v}(1/2)(\chi_{2v}(-1/2))^{-1}) = L(2, \chi_{1v}\chi_{2v}^{-1})$.

Let S_{χ_i} be the set of finite places where χ_i is ramified; then put $S_\Sigma = S_{\chi_1} \cup S_{\chi_2}$. Let $v \in S_\Sigma$. Applying Lemma 4.4, we take for $w^{\mathrm{new}}_{\Sigma_v}$ the unique $K_{01}(\mathfrak{p}^n)$-fixed vector normalized such that $w^{\mathrm{new}}_{\Sigma_v}(t^*) = 1$. Since the space of new-vectors is one-dimensional, there exists $A_v \in \mathbb{C}^\times$ such that $w_{f_v^{\mathrm{new}}} = A_v w^{\mathrm{new}}_{\Sigma_v}$. Hence, $A_v = w_{f_v}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The precise value of A_v is given by the following

Proposition 4.6 *Let f_v^{new} be the new vector in the induced representation Σ_v as in Proposition 4.3. Then*

$$A_v \;=\; w_{f_v^{\mathrm{new}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) \;=\; \begin{cases} q_v^{-n_2/2}\, \chi_{2v}(\varpi^{-n_2})\, \mathcal{G}(\chi_{2v}), & \textit{if } \chi_{2v} \textit{ is ramified,} \\ \mathrm{Vol}(\mathcal{O}_v), & \textit{if } \chi_{2v} \textit{ is unramified.} \end{cases}$$

Proof. Before we begin, let's recall the following well-known fact about local Gauß sums: if the conductor of ψ_v is $\mathcal{O}_v$, then

$$\int_{\mathcal{O}_v^\times} \psi_v(a\varepsilon)\chi_v(\varepsilon) d^\times\varepsilon \;=\; \begin{cases} \chi_v^{-1}(a\varpi^e)\mathcal{G}(\chi_v), & \text{if } \mathbf{val}_v(a) = -e, \\ 0, & \text{otherwise,} \end{cases}$$

where $e = \mathrm{cond}(\chi_v)$. Thus, by the definition of the Whittaker function we have

$$\begin{aligned} w_{f_v^{\mathrm{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \int_{N(F_v)} f\left(\mathrm{w}_\circ^{-1} u \left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)\right) \psi_v(u) du \\ &= \int_{F_v^\times} f\left(\mathrm{w}_\circ^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \psi_v(x) dx \;=\; \int_{F_v^\times} f\left(\left(\begin{smallmatrix} 0 & -1 \\ t & x \end{smallmatrix}\right)\right) \psi_v(x) dx. \end{aligned}$$

Make the substitution $x \longmapsto tx$, and use $\left(\begin{smallmatrix} 0 & -1 \\ t & tx \end{smallmatrix}\right) = \left(\begin{smallmatrix} x^{-1} & -1 \\ 0 & tx \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)$ to rewrite the last integral as

$$\begin{aligned} |t|_v \int_{F_v^\times} & f\left(\left(\begin{smallmatrix} x^{-1} & -1 \\ 0 & tx \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\right) \psi_v(tx) dx \\ &= |t|_v \int_{F_v^\times} \chi_{1v}(x)^{-1}\chi_{2v}(tx)|x^{-2}t^{-1}|_v f\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\psi_v(tx)dx \\ &= \sum_{n\in\mathbb{Z}} \int_{\varpi_v^n \mathcal{O}_v^\times} \chi_{1v}(x)^{-1}\chi_{2v}(tx)|x^{-2}|_v f\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\psi_v(tx)dx. \end{aligned}$$

Now to compute $w_{f_v^{\mathrm{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$ using the very last expression, we consider three cases:

1. χ_{1v} **and** χ_{2v} **are both ramified.** In this case, by Proposition 4.3, we have $f\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right) = 0$ for all x such that $\mathbf{val}(x^{-1}) \neq n_2$. Hence only the summand for $n = -n_2$ survives to get:

$$\begin{aligned} w_{f_v^{\mathrm{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \int_{\varpi_v^{-n_2}\mathcal{O}_v^\times} |x|_v^{-2}\chi_{1v}(x)^{-1}\chi_{2v}(tx)\chi_{1v}(x)|x|_v^{\frac{1}{2}}\psi_v(tx)dx \\ &= \int_{\varpi_v^{-n_2}\mathcal{O}_v^\times} |x|_v^{\frac{-3}{2}}\chi_{2v}(tx)\psi_v(tx)dx \quad (\text{put } \; x = \varpi_v^{-n_2}y) \\ &= |\varpi_v^{-n_2}|_v^{\frac{-3}{2}} \int_{\mathcal{O}_v^\times} \chi_{2v}(t\varpi_v^{-n_2}y)\psi_v(t\varpi_v^{-n_2}y)|\varpi_v^{-n_2}|_v dy \\ &= |\varpi_v|_v^{\frac{n_2}{2}} \chi_{2v}(\varpi_v^{-n_2}) \int_{\mathcal{O}_v^\times} \chi_{2v}(ty)\psi_v(\varpi_v^{-n_2}ty)d^\times y. \end{aligned}$$

   Recall that on $\mathcal{O}_v^\times, dy = d^\times y$. Note that

$$\int_{\mathcal{O}_v^\times} \chi_{2v}(y)\psi_v(\varpi_v^{-n_2}ty)d^\times y \neq 0 \Leftrightarrow \mathrm{val}(\varpi_v^{-n_2}t) = -n_2 \Leftrightarrow t \in \mathcal{O}_v^\times.$$

Put $ty = z$ to get

$$\begin{aligned} w_{f_v^{\text{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= |\varpi_v|_v^{\frac{n_2}{2}} \chi_{2v}(\varpi_v^{-n_2}) \int_{\mathcal{O}_v^\times} \chi_{2v}(z)\psi_v(\varpi_v^{-n_2} z) d^\times z \\ &= q_v^{-n_2/2} \chi_{2v}(\varpi^{-n_2}) \mathcal{G}(\chi_{2v}). \end{aligned}$$

2. χ_{1v} **is unramified and** χ_{2v} **is ramified.** In this case, by Proposition 4.3, we have $f\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right) = 0$, for $n > -n_2$. Hence we get

   $w_{f_v^{\text{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$
   $= \sum_{n \le -n_2} \int_{\varpi_v^n \mathcal{O}_v^\times} |x|_v^{-2} \chi_{1v}(x)^{-1} \chi_{2v}(tx) \chi_{1v}(\varpi_v^{-n_2}) |\varpi_v|_v^{-n_2/2} \psi_v(tx) dx$
   (now put $x = \varpi_v^n y$);
   $= \sum_{n \le -n_2} |\varpi_v|_v^{-n-n_2/2} \chi_{1v}(\varpi_v^{-n-n_2}) \chi_{2v}(\varpi_v^n) \int_{\mathcal{O}_v^\times} \chi_{2v}(ty) \psi_v(t\varpi_v^n y) d^\times y.$

   For the inner integral we have:

$$\int_{\mathcal{O}_v^\times} \chi_{2v}(y) \psi_v(\varpi_v^n t y) d^\times y \neq 0 \Leftrightarrow \text{val}(\varpi_v^n t) = -n_2.$$

   Let's take $t \in \mathcal{O}_v^\times$, then only the summand for $n = -n_2$ will be nonzero, and we get:

$$\begin{aligned} w_{f_v^{\text{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= q_v^{-n_2/2} \chi_{2v}(\varpi_v^{-n_2}) \int_{\mathcal{O}_v^\times} \chi_{2v}(ty) \psi_v(t\varpi_v^{-n_2} y) d^\times y \\ &= q_v^{-n_2/2} \chi_{2v}(\varpi_v^{-n_2}) \mathcal{G}(\chi_{2v}). \end{aligned}$$

3. χ_{1v} **is ramified and** χ_{2v} **is unramified.** In this case, by Proposition 4.3, we have nonzero summands corresponding to $n \geq 0$:

$$\begin{aligned} w_{f_v^{\text{ess}}}\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) &= \sum_{n \ge 0} \int_{\varpi_v^n \mathcal{O}_v^\times} |x|_v^{-2} \chi_{1v}(x)^{-1} \chi_{2v}(tx) \chi_{1v}(x) \chi_{2v}(x)^{-1} |x|_v^2 \psi_v(tx) dx \\ &= \sum_{n \ge 0} \int_{\varpi_v^n \mathcal{O}_v^\times} \chi_{2v}(t) \psi_v(tx) dx. \end{aligned}$$

Now we take $t \in \mathcal{O}_v^\times$ so that $\chi_{2v}(t) = 1$ and $\psi_v(tx) = 1$ for $x \in \varpi_v^n \mathcal{O}_v^\times$ and $n \geq 0$; this gives:

$$\sum_{n \geq 0} \int_{\varpi_v^n \mathcal{O}_v^\times} dx = \mathrm{Vol}(\mathcal{O}_v).$$

□

For future reference, let's define

$$A_\Sigma \ := \ \prod_{v \in S_\Sigma} A_v. \tag{4.7}$$

4.1.4 Integral representation of $L_f(\frac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2))$

Let's go back to the period integral in Equation (4.2) and its expression as a product of the local zeta integrals involving Whittaker vectors:

$$I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) \ = \ \prod_v \Psi(s, w_{\Pi_v}, w_{E_v}). \tag{4.8}$$

We now make a judicious choice of Whittaker vectors and compute the zeta integrals as follows:

(1) If $v \notin S_\Sigma \cup S_\infty$, take $w_{\Pi_v} = w_{\Pi_v}^{\mathrm{ess}}$ which is the essential vector as in [21], and $w_{E_v} = w_{f_v^{\mathrm{sp}}}$, then we have:

$$\begin{aligned}\Psi(s, w_{\Pi_v}^{\mathrm{ess}}, w_{f_v^{\mathrm{sp}}}) &= \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w_{\Pi_v}^{\mathrm{ess}}(\iota(g_v)) w_{f_v^{\mathrm{sp}}}(g_v) |\det(g_v)|_v^{s-\frac{1}{2}} dg_v \\ &= L(2, \chi_{1v}\chi_{2v}^{-1})^{-1} \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w_{\Pi_v}^{\mathrm{ess}}(\iota(g_v)) w_v^\circ(g_v) \\ &\qquad \times \ |\det(g_v)|_v^{s-\frac{1}{2}} dg_v \\ &= L(2, \chi_{1v}\chi_{2v}^{-1})^{-1} L(s, \Pi_v \times \Sigma(\chi_{1v}, \chi_{2v})).\end{aligned}$$

Let's define

$$L_\Sigma \ := \ L_{S_\Sigma}(2, \chi_1\chi_2^{-1})^{-1} \ = \ \prod_{v \in S_\Sigma} L_v(2, \chi_{1v}\chi_{2v}^{-1})^{-1}. \tag{4.9}$$

(2) If $v \in S_\Sigma$, take $w_{E_v} = w_{f_v^{\text{new}}}$, and let w_{Π_v} be the unique Whittaker function whose restriction to $\iota(\mathrm{GL}_2(F_v))$ is supported on the double coset $N_2(F_v)t^*K_{01}(\mathrm{cond}(\Sigma_v))$, and on this double coset it's given by $w_{\Pi_v}(\iota(nt^*k)) = \psi(n)$, for all $n \in N_2(F_v)$ and all $k \in K_{01}(\mathrm{cond}(\Sigma_v))$.The existence and uniqueness of w_{Π_v} follows from Kirillov theory ([1, Section 5]). So,

$$\begin{aligned}
\Psi(s, w_{\Pi_v}, w_{f_v^{\text{ess}}}) &= \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w_{\Pi_v}(\iota(g_v)) w_{f_v^{\text{new}}}(g_v) |\det(g_v)|_v^{s-\frac{1}{2}} dg_v \\
&= A_v \int_{N_2(F_v)\backslash \mathrm{GL}_2(F_v)} w_{\Pi_v}(\iota(g_v)) w_{\Sigma_v}^{\text{new}}(g_v) |\det(g_v)|_v^{s-\frac{1}{2}} dg_v \\
&= A_v \mathrm{Vol}(K_{01}(\mathrm{cond}(\Sigma_v))).
\end{aligned}$$

Let's define

$$V_\Sigma := \prod_{v \in S_\Sigma} \mathrm{Vol}(K_{01}(\mathrm{cond}(\Sigma_v))). \tag{4.10}$$

(3) If $v \in S_\infty$, let w_{Π_v} and w_{E_v} be arbitrary nonzero vectors. (Later these will be certain 'cohomological vectors'.)

Let's note that the function φ_{χ_1,χ_2} in the induced space $\Sigma(\chi_1, \chi_2)$ is taken accordingly:

$$\varphi_{\chi_1,\chi_2} = \varphi_\infty \otimes \varphi_f, \quad \varphi_f = \otimes_{v \notin S_\Sigma} f_v^{\text{sp}} \otimes_{v \in S_\Sigma} f_v^{\text{new}}, \tag{4.11}$$

with φ_∞ some cohomogical vector. Similarly, the cusp form ϕ_Π is chosen as:

$$\phi_\Pi = \phi_\infty \otimes \phi_f, \quad \phi_f = \otimes_{v \notin S_\infty} \phi_v, \quad \phi_v \text{ corresponds to } w_{\Pi_v}, \tag{4.12}$$

with ϕ_∞ some cohomogical vector.

With the above choice of Whittaker vectors in Equation (4.8) becomes (after multiplying and dividing by suitable local factors and after using the definitions in Equations (4.7), (4.9) and (4.10)):

$I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) \;=$

$$\prod_{v \in S_\infty} \Psi_v(s, w_{\Pi_v}, w_{E_v}) \cdot \frac{A_\Sigma \cdot V_\Sigma \cdot L_\Sigma}{\prod_{v \in S_\Sigma} L(s, \Pi_v \times \Sigma(\chi_{1v}, \chi_{2v}))} \cdot \frac{L_f(s, \Pi \times \Sigma(\chi_1, \chi_2))}{L_f(2, \chi_1\chi_2^{-1})}.$$

For the factors for $v \in S_\infty$, suppose $s = 1/2$ is critical (as we will take a little later on), then by definition of criticality, $L(\frac{1}{2}, \Pi_v \times \Sigma_v)$ is finite. Also $\frac{\Psi(s,w_{\Pi_v},w_{f_v})}{L_v(s,\Pi_v \times \Sigma_v)}$ is holomorphic for all $s \in \mathbb{C}$, hence

$$\Psi(\tfrac{1}{2}, w_{\Pi_v}, w_{f_v}) \;:=\; \left(\frac{\Psi(s,w_{\Pi_v},w_{f_v})}{L_v(s,\Pi_v \times \Sigma_v)}\right)|_{s=1/2} \cdot L(\tfrac{1}{2}, \Pi_v \times \Sigma_v)$$

is finite. Furthermore, the local L-factors are nonzero and the finite part of a global L-function $L_f(s, \Pi \times \Sigma(\chi_1, \chi_2))$ has an analytic continuation for all s. Hence we get, at $s = \frac{1}{2}$:

$$I(\tfrac{1}{2}, \phi_\Pi, E(\varphi_{\chi_1,\chi_2})) = \prod_{v \in S_\infty} \Psi_v(\tfrac{1}{2}, w_{\Pi_v}, w_{E_v}) \frac{A_\Sigma \cdot V_\Sigma \cdot L_\Sigma}{L_{S_\Sigma}(\frac{1}{2}, \Pi \times \Sigma)} \cdot \frac{L_f(\frac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2))}{L_f(2, \chi_1\chi_2^{-1})}, \tag{4.13}$$

where, $L_{S_\Sigma}(\frac{1}{2}, \Pi \times \Sigma) = \prod_{v \in S_\Sigma} L_v(\frac{1}{2}, \Pi_v \times \Sigma(\chi_{1v}, \chi_{2v}))$.

4.2 Special values of L-functions on $\mathrm{GL}_3 \times \mathrm{GL}_1$

4.2.1 Local Langlands correspondence for $\mathrm{GL}_3(\mathbb{F})$

In this subsection, we recall the dictionary to attach an L-function to a given irreducible admissible representation of $\mathrm{GL}_3(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, using Langlands classification. (See [24] for details and future reference.) It says that there is a well-defined bijection between the set of all equivalence classes of n-dimensional semisimple complex representations of Weil group of $\mathbb{F}$, denoted $W_\mathbb{F}$, and the set of all equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{F})$. We will discuss real and complex case separately:

$$\mathbb{F} = \mathbb{R}$$

The building blocks for irreducible admissible representations of $\mathrm{GL}_3(\mathbb{R})$ are the representations of $\mathrm{GL}_1(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})$. Thus the building blocks will be:

$$\mathbb{1} \otimes |\ |_{\mathbb{R}}^t, \quad \text{or} \quad \mathrm{sgn} \otimes |\ |_{\mathbb{R}}^t \rightsquigarrow \text{ for } \mathrm{GL}_1(\mathbb{R}),$$

and

$$D_l \otimes |\det|_{\mathbb{R}}^t \rightsquigarrow \text{ for } \mathrm{GL}_2(\mathbb{R}),$$

for some $t \in \mathbb{C}$.

On the other hand, the Weil group of $\mathbb{R}$, denoted as $W_{\mathbb{R}}$, is the non split extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2\mathbb{Z}$ given by

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times,$$

where $j^2 = -1$ and $jcj^{-1} = \bar{c}$.

The one dimensional representations ϕ of $W_{\mathbb{R}}$ are parametrized by a sgn and a complex parameter t as follows:

$$(+, t) : \phi(z) = |z|_{\mathbb{R}}^t \text{ and } \phi(j) = +1,$$

$$(-, t) : \phi(z) = |z|_{\mathbb{R}}^t \text{ and } \phi(j) = -1.$$

Furthermore, the irreducible two dimensional semi-simple representation ϕ of $W_{\mathbb{R}}$, up to equivalence, is classified by a pair $(l,\ t)$ with $l \in \mathbb{Z},\ l \geq 1$ and $2t \in \mathbb{C}$.

Lemma 4.14 *Every finite dimensional semi-simple representation ϕ of $W_{\mathbb{R}}$ is fully reducible, and each irreducible representation has dimension one or two.*

Now let ϕ be an 3-dimensional semi simple complex representation of $W_{\mathbb{R}}$. By the above Lemma, ϕ is fully reducible, and on the one and two–dimensional building blocks, the correspondence is given by

$$(+, t) \longrightarrow \mathbb{1} \otimes |\ |_{\mathbb{R}}^t$$

$$(-,t) \longrightarrow \mathrm{sgn} \otimes |\ |_{\mathbb{R}}^{t}$$

$$(l,\ t) \longrightarrow D_l \otimes |\det|_{\mathbb{R}}^{t}.$$

To each finite dimensional semi-simple complex representation ϕ of the Weil group of $\mathbb{R}$, is associated a local L factor with certain nice properties. The formula is:

$$L(s,\phi) = \begin{cases} L(s,\phi_1)L(s,\phi_2), & \text{if } \ \phi = \phi_1 \oplus \phi_2 \\ \\ \pi^{-(s+t)/2}\Gamma(\frac{s+t}{2}), & \text{if } \ \phi \mapsto (+,t) \\ \\ \pi^{-(s+t+1)/2}\Gamma(\frac{s+t+1}{2}), & \text{if } \ \phi \mapsto (-,t) \\ \\ 2(2\pi)^{-\left(s+t+\frac{l}{2}\right)}\Gamma\left(s+t+\frac{l}{2}\right), & \text{if } \ \phi \mapsto (l,\ t). \end{cases}$$

$$\mathbb{F} = \mathbb{C}$$

For $z \in \mathbb{C}$, let $[z] = z/|z|$. The building blocks for irreducible addmissible representations of $\mathrm{GL}_3(\mathbb{C})$ are the representations of $\mathrm{GL}_1(\mathbb{C})$ given by

$$z \longmapsto [z]^l |z|_{\mathbb{C}}^t,$$

where $|z|_{\mathbb{C}} = z\bar{z}$ and $l \in \mathbb{Z}, t \in \mathbb{C}$.

On the other hand, the Weil group of $\mathbb{C}$ is given by $W_{\mathbb{C}} = \mathbb{C}^{\times}$. Since $\mathbb{C}^{\times}$ is abelian, such a representation ϕ is diagonalizable and hence is the direct sum of one–dimensional representations, each classified by a pair $(l,\ t)$. Thus the correspondence is given by

$$(l,\ t) \longmapsto [z]^l.|z|^t.$$

The local L-factor corresponding to a one dimensional representation ϕ of $W_{\mathbb{C}}$ is

$$L(s,\phi) = 2(2\pi)^{-\left(s+t+\frac{|l|}{2}\right)}\Gamma\left(s+t+\frac{|l|}{2}\right), \text{ if } \phi \mapsto (l,\ t).$$

For ϕ reducible, $L(s,\phi)$ is the product of the L-factors of the irreducible constituents of ϕ.

4.2.2 Critical set for L-functions

Consider the Rankin–Selberg L-function $L(s, \Pi \otimes \chi)$ where Π is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$, and χ is an algebraic Hecke character of $F^\times \setminus \mathbb{A}_F^\times$.

Definition: We say an integer m is critical for $L(s, \Pi \otimes \chi)$ if both

$$L_\infty(s, \Pi_\infty \otimes \chi_\infty) \text{ and } L_\infty(1-s, \Pi_\infty^\vee \otimes \chi_\infty^\vee)$$

are regular, that is, both the L-factors at infinity on either side of the functional equation have no poles at $s = m$, where

$$L_\infty(s, \Pi_\infty \otimes \chi_\infty) = \prod_{v \in S_\infty} L_v(s, \Pi_v \otimes \chi_v).$$

We will find critical set for $L(s, \Pi \otimes \chi)$ separately for totally real field and CM field.

F : totally real

In Section 3.3.1, we have seen that without loss of generality we can take Π as follows: $\Pi \in \mathrm{Coh}(G_3, \mu)$ such that for each $v \in S_r$, $\mu_v = (n_v,\ 0,\ -n_v)$; with $n_v \geq 0$. Further let χ^0 be a finite order algebraic Hecke character. For each $v \in S_r$, χ_v^0 is character from $\mathbb{R}^\times$ to $\mathbb{R}^\times$ of finite order. Let us denote $\chi_v^0 = \varepsilon_{\chi_v} := (\mathrm{sgn})^{e_{\chi_v}}$ with $e_{\chi_v} \in \{0, 1\}$.

Since F is totally real number field, $S_\infty = S_r$. As in Proposition 3.9, for each $v \in S_r$ we have

$$\Pi_v = \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})}(D_{l_{1v}}|\det|^{w/2} \otimes \varepsilon_{\Pi_v}|\det|^{w/2}),$$

where $l_{1v} = 2\mu_1^v + 2 = 2n_v + 2$, and $w = 2\mu_2^v = 0$. Using these values we get

$$\Pi_v \otimes \chi_v = \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})}(D_{2n_v+2} \otimes \varepsilon_{\Pi_v}\varepsilon_{\chi_v}).$$

Reduction to the case of $\varepsilon_{\Pi_v} = \mathbb{1}$: Take $\Pi \in \mathrm{Coh}(G_3, \mu)$ as before, and fix a nontrivial quadratic character η of F such that $\eta_v = \varepsilon_{\Pi_v}$ for all $v \in S_\infty$.

(Such an η exists; consider the character attached to a quadratic extension obtained by adjoining the square root of an element that is negative for a prescribed set of embeddings–this element may be produced using weak-approximation in F.) Then, $\Pi\otimes\eta$ also has cohomology with respect to μ, and it is easy to see that $\varepsilon_{\Pi_v\otimes\eta_v} = \mathbb{1}$. Furthermore, to study the critical values of $L(s,\Pi\otimes\chi)$, it suffices to consider $L(s,(\Pi\otimes\eta)\otimes(\chi\otimes\eta))$. Henceforth, we will assume:

1. $\mu = (\mu_v)_{v\in S_\infty}$, $\mu_v = (n_v, 0, -n_v)$ with $n_v \geq 0$, and
2. $\Pi \in \mathrm{Coh}(G_3,\mu)$, and $\varepsilon_{\Pi_v} = \mathbb{1}$ for all $v \in S_\infty$.

Thus after the above reduction, we get for each $v \in S_r$

$$\Pi_v \otimes \chi_v = \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})}(D_{2n_v+2} \otimes \varepsilon_{\chi_v}).$$

Case 1. $\chi_v = \mathbb{1}$ for all $v \in S_r$.
This case may also be described as $e_{\chi_v} = 0$ for all $v \in S_r$. We have

$$\Pi_v \otimes \chi_v = \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})}(D_{2n_v+2} \otimes \mathbb{1}).$$

Now for each $v \in S_r$, dual representation of a pair $\Pi\otimes\chi$ is given by

$$\Pi_v^\vee \otimes \chi_v^\vee = \Pi_v^\vee \otimes \bar{\chi}_v = \Pi_v \otimes \chi_v,$$

as discrete series representation is self–dual.
Using Section 4.2.1, by the local Langlands correspondence, the associated L-factors are:

$$L_\infty(s,\Pi_\infty\otimes\chi_\infty) = \prod_{v\in S_r}\left(2(2\pi)^{-(s+\frac{2n_v+2}{2})}\Gamma\left(s+\frac{2n_v+2}{2}\right)\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)$$
$$\approx \prod_{v\in S_r}\Gamma(s+n_v+1)\Gamma\left(\frac{s}{2}\right),$$

$$L_\infty(1-s,\Pi_\infty^\vee\otimes\chi_\infty^\vee) \approx \prod_{v\in S_r}\Gamma(1-s+n_v+1)\Gamma\left(\frac{1-s}{2}\right),$$

where, by $\approx$, we mean up to nonzero constants and exponential functions which are holomorphic and nonvanishing everywhere, and hence irrelevant for computing the critical points.

It is known that for an integer m, $\Gamma(m)$ is finite if and only if $m \geq 1$, i.e., $m \in \{1, 2, 3, ...\}$. Hence $s = m$ is critical if for each $v \in S_r$

$$m + n_v + 1 \geq 1,\ \tfrac{m}{2} \geq 1;\quad 2 - m + n_v \geq 1,\ \tfrac{1-m}{2} \geq 1.$$

Now it is an easy exercise to see that:

$$\text{Critical set for } L(s, \Pi \times \chi) = \{1 - n_{\text{ev}}, \ldots, -3, -1, 2, 4, \ldots, n_{\text{ev}}\}, \tag{4.15}$$

where,

$$n_{\text{ev}} = 2\left[\frac{n+1}{2}\right]$$
$$= \text{the largest even positive integer less than or equal to } n + 1,$$

and $n = \min_{v \in S_r}\{n_v\}$.

Note that if $n = 0$ (this is the case, for example if $\mu = 0$, i.e., the case of constant coefficients for the cohomology of GL_3) then the critical set is empty.

Case 2. $\chi_v = \text{sgn}$ for all $v \in S_r$.

It may also be described as $e_{\chi_v} = 1$ for all $v \in S_\infty$. In this case we have $\Pi_v \otimes \chi_v = \mathrm{Ind}_{P_{(2,1)}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})}(D_{2n_v+2} \otimes \text{sgn})$ and again using Section 4.2.1 the associated L-factors are:

$$L_\infty(s, \Pi_\infty \times \chi_\infty) \approx \prod_{v \in S_r} \Gamma(s + n_v + 1)\Gamma\left(\frac{s+1}{2}\right),$$

$$L_\infty(1 - s, \Pi_\infty^\vee \times \chi_\infty^\vee) \approx \prod_{v \in S_r} \Gamma(1 - s + n_v + 1)\Gamma\left(\frac{2-s}{2}\right).$$

Then an integer m is critical if the following holds:

$$m + n_v + 1 \geq 1,\ \tfrac{m+1}{2} \geq 1;\quad 2 - m + n_v \geq 1,\ \tfrac{2-m}{2} \geq 1.$$

It is an easy exercise now to deduce that:

$$\text{Critical set for } L(s, \Pi \times \chi) = \{1 - n_{\mathrm{od}}, \ldots, -2, 0, 1, 3, \ldots, n_{\mathrm{od}}\}, \quad (4.16)$$

where,

$$n_{\mathrm{od}} = 2\left[\frac{n}{2}\right] + 1 = \text{the largest odd integer less than or equal to } n+1.$$

Note that in this case, the critical set is always nonempty.

Case 3: There exist two places $v_1, v_2 \in S_\infty$ such that $\varepsilon_{\Pi_{v_1}} = \varepsilon_{\chi_{v_1}}$ and $\varepsilon_{\Pi_{v_2}} \neq \varepsilon_{\chi_{v_2}}$.

Then in the expression for $L_\infty(s, \Pi_\infty \times \chi_\infty)$ we would have as a factor: $\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$ and it is easy to see that in this situation there are no critical points; whence, we will not consider this case.

F : CM field

As in Proposition 3.13, let $\mu \in X_0^+(T_3)$ be a "parallel" weight written as $\mu = (\mu_v)_{v \in S_\infty}$ with $\mu_v = (n_1, 0, n_2; -n_2, 0, -n_1)$ for integers n_1, n_2 such that $n_1 \geq 0 \geq n_2$, and suppose $\Pi \in \mathrm{Coh}(G_3, \mu)$. Let χ be an algebraic Hecke character of $F^\times \setminus \mathbb{A}_F^\times$ of type(A_0) such that $\chi_\infty(z_\infty) = \prod_{v \in S_\infty} \left(\frac{z_v}{|z_v|}\right)^{-f}$ for some $f \in \mathbb{Z}$. Comparing with the general form of an algebraic Hecke character, we get $f \equiv 0 \pmod 2$, that is, f is even. Put $f = 2t$ for some $t \in \mathbb{Z}$. Hence we have $\chi_\infty(z) = z^{-t}\bar{z}^t$.

For each $v \in S_\infty$, we have

$$\Pi_v = \mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}_3(\mathbb{C})}(z^{n_1+1}\bar{z}^{-n_1-1} \otimes \mathbb{1} \otimes z^{n_2-1}\bar{z}^{-n_2+1}).$$

Then, on tensoring with χ_v we get

$$\Pi_v \otimes \chi_v = \mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}_3(\mathbb{C})}(z^{a_1}\bar{z}^{b_1} \otimes z^{a_2}\bar{z}^{b_2} \otimes z^{a_3}\bar{z}^{b_3});$$

with the cuspidal parameters

$$a = (a_1, a_2, a_3) = (n_1 + 1 - t,\ -t,\ n_2 - 1 - t),$$

and

$$b = (b_1, b_2, b_3) = (-n_1 - 1 + t,\ t,\ -n_2 + 1 + t).$$

Again using Section 4.2.1, the associated local L-factor is:

$$\begin{aligned}
L(s, \Pi_v \otimes \chi_v) &= \prod_{i=1}^{3} L(s, z^{a_i} \bar{z}^{b_i}) \\
&= \prod_{i=1}^{3} 2(2\pi)^{-\left(s+\frac{a_i+b_i}{2}+\frac{|a_i-b_i|}{2}\right)} \Gamma\left(s + \frac{a_i + b_i}{2} + \frac{|a_i - b_i|}{2}\right) \\
&\approx \prod_{i=1}^{3} \Gamma\left(s + \frac{a_i + b_i}{2} + \frac{|a_i - b_i|}{2}\right) \\
&\approx \Gamma\left(s + \frac{|2n_1 + 2 - 2t|}{2}\right) \Gamma\left(s + \frac{|-2t|}{2}\right) \\
&\qquad\qquad \Gamma\left(s + \frac{|2n_2 - 2 - 2t|}{2}\right) \\
&= \Gamma(s + |n_1 + 1 - t|)\Gamma(s + |t|)\Gamma(s + |n_2 - 1 - t|),
\end{aligned}$$

where, by $\approx$, we mean up to nonzero constants and exponential functions.

Similarly, look at the dual representation

$$\Pi_v^\vee \otimes \chi_v^\vee = \mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}_3(\mathbb{C})}(z^{c_1} \bar{z}^{d_1} \otimes z^{c_2} \bar{z}^{d_2} \otimes z^{c_3} \bar{z}^{d_3});$$

with the cuspidal parameters

$$c = (c_1, c_2, c_3) = (-n_2 + 1 + t,\ t,\ -n_1 - 1 + t),$$

and

$$d = (d_1, d_2, d_3) = (n_2 - 1 - t,\ -t,\ n_1 + 1 - t).$$

Now one can easily check that

$L(1-s,\Pi_v^\vee\otimes\chi_v^\vee)$

$=L(1-s,\Pi_v\otimes\chi_v)=\Gamma(1-s+|n_1+1-t|)\Gamma(1-s+|t|)\Gamma(1-s+|n_2-1-t|).$

We will separate this into cases depending on whether t is positive or negative.

Case 1: $t>0$ for all $v\in S_\infty$.

This implies $n_2-1-t\le 0$ as n_2 is non-positive.

Case 1a. $n_1+1-t\ge 0$.

Using Section 4.2.1, the associated L-factors are:

$$L_\infty(s,\Pi_\infty\otimes\chi_\infty)\approx\prod_{v\in S_c}\Gamma(s+n_1+1-t)\Gamma(s+t)\Gamma(s-n_2+1+t),$$

and

$L_\infty(1-s,\Pi_\infty^\vee\otimes\chi_\infty^\vee)$

$$\approx\prod_{v\in S_c}\Gamma(1-s+n_1+1-t)\Gamma(1-s+t)\Gamma(1-s-n_2+1+t).$$

By the definition of criticality of Γ functions, an integer $m\in\mathbb{Z}$ is critical if the following inequalities hold for all $v\in S_c$:

$$m+n_1+1-t\ge 1,\quad m+t\ge 1,\quad m-n_2+1+t\ge 1;$$

and

$$1-m+n_1+1-t\ge 1,\quad 1-m+t\ge 1,\quad 1-m-n_2+1+t\ge 1.$$

On solving above inequalities we get:

$$m\ge 1-t;\ m\ge t-n_1,\ \text{ and }\ m\le t;\ m\le 1-t+n_1.$$

Consider following two cases:

- If $n_1\le 2t-1$ for all $v\in S_c$. Then it is easy to check that:

  Critical set for $L(s,\Pi\times\chi)\ =\ \{m\in\mathbb{Z}\mid t-n_1\le m\le n_1+1-t\}$.

  In this case critical set is non-empty only if $n_1\ge t$.

- If $n_1 \geq 2t$ for all v. Again it is an easy exercise now to see that:

$$\text{Critical set for } L(s, \Pi \times \chi) \ = \ \{m \in \mathbb{Z} \mid 1-t \leq m \leq t\}.$$

  Note that under the assumption t is positive, the critical set is always non-empty.

Case 1b. $n_1 + 1 - t \leq 0$ for all $v \in S_c$.
Recall $n_2 - 1 - t \leq 0$. For each $v \in S_c$ the associated local L-factors is:

$$L(s, \Pi_v \otimes \chi_v) \approx \Gamma(s - n_1 - 1 + t)\Gamma(s + t)\Gamma(s - n_2 + 1 + t),$$

and

$$L(1-s, \Pi_v^\vee \otimes \chi_v^\vee) \approx \Gamma(1-s-n_1-1+t)\Gamma(1-s+t)\Gamma(1-s-n_2+1+t).$$

An integer m is critical if

$$m \geq \{2 + n_1 - t;\ 1 - t;\ n_2 - t\},$$

and

$$m \leq \{t - n_1 - 1;\ \ t;\ t + 1 - n_2\}.$$

Hence it is easy to see that:

$$\text{Critical set for } L(s, \Pi \times \chi) \ = \ \{m \in \mathbb{Z} \mid 2 + n_1 - t \leq m \leq t - n_1 - 1\}.$$

Clearly it is non-empty only if $n_1 \leq t - 2$.

On summarizing Case 1 we get: If t is strictly positive,

critical set for $L(s, \Pi \times \chi)$

$$= \ m \in \begin{cases} [2 + n_1 - t,\ t - n_1 - 1] & \text{if} \quad 0 \leq n_1 \leq t - 2, \\ [t - n_1,\ n_1 + 1 - t] & \text{if} \quad t \leq n_1 \leq 2t - 1, \\ [1 - t,\ t] & \text{if} \quad n_1 \geq 2t. \end{cases} \tag{4.17}$$

Note that for $n_1 = t-1$ there are no critical points.

Case 2: $t < 0$ for all $v \in S_c$.
This implies $n_1 + 1 - t \geq 0$ as n_1 is non-negative.

Case 2a. $n_2 - 1 - t \leq 0$ for all v. The associated L-factors at infinity are:

$$L_\infty(s, \Pi_\infty \otimes \chi_\infty) \approx \prod_{v \in S_c} \Gamma(s + n_1 + 1 - t)\Gamma(s - t)\Gamma(s - n_2 + 1 + t),$$

and
$L_\infty(1 - s, \Pi_\infty^\vee \otimes \chi_\infty^\vee)$

$$\approx \prod_{v \in S_c} \Gamma(1 - s + n_1 + 1 - t)\Gamma(1 - s - t)\Gamma(1 - s - n_2 + 1 + t).$$

An integer $m \in \mathbb{Z}$ is critical if for all $v \in S_c$ the following inequalities hold:

$$m + n_1 + 1 - t \geq 1, \quad m - t \geq 1, \quad m - n_2 + 1 + t \geq 1;$$

and

$$1 - m + n_1 + 1 - t \geq 1, \quad 1 - m - t \geq 1, \quad 1 - m - n_2 + 1 + t \geq 1.$$

To simplify above inequalities we get:

$$m \geq 1 + t;\ m \geq n_2 - t, \ \text{ and } \ m \leq -t;\ m \leq 1 + t - n_2.$$

Similar to case 1 consider the following two cases

- If $2t + 1 \leq n_2$ for all $v \in S_c$. Then it is easy to conclude that:

  Critical set for $L(s, \Pi \times \chi) \ = \ \{m \in \mathbb{Z} \mid n_2 - t \leq m \leq 1 + t - n_2\}$,

  which is non-empty if and only if $n_2 \leq t$ for all $v \in S_\infty$.

- If $n_2 \leq 2t$ for all v. Again it is an easy exercise now to see that:

$$\text{Critical set for } L(s, \Pi \times \chi) \;=\; \{m \in \mathbb{Z} \mid 1 + t \leq m \leq -t\}.$$

In this case critical set is always non-empty as $t > 0$.

Case 2b. $n_2 - 1 - t \geq 0$ for all $v \in S_c$.
Recall $n_1 + 1 - t \geq 0$. The associated local L-factors are:

$$L(s, \Pi_v \otimes \chi_v) \approx \Gamma(s + n_1 + 1 - t)\Gamma(s - t)\Gamma(s + n_2 - 1 - t),$$

and

$$L(1-s, \Pi_v^\vee \otimes \chi_v^\vee) \approx \Gamma(1-s+n_1+1-t)\Gamma(1-s-t)\Gamma(1-s+n_2-1-t).$$

An integer $m \in \mathbb{Z}$ is critical if for all $v \in S_c$ the following inequalities hold :

$$m \geq \{-n_1 - 1 + t;\ 1 + t;\ 2 + t - n_2\};$$

and

$$m \leq \{n_1 + 1 - t;\ -1 - t;\ n_2 - 1 - t\}.$$

It is an easy exercise now to see that:

$$\text{Critical set for } L(s, \Pi \times \chi) \;=\; \{m \in \mathbb{Z} \mid 2 + t - n_2 \leq m \leq n_2 - 1 - t\}.$$

Observe that critical set is non-empty only if $t + 2 \leq n_2$.

On summarizing Case 2 we get: If t is strictly negative,

critical set for $L(s, \Pi \times \chi)$

$$= \; m \in \begin{cases} [2 - n_2 + t,\ n_2 - 1 - t] & \text{if} \quad t + 2 \leq n_2 \leq 0, \\ [n_2 - t,\ 1 + t - n_2] & \text{if} \quad 2t + 1 \leq n_2 \leq t, \\ [1 + t,\ -t] & \text{if} \quad n_2 \leq 2t. \end{cases} \tag{4.18}$$

Note that for $n_2 = t + 1$ critical set is empty.

Case 3: $t = 0$.
Then in the expression for $L_\infty(s, \Pi_\infty \times \chi_\infty)$ and $L_\infty(1-s, \Pi_\infty^\vee \times \chi_\infty^\vee)$ we would have a factor: $\Gamma(s)$ and $\Gamma(1-s)$ respectively and it is easy to see that in this situation there are no critical points; whence, we will not consider this case.

4.3 Interlacing of weights

Before stating our next proposition, let's recall the following well-known branching rule (a condition on the coefficients system) for finite-dimensional representations:

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{C})}(\mathcal{M}_\lambda \otimes \mathcal{M}_\mu, \mathbb{C}) \neq 0 \Longleftrightarrow \mu \succ \lambda^\vee$$

where $\mu \succ \lambda^\vee$ means $\mu_v \succ \lambda_v^\vee$ for all $v \in S_\infty$ and

$$\mu_v \succ \lambda_v^\vee \equiv \begin{cases} \mu_{1v} \geq -\lambda_{2v} \geq \mu_{2v} \geq -\lambda_{1v} \geq \mu_{3v} & \text{if } F \text{ is totally real} \\ \\ \mu_1 \geq -\lambda_2 \geq \mu_2 \geq -\lambda_1 \geq \mu_3; & \\ \mu_1^* \geq -\lambda_2^* \geq \mu_2^* \geq -\lambda_1^* \geq \mu_3^* & \text{if } F \text{ is a CM field.} \end{cases}$$

Proposition 4.19 (F is totally real) *Let μ and Π be as in Remark 4.2.2, and let χ be a finite order character of $\mathbb{A}_F^\times/F^\times$. We fix once and for all, a totally odd quadratic Hecke character ξ of F, and make the following choices for Hecke characters $\chi_i = \|\ \|^{d_i}\ \chi_i^0$, with integers d_i and finite order characters χ_i^0:*

Case 1. $\varepsilon_{\chi_v} = 1\!\!1$ *for all* $v \in S_\infty$.

Case 1a. $m \in \{2, 4, \ldots, n_{\mathrm{ev}}\}$, $d_1 = m-1$, $d_2 = -1$, $\chi_1^0 = \chi$, *and* $\chi_2^0 = 1\!\!1$; *put* $\lambda_v = (1, 1-m)$.

Case 1b. *$m \in \{1-n_{\mathrm{ev}}, \ldots, -3, -1\}$, $d_1 = 1$, $d_2 = m$, $\chi_1^0 = \mathbb{1}$, and $\chi_2^0 = \chi$; put $\lambda_v = (-m, -1)$.*

Case 2. *$\varepsilon_{\chi_v} = \mathrm{sgn}$ for all $v \in S_\infty$.*

Case 2a. *$m \in \{1, 3, \ldots, n_{\mathrm{od}}\}$, $d_1 = m-1$, $d_2 = 0$, $\chi_1^0 = \chi$, and $\chi_2^0 = \xi$; put $\lambda_v = (0, 1-m)$.*

Case 2b. *$m \in \{1-n_{\mathrm{od}}, \ldots, -2, 0\}$, $d_1 = 0$, $d_2 = m$, $\chi_1^0 = \xi$, and $\chi_2^0 = \chi$; put $\lambda_v = (-m, 0)$.*

Then, in all the above four cases, we have

1. *$L(\frac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2))$ is critical;*
2. *$H^1\left(\mathfrak{gl}_2, \mathrm{SO}(2)\mathbb{R}_+^\times; \Sigma(\chi_1,\chi_2)_v \otimes \mathcal{M}_{\lambda_v}\right) = \mathbb{C}\mathbb{1}$ as an $\mathrm{O}(2)/\mathrm{SO}(2)$-module;*
3. *$\mu \succ \lambda^\vee$.*

Proof. The proof is a routine check in each case and we will only briefly present the key details:

Case 1a. For the L-value we see that

$$\begin{aligned} L(\tfrac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2)) &= L(1+d_1, \Pi \otimes \chi_1^0) L(d_2, \Pi \otimes \chi_2^0) \\ &= L(m, \Pi \otimes \chi) L(-1, \Pi), \end{aligned}$$

and both the L-values on the right hand side are critical by Equation (4.15). The induced representation may be written as

$$\Sigma(\chi_1, \chi_2)_v = \xi_{m-1}(\mathrm{sgn})(1/2) \times \xi_{-1}(\mathrm{sgn})(-1/2),$$

which has nontrivial cohomology with respect to $\lambda_v = (1, 1-m)$; see Proposition 3.18. Clearly $\lambda \in X_0^+(T_2)$. Further for interlacing condition $\mu \succ \lambda^\vee$ we want:

$$n \geq m-1 \geq 0 \geq -1 \geq -n$$

which follows from the given range of critical set.

Case 1b. For the L-value we have

$$L(\tfrac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2)) = L(2, \Pi)L(m, \Pi \otimes \chi),$$

and for the induced representation we have

$$\Sigma(\chi_1, \chi_2)_v = \xi_1(\mathrm{sgn})(1/2) \times \xi_m(\mathrm{sgn})(-1/2),$$

which has nontrivial cohomology with respect to $\lambda_v = (-m, -1)$. One can easily check that $\mu \succ \lambda^\vee$ which is $n \geq 1 \geq 0 \geq m \geq -n$.

Case 2a. For the L-value we have

$$L(\tfrac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2)) = L(m, \Pi \otimes \chi)L(0, \Pi \otimes \xi),$$

and for the induced representation we have

$$\Sigma(\chi_1, \chi_2)_v = \xi_{m-1}(\mathrm{sgn})(1/2) \times (\mathrm{sgn})(-1/2),$$

which has nontrivial cohomology with respect to $\lambda_v = (0, 1-m)$. Clearly $\lambda \in X_0^+(T_2)$ and $\mu \succ \lambda^\vee$.

Case 2b. For the L-value we have

$$L(\tfrac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2)) = L(1, \Pi \otimes \xi)L(m, \Pi \otimes \chi),$$

and for the induced representation we have

$$\Sigma(\chi_1, \chi_2)_v = (\mathrm{sgn})(1/2) \times \xi_m(\mathrm{sgn})(-1/2).$$

□

Proposition 4.20 (F is a CM field) *Let $\Pi \in \mathrm{Coh}(G_3, \mu)$ such that $\mu = (\mu_v)_{v \in S_r}$, $\mu_v = (n_1,\ 0,\ n_2; -n_2,\ 0,\ -n_1)$, with $n_1 \geq 0$ and $n_2 \leq 0$. Let χ be a character of $F^\times \setminus \mathbb{A}_F^\times$ such that $\chi_\infty(z_\infty) = \prod_{v \in S_\infty}(z_v/|z_v|)^{-2t}$ for some $t \in \mathbb{Z}$. We fix once and for all, a unitary algebraic Hecke character ϕ of*

F with 'parallel' weight such that $\phi_\infty(z) = \left(\frac{z}{|z|}\right)^2$, *and make the following choice for Hecke characters*

$$\chi_i = \|\cdot\|^{\sigma_i}\, \chi_i^1,$$

for some $\sigma_i \in \frac{1}{2}\mathbb{Z}$ *and characters* χ_i^1 *are such that* $\chi_{i\infty}^1(z_\infty) = \prod_{v \in S_\infty}(z_v/|z_v|)^{-f_i}$ *and* $f_i \equiv 2\sigma_i$ *(mod 2):*

Case 1. *t is strictly positive, that is,* $t > 0$; $n_1 \geq 1$ *and* $n_2 \leq -2t$ $\forall\, v \in S_\infty$. *Also an integer m satisfies Equation (4.17). Choose* $\chi_1^1 = \phi$, $\sigma_1 = -1$, $\chi_2^1 = \chi$, *and* $\sigma_2 = m$; *put*

$$\lambda_v = (t - m,\ 0;\ 1,\ 1 - t - m).$$

Case 2. *t is strictly negative, i.e.,* $t < 0$; $n_2 \leq -1$ *and* $n_1 \geq -2t$ $\forall\, v \in S_\infty$. *Furthermore, an integer m satisfies Equation (4.18). Choose* $\chi_1^1 = \chi$, $\sigma_1 = m - 1$, $\chi_2^1 = \phi^{-1}$, *and* $\sigma_2 = 1$; *put*

$$\lambda_v = (0,\ t - m + 1;\ -t - m,\ -1).$$

Then, in all the cases, we have

1. $L(\frac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2))$ *is critical;*
2. $H^1(\mathfrak{gl}_2, \mathrm{U}(2)\mathbb{C}^\times; \Sigma(\chi_{1v}, \chi_{2v}) \otimes \mathcal{M}_{\lambda_v}) = \mathbb{C}\mathbb{1}$ *as an* $\mathrm{U}(2)/\mathrm{SU}(2)$*-module;*
3. $\mu \succ \lambda^{\mathsf{v}}$.

Proof. The details of the proof in each case are as follows:

Case 1 Clearly $\chi_1^1 = \phi$ implies $f_1 = -2$ and $\chi_2^1 = \chi$ implies $f_2 = 2t$. Then for the L-value we see that

$$\begin{aligned} L(\tfrac{1}{2}, \Pi \otimes \Sigma(\chi_1, \chi_2)) &= L(1, \Pi \otimes \chi_1) L(0, \Pi \otimes \chi_2) \\ &= L(1 + \sigma_1, \Pi \otimes \chi_1^1) L(\sigma_2, \Pi \otimes \chi_2^1) \\ &= L(0, \Pi \otimes \phi) L(m, \Pi \otimes \chi), \end{aligned}$$

and both the L-values on the right hand side are critical by Equation (4.17). Observe that $\Pi\otimes\phi$ is unitary and hence using [20], $L(0,\Pi\otimes\phi)\neq 0$. The induced representation may be written as

$$\Sigma(\chi_1,\chi_2)_v = {}^{\mathrm{a}}\mathrm{Ind}_{B_2(\mathbb{C})}^{\mathrm{GL}_2(\mathbb{C})}(\xi_{(1,\ -1)}\otimes\xi_{(-t+m-1,\ t+m-1)})$$

which has nontrivial cohomology with respect to

$$\lambda_v = (t-m,\ 0;\ 1,\ 1-t-m);$$

see Proposition 3.18. Clearly for the sub-cases in critical set $\lambda\in X^+(T_2)$. For the interlacing of weights we want

$$n_1\geq 0\geq 0\geq m-t\geq n_2;$$

and

$$n_2\leq 1-t-m\leq 0\leq 1\leq n_1,$$

which immediately follows from the condition on n_1 and n_2 in terms of t.

Case 2. Clearly from the above substitutions $\chi_1^1=\chi$ and $\chi_2^1=\phi^{-1}$ we have $f_1=2t$ and $f_2=2$. For the L-value we see that

$$\begin{aligned}L(\tfrac{1}{2},\Pi\otimes\Sigma(\chi_1,\chi_2)) &= L(1+\sigma_1,\Pi\otimes\chi_1^1)L(\sigma_2,\Pi\otimes\chi_2^1)\\ &= L(m,\Pi\otimes\chi)L(1,\Pi\otimes\phi^{-1}),\end{aligned}$$

and both the L-values on the right hand side are critical by Equation (4.18). Clearly $\Pi\otimes\phi^{-1}$ is unitary and hence $L(1,\Pi\otimes\phi^{-1})\neq 0$ by [20]. The induced representation may be written as

$$\Sigma(\chi_1,\chi_2)_v = {}^{\mathrm{a}}\mathrm{Ind}_{B_2}^{\mathrm{GL}_2}(\xi_{(-t+m,t+m)}\otimes\xi_{(-1,1)})$$

which has nontrivial cohomology with respect to

$$\lambda_v = (0,\ t-m+1;\ -t-m,\ -1);$$

again see Proposition 3.18. In this case also it is easy to check that $\lambda \in X^+(T_2)$. For the interlacing of weights we want

$$n_1 \geq m - t - 1 \geq 0 \geq 0 \geq n_2;$$

and

$$n_2 \leq -1 \leq 0 \leq -(t+m) \leq n_1.$$

From the given conditions on n_1 and n_2 in terms of t it is easy to check that all the inequalities are satisfied.

□

Chapter 5

Cohomological interpretation of the integral

In this chapter, we interpret the period integral $I(s, \phi_\Pi, E(\varphi_{\chi_1,\chi_2}))$ in terms of Poincaré duality. More precisely, the vector w_{Π_f} will correspond to a cohomology class $\vartheta_{\Pi,\epsilon_\Pi}$ in degree b_3^F (the bottom degree of the cuspidal range for G_3) on a locally symmetric space denoted by $S_3(K_f)$ for GL_3, and similarly $\varphi_f \in \Sigma(\chi_1, \chi_2)_f$ will correspond to a class ϑ_Σ in degree b_2^F. The class $\vartheta_{\Pi,\epsilon_\Pi}$, after dividing by a certain period, has good rationality properties. Pull back ϑ_Π along the proper map $\iota : \tilde{S}_2 \longrightarrow S_3$, and wedge (or cup) with ϑ_Σ, to give a top degree class on $\tilde{S}_2$ with coefficients in a tensor product sheaf. Now if $s = 1/2$ is critical which is the same as saying the constituent sheaves are compatible (which is the case when the weights interlace: $\mu \succ \lambda^\vee$), then we get a top-degree class on $\tilde{S}_2$ with constant coefficients. Apply Poincaré duality, i.e., fix an orientation on $\tilde{S}_2$ and integrate. One realizes then that this is essentially the above period integral. Interpreting the integral, and hence the L-value it represents, as a cohomological pairing permits us to study arithmetic properties of such special values, since this pairing is Galois equivariant. We now make all this precise.

5.1 The cohomology classes

Recall from Propositions 3.9 and 3.13, given any $\Pi \in \mathrm{Coh}(G_3, \mu)$ and for the signature ϵ_Π for Π, the cohomology group

$$H^{b_3^F}(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_\mu)(\epsilon_\Pi) \neq 0,$$

and is one-dimensional. Fix a basis $[\Pi_\infty]$ of this one-dimensional space, and this gives us the following comparison isomorphism (see [33]):

$$\mathcal{F}_{\Pi_f} \equiv \mathcal{F}_{\Pi_f, \epsilon_\Pi, [\Pi_\infty]} : \mathcal{W}(\Pi_f) \longrightarrow H^{b_3^F}(\mathfrak{g}_{3,\infty}, K^0_{3,\infty}; V_\Pi \otimes \mathcal{M}_\mu)(\epsilon_\Pi). \tag{5.1}$$

The isomorphism $\mathcal{F}_{\Pi_f}$ is a $G_3(\mathbb{A}_f)$-equivariant map between irreducible modules, both of which have $\mathbb{Q}(\Pi)$ structures that are unique up to homotheties; we can adjust the map by a scalar–which is the period–so as to preserve rational structures; for more details see [33]. There is a nonzero complex number $p^{\epsilon_\Pi}(\Pi)$ attached to the datum $(\Pi_f, \epsilon_\Pi, [\Pi_\infty])$ such that the normalized map,

$$\mathcal{F}^\circ_{\Pi_f} := p^{\epsilon_\Pi}(\Pi)^{-1} \mathcal{F}_{\Pi_f}$$

is $\mathrm{Aut}(\mathbb{C})$–equivariant, i.e., the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}^\circ_{\Pi_f} : \mathcal{W}(\Pi_f) & \longrightarrow & H^{b_3^F}(\mathfrak{g}_{3,\infty}, \ K^0_{3,\infty}; V_\Pi \otimes \mathcal{M}_{\mu,\mathbb{C}})(\epsilon_\Pi) \\ \Big\downarrow \sigma & & \Big\downarrow \sigma \\ \mathcal{F}^\circ_{{}^\sigma\Pi_f} : \mathcal{W}({}^\sigma\Pi_f) & \longrightarrow & H^{b_3^F}(\mathfrak{g}_{3,\infty}, \ K^0_{3,\infty}; V_{{}^\sigma\Pi} \otimes \mathcal{M}_{{}^\sigma\mu,\mathbb{C}})(\epsilon_\Pi). \end{array}$$

The complex number $p^{\epsilon_\Pi}(\Pi)$ is well-defined up to multiplication by elements of $\mathbb{Q}(\Pi)^\times$. The collection $\{p^{\epsilon_\Pi}({}^\sigma\Pi) \ : \ \sigma \in \mathrm{Aut}(\mathbb{C})\}$ is well-defined in $(\mathbb{Q}(\Pi) \otimes \mathbb{C})^\times / \mathbb{Q}(\Pi)^\times$. In terms of the un-normalized maps, we can write the above commutative diagram as

$$\sigma \circ \mathcal{F}_{\Pi_f} = \left(\frac{\sigma(p^{\epsilon_\Pi}(\Pi))}{p^{\epsilon_\Pi}({}^\sigma\Pi)} \right) \mathcal{F}_{{}^\sigma\Pi_f} \circ \sigma.$$

Now define the cohomology class attached to the global Whittaker vector w_{Π_f} as,

$$\vartheta_{\Pi,\epsilon_\Pi} := \mathcal{F}_{\Pi_f}(w_{\Pi_f}), \quad \text{and} \quad \vartheta^\circ_{\Pi,\epsilon_\Pi} = p^{\epsilon_\Pi}(\Pi)^{-1} \vartheta_{\Pi,\epsilon_\Pi}. \tag{5.2}$$

Let K_f be an open compact subgroup of $G_3(\mathbb{A}_f)$ which fixes w_{Π_f} and such that, $\Sigma(\chi_1, \chi_2)_f$ has vectors fixed under $R_f := \iota^* K_f$. Then

$$\vartheta_{\Pi,\epsilon_\Pi} \in H^{b_3^F}(\mathfrak{g}_{3,\infty}, K_{3,\infty}^\circ; V_\Pi^{K_f} \otimes \mathcal{M}_\mu)(\epsilon_\Pi)$$

and via certain standard isomorphisms [33, Section 3.3], we may identify the class in $H^{b_3^F}_{\mathrm{cusp}}(S_3(K_f), \widetilde{\mathcal{M}}_\mu)(\widetilde{\Pi}_f)$, where $\widetilde{\Pi}_f := \Pi_f \otimes \epsilon_\Pi$ is a representation of $G_3(\mathbb{A}_f) \otimes \pi_0(K_{3,\infty})$. Furthermore, since cuspidal cohomology injects into cohomology with compact support, we get $\vartheta_{\Pi,\epsilon_\Pi} \in H^{b_3^F}_{\mathrm{c}}(S_3(K_f), \widetilde{\mathcal{M}}_\mu)$.

On the other hand, recall the map in Equation (3.22):

$$\mathcal{F}_{\Sigma_f} \; : \; \Sigma(\chi_1, \chi_2)_f^{R_f} \;\longrightarrow\; H^{b_2^F}(S_2(R_f), \widetilde{\mathcal{M}}_\lambda),$$

which is $\mathrm{Aut}(\mathbb{C})$-equivariant, that is, $\sigma \circ \mathcal{F}_\Sigma = \mathcal{F}_{{}^\sigma\Sigma} \circ \sigma$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Define the class

$$\vartheta_\Sigma^\circ \;:=\; \mathcal{F}_\Sigma(\varphi_f), \tag{5.3}$$

where φ_f is defined in Equation (4.11). Using the canonical map ϕ^* (the map induced by ϕ in cohomology)

$$H^{b_2^F}(S_2(R_f), \widetilde{\mathcal{M}}_\lambda) \xrightarrow{\phi^*} H^{b_2^F}(\tilde{S}_2(R_f), \widetilde{\mathcal{M}}_\lambda)\ ,$$

we get $\phi^* \vartheta_\Sigma$ in $H^{b_2^F}(\tilde{S}_2(R_f), \widetilde{\mathcal{M}}_\lambda)$.

For the open compact subgroups K_f of $\mathrm{GL}_3(\mathbb{A}_f)$ and $R_f = \iota^*(K_f)$ of $\mathrm{GL}_2(\mathbb{A}_f)$, the map ι, being a proper map, induces a map between the cohomology with compact supports:

$$\iota^* : H_{\mathrm{c}}^\bullet(S_3(K_f), \widetilde{\mathcal{M}}_\mu) \longrightarrow H_{\mathrm{c}}^\bullet(\tilde{S}_2(R_f), \iota^* \widetilde{\mathcal{M}}_\mu).$$

Now consider the following diagram:

$$\begin{array}{ccc}\mathcal{W}(\Pi_f) \times \Sigma(\chi_1, \chi_2)_f & \longrightarrow & H_c^{b_3^F}(S_3(K_f), \widetilde{\mathcal{M}}_\mu) \times H^{b_2^F}(S_2(R_f), \widetilde{\mathcal{M}}_\lambda) \\ & & \downarrow \iota^* \times \phi^* \\ & & H_c^{b_3^F}(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu) \times H^{b_2^F}(\tilde{S}_2(R_f), \phi^*\widetilde{\mathcal{M}}_\lambda) \\ & & \downarrow \wedge \\ & & H_c^{b_3^F + b_2^F}(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu \times \phi^*\widetilde{\mathcal{M}}_\lambda).\end{array}$$

Numerical coincidence:

1. *F is totally real:* Using Propositions 3.9 and 3.18, we have $b_3^F = 2d_F$ and $b_2^F = d_F$. Then we have

$$b_3^F + b_2^F = 2d_F + d_F = 3d_F = 3r_1$$

$$= d_F \ \dim(\mathrm{GL}_2(\mathbb{R})^0/\mathrm{SO}(2)) = \dim(\tilde{S}_2(R_f)).$$

2. *F is a CM field:* Similarly using Propositions 4.20 and 3.18 in this case we get

$$b_3^F + b_2^F = 3d_0 + d_0 = 4d_0 = 4r_2$$

$$= d_0 \ \dim(\mathrm{GL}_2(\mathbb{C})/\mathrm{U}(2)) = \dim(\tilde{S}_2(R_f)).$$

Hence in both the cases

$$\vartheta_{\Pi,\epsilon_\Pi} \wedge \vartheta_\Sigma \in H_c^{\dim(\tilde{S}_2(R_f))}(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu \times \phi^*\widetilde{\mathcal{M}}_\lambda).$$

Compatibility of sheaves

We now assume the hypotheses of Proposition 4.19 for totally real case and of Proposition 4.20 for CM case. The interlacing condition of weights $\mu \succ \lambda^\vee$, gives the branching rule for finite-dimensional representations as

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{C})}(\iota^*\mathcal{M}_\mu, \mathcal{M}_{\lambda^\vee}) \neq 0$$

which gives a non-trivial pairing, $\langle\cdot,\cdot\rangle : \iota^*\mathcal{M}_\mu \times \mathcal{M}_\lambda \longrightarrow \mathbb{C}$ which in turn induces a pairing at the level of sheaves:

$$\langle\cdot,\cdot\rangle : \iota^*\widetilde{\mathcal{M}}_\mu \times \phi^*\widetilde{\mathcal{M}}_\lambda \longrightarrow \underline{\mathbb{C}},$$

where $\underline{\mathbb{C}}$ is the constant sheaf corresponding to $\mathbb{C}$. Now by composing this map with the $\wedge$-map gives

$$\langle\cdot,\cdot\rangle \circ \wedge : H_{\mathrm{c}}^{\dim(\tilde{S}_2(R_f))}(\tilde{S}_2(R_f), \iota^*\widetilde{\mathcal{M}}_\mu \times \phi^*\widetilde{\mathcal{M}}_\lambda) \longrightarrow H_c^{\dim(\tilde{S}_2(R_f))}(\tilde{S}_2(R_f), \underline{\mathbb{C}}).$$

5.2 The global pairing

We now have a top-degree class on an orientable manifold. We fix an orientation, compatibly on all the connected components; this was called the Harder-Mahnkopf cycle in [30, Section 3.2.3] (see also [29, Section 2.5.3.3]), and defined therein as

$$C(R_f) = \frac{1}{\mathrm{Vol}(R_f)} \sum_{x \in \mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_{>0} \det(R_f)} [\vartheta_{x,R_f}].$$

The action of $\delta_2 = (-1,1)$ on this cycle $C(R_f)$ is given by $r_{\delta_2}^* C(R_f) = (-1)C(R_f)$. Here $\delta_2 \in \mathrm{O}(2)/\mathrm{SO}(2)$ and this action is relevant only in the totally real case; in the CM case since $\mathrm{U}(2)$ is connected, $\pi_0(K_{2,\infty})$ is trivial. We can define the global pairing as:

$$\langle \vartheta_{\Pi,\epsilon_\Pi}, \vartheta_\Sigma \rangle_{C(R_f)} = \int_{C(R_f)} \iota^*\vartheta_{\Pi,\epsilon_\Pi} \wedge \phi^*\vartheta_\Sigma. \tag{5.4}$$

To evaluate this global pairing, we will write the cohomology classes as differential forms, and as in [30, Section 3.2.5], but before we evaluate the global pairing we will need to discuss an analogous pairing involving the $(\mathfrak{g}_\infty, K_\infty^0)$-classes at infinity.

The pairing at infinity

(See [29, Section 2.5.3.6].) Recall, that we have fixed $[\Pi_\infty]$ a basis of the one-dimensional space $H^{b_3^F}(\mathfrak{g}_{3,\infty},\ K^0_{3,\infty}; \Pi_\infty \otimes \mathcal{M}_{\mu,\mathbb{C}})(\epsilon_\Pi)$, and similarly, we have $[\Sigma_\infty]$ generating the one-dimensional space $H^{b_2^F}(\mathfrak{g}_{2,\infty},\ K^0_{2,\infty}; \Sigma_\infty(\chi_1,\chi_2) \otimes \mathcal{M}_{\lambda,\mathbb{C}})$. Define

$$d_n^F = \dim(\mathfrak{g}_{n,\infty}/\mathfrak{k}_{n,\infty}).$$

To compute the pairing at infinity, we choose a basis $\{\mathrm{y}_j : 1 \le j \le d_2^F\}$ of $(\mathfrak{g}_{2,\infty}/\mathfrak{k}_{2,\infty})^*$ such that $\{\mathrm{y}_j : 1 \le j \le d_2^F - 1\}$ is a basis of $(\mathfrak{g}_{2,\infty}/\mathrm{Lie}(C_{2,\infty}))^*$. Next, fix a basis $\{\mathrm{x}_i : 1 \le i \le d_3^F\}$ of $(\mathfrak{g}_{3,\infty}/\mathfrak{k}_{3,\infty})^*$, such that $\iota^*\mathrm{x}_j = \mathrm{y}_j$ for all $1 \le j \le d_2^F - 1$, and $\iota^*\mathrm{x}_j = 0$ if $j \ge d_2^F$. We further note that $\mathrm{y}_1 \wedge \mathrm{y}_2 \wedge \cdots \wedge \mathrm{y}_{d_2^F}$ corresponds to a $G_2(\mathbb{R})^0$-invariant measure on $\tilde{S}_2(R_f)$. Let $\{m_\alpha\}$ (resp., $\{m_\beta\}$) be a $\mathbb{Q}$-basis for $\mathcal{M}_\mu$ (resp., $\mathcal{M}_\lambda$).
The class $[\Pi_\infty]$ is represented by a $K^0_{3,\infty}$-invariant element in $\wedge^{b_3^F}(\mathfrak{g}_{3,\infty}/\mathfrak{k}_{3,\infty})^* \otimes \mathcal{W}(\Pi_\infty) \otimes \mathcal{M}_{\mu,\mathbb{C}}$ which can be written as

$$[\Pi_\infty] = \sum_{\mathrm{i}=i_1<\cdots<i_{b_3^F}} \sum_\alpha \mathrm{x}_\mathrm{i} \otimes w_{\infty,\mathrm{i},\alpha} \otimes m_\alpha, \tag{5.5}$$

where $w_{\infty,\mathrm{i},\alpha} \in \mathcal{W}(\Pi_\infty, \psi_\infty)$. Similarly, $[\Sigma_\infty]$ is represented by a $K^0_{2,\infty}$-invariant element in

$$\wedge^{b_2^F}(\mathfrak{g}_{2,\infty}/\mathfrak{k}_{2,\infty})^* \otimes \Sigma_\infty(\chi_1,\chi_2) \otimes \mathcal{M}_\lambda$$

which we write as

$$[\Sigma_\infty] = \sum_{\mathrm{j}=j_1<\cdots<j_{b_2^F}} \sum_\beta \mathrm{y}_\mathrm{j} \otimes \varphi_{\infty,\mathrm{j},\beta} \otimes m_\beta. \tag{5.6}$$

Let $w_{\infty,\mathrm{j},\beta} \in \mathcal{W}(\Sigma_\infty(\chi_1,\chi_2), \psi_\infty^{-1})$ be the Whittaker vector corresponding to $\varphi_{\infty,\mathrm{j},\beta}$. We now define a pairing at infinity by

$$\langle [\Pi_\infty],\ [\Sigma_\infty] \rangle \ = \ \sum_{\mathrm{i},\mathrm{j}} s(\mathrm{i},\mathrm{j}) \sum_{\alpha,\beta} \langle m_\alpha, m_\beta \rangle \Psi_\infty(1/2, w_{\infty,\mathrm{i},\alpha}, w_{\infty,\mathrm{j},\beta}), \tag{5.7}$$

where $s(\mathrm{i},\mathrm{j}) \in \{0,-1,1\}$ is defined by $\iota^*\mathrm{x}_\mathrm{i} \wedge \mathrm{y}_\mathrm{j} = s(\mathrm{i},\mathrm{j})\mathrm{y}_1 \wedge \mathrm{y}_2 \wedge \cdots \wedge \mathrm{y}_{d_2^F}$. Recall that the zeta integral at infinity $\Psi_\infty(1/2, w_{\infty,\mathrm{i},\alpha}, w_{\infty,\mathrm{j},\beta})$ is defined only after meromorphic continuation. However, the assumption that $s = 1/2$ is critical ensures that they are all finite, hence $\langle[\Pi_\infty], [\Sigma_\infty]\rangle$ is finite.

Lemma 5.8 $\langle[\Pi_\infty], [\Sigma_\infty]\rangle \neq 0$.

Proof. It is easy to see that

$$\langle[\Pi_\infty], [\Sigma_\infty]\rangle = \prod_{v \in S_\infty} \langle[\Pi_v], [\Sigma_v]\rangle;$$

hence it is enough to prove non-vanishing locally for every $v \in S_\infty$. We will consider the totally real and CM cases separately:

- *If F is a totally real field:* As in Section 3.4.2, for the discrete series representation, it is well known that

  $$H^1(\mathfrak{gl}_2, \mathrm{SO}(2)\mathbb{R}_+^\times; D_{\lambda^\vee} \otimes \mathcal{M}_\lambda) \cong \mathbb{C}[D_\lambda]^+ \oplus \mathbb{C}[D_\lambda]^-.$$

  Recall from Equation (3.19) that $[D_\lambda]^+$ maps to $[\Sigma_v]$ under the map denoted $i^\bullet$ therein, and this map also kills $[D_\lambda]^-$. One can conclude that

  $$\langle[\Pi_v], [\Sigma_v]\rangle \ = \ \langle[\Pi_v], [D_\lambda]^+\rangle.$$

  Now Kasten and Schmidt [22] have proved $\langle[\Pi_v], [D_\lambda]^+\rangle \neq 0$, which proves the lemma.

- *If F is CM field:* Using Binyong Sun [41, Theorem C], after converting to our notations, we have

  $$H^{3d_0}(\mathfrak{gl}_2(\mathbb{C}), \mathrm{U}(2)\mathbb{C}^\times; \Pi_v \otimes \mathcal{M}_{\mu_v}^\vee \otimes \sigma_v \otimes \mathcal{M}_{\nu_v}^\vee) \neq 0.$$

  In other words

  $$\langle[\Pi_v], [\sigma_v]\rangle \neq 0,$$

  where $[\sigma_v] = H^{d_0}(\mathfrak{gl}_2(\mathbb{C}), \mathrm{U}(2)\mathbb{C}^\times; \sigma_v \otimes \mathcal{M}_{\nu_v}^\vee)$ is a generator of one-dimensional cohomology group. Here $\sigma_v \in \mathrm{Coh}(G_3, \nu)$ with $\nu = (\nu_v)_{v \in S_\infty}$ such that if

  $$\nu_v = (\nu_1, \nu_2;\ \nu_1^*, \nu_2^*)$$

then

$$\sigma_v = \mathrm{Ind}_{B_2(\mathbb{C})}^{\mathrm{GL}_2(\mathbb{C})}(z^{a_1}\bar{z}^{b_1} \otimes z^{a_2}\bar{z}^{b_2}),$$

where $(a_1, a_2) = \left(\nu_1 + \frac{1}{2}, \nu_2 - \frac{1}{2}\right)$ and $(b_1, b_2) = \left(\nu_2^* - \frac{1}{2}, \nu_1^* + \frac{1}{2}\right)$. On comparing σ_v with our $G_2(F_v)$-representation $\Sigma(\chi_{1v}, \chi_{2v})$ (see Section 3.4 for the definition of $\Sigma(\chi_{1v}, \chi_{2v})$) we get

$$\frac{2\sigma_1 - f_1 + 1}{2} = \nu_1 + \frac{1}{2}; \quad \frac{2\sigma_1 + f_1 + 1}{2} = \nu_2^* - \frac{1}{2}$$

and

$$\frac{2\sigma_2 - f_2 - 1}{2} = \nu_2 - \frac{1}{2}; \quad \frac{2\sigma_2 + f_2 - 1}{2} = \nu_1^* + \frac{1}{2}.$$

Simplifying these equations we get:

$$\begin{aligned}
\nu_v &= (\nu_1, \nu_2;\ \nu_1^*, \nu_2^*) \\
&= \left(\frac{2\sigma_1 - f_1}{2}, \frac{2\sigma_2 - f_2}{2};\ \frac{2\sigma_1 + f_1}{2} + 1, \frac{2\sigma_2 + f_2}{2} - 1\right) \\
&= (-\lambda_2, -\lambda_1;\ -\lambda_2^*, -\lambda_1^*) \\
&= \lambda_v^\vee
\end{aligned}$$

This implies $\sigma_v \otimes \mathcal{M}_{\nu_v}^\vee = \sigma_v \otimes \mathcal{M}_{\nu_v^\vee} = \Sigma_v \otimes \mathcal{M}_{\lambda_v}$. Hence we get

$$\langle [\Pi_v], [\Sigma_v] \rangle = \langle [\Pi_v], [\sigma_v] \rangle \neq 0,$$

which proves the lemma. □

In both cases we are now justified in making the definition:

$$P_\infty(\mu, \lambda) \ := \ \frac{1}{\langle [\Pi_\infty], [\Sigma_\infty] \rangle}. \tag{5.9}$$

5.3 L-value as a global pairing of cohomology classes

Using Equations (5.1) and (5.5) we write:

$$\vartheta_{\Pi, \epsilon_\Pi} = \sum_{\mathrm{i}} \sum_{\alpha} \mathrm{x_i} \otimes \phi_{\mathrm{i},\alpha} \otimes m_\alpha,$$

where the cusp form $\phi_{\mathrm{i},\alpha}$ in the ψ -Whittaker model of Π, looks like $w_{\Pi_f} \otimes w_{\infty,\mathrm{i},\alpha}$; recall from Equation (4.12) that w_{Π_f} corresponds to ϕ_f. Similarly, using Equations (3.22) and (5.6), we may write

$$\vartheta^{\circ}_{\Sigma} = \sum_{\mathrm{j}} \sum_{\beta} \mathrm{y}_{\mathrm{j}} \otimes E_{\mathrm{j},\beta} \otimes m_{\beta},$$

where the Eisenstein series $E_{\mathrm{j},\beta}$ is constructed by taking Eisenstein summation for the function $\varphi_f \otimes \varphi_{\infty,\mathrm{j},\beta}$ in the full induced representation $\Sigma(\chi_1, \chi_2)$, with φ_f as is in Equation (4.11). We get the global pairing

$$\begin{aligned} \langle \vartheta_{\Pi,\epsilon_\Pi}, \vartheta^{\circ}_{\Sigma} \rangle_{C(R_f)} &= \textstyle\sum_{\mathrm{i,j}} \sum_{\alpha,\beta} s(\mathrm{i},\mathrm{j}) \langle m_\alpha, m_\beta \rangle \int_{\tilde{S}_2(R_f)} \phi_{\mathrm{i},\alpha}(\iota(g)) E_{\mathrm{j},\beta}(g)\, dg \\ &= \mathrm{vol}(R_f) \textstyle\sum_{\mathrm{i,j}} \sum_{\alpha,\beta} s(\mathrm{i},\mathrm{j}) \langle m_\alpha, m_\beta \rangle I(\tfrac{1}{2}, \phi_{\mathrm{i},\alpha}, E_{\mathrm{j},\beta}) \end{aligned}$$

The second equality is because of ϕ_f and φ_f are both R_f invariant, and also by our normalization of measure in Section 2.1 that $\mathrm{vol}(\mathrm{SO}(n)) = 1$ and $\mathrm{vol}(\mathrm{U}(n)) = 1$. Using Equation (4.13) we get:

$$\langle \vartheta_{\Pi,\epsilon_\Pi}, \vartheta^{\circ}_{\Sigma} \rangle_{C(R_f)} = \frac{1}{P_\infty(\mu,\lambda)} \cdot \frac{\mathrm{vol}(R_f) A_\Sigma \cdot V_\Sigma \cdot L_\Sigma}{L_{S_\Sigma}(\frac{1}{2}, \Pi \times \Sigma)} \cdot \frac{L_f(\frac{1}{2}, \Pi \times \Sigma(\chi_1,\chi_2))}{L_f(2, \chi_1\chi_2^{-1})}. \tag{5.10}$$

Dividing by the period $p^{\epsilon_\Pi}(\Pi)$ to get the rational class $\vartheta^{\circ}_{\Pi,\epsilon_\Pi}$ now proves the following:

Theorem 5.11 *Let* $\Pi \in Coh(G_3, \mu)$ *with* $\mu \in X_0^+(T_3)$, *and* $\Sigma(\chi_1, \chi_2)$ *be the induced representation of* $\mathrm{GL}_2(\mathbb{A}_{\mathbb{F}})$ *as in Section 3.4. Separate the cases as follows:*

F is totally real. *Let* $\lambda \in X_0^+(T_2)$ *be the dominant integral 'parallel' weight determined by* $d_1, d_2, \chi_1^{\circ}, \chi_2^{\circ}$ *as:* $\lambda = (\lambda_v)_{v \in S_r}$, *where each* $\lambda_v = (-d_2, -d_1)$ *such that* $e_{iv} \not\equiv d_i \pmod 2$, *and*

F is a CM field. *Let* $\lambda \in X^+(T_2)$ *be the dominant integral 'parallel' weight determined by* $\sigma_1, \sigma_2, \chi_1^1, \chi_2^1$ *as:* $\lambda = (\lambda_v)_{v \in S_c}$, *where each* $\lambda_v = (\lambda_v^{\iota},\ \lambda_v^{\bar\iota})$; *with* $\lambda_v^{\iota} = \left(\frac{f_2}{2} - \sigma_2,\ \frac{f_1}{2} - \sigma_1\right)$, *and* $\lambda_v^{\bar\iota} = \left(-\left(\frac{f_1}{2} + \sigma_1\ + 1\right),\ 1 - \frac{f_2}{2} - \sigma_2\right)$.

Assume that $s = 1/2$ is critical for $L(s, \Pi \times \Sigma(\chi_1, \chi_2))$ and that $\mu \succ \lambda^\vee$. Then there exist nonzero complex numbers $P_\infty(\mu, \lambda)$ and $p^{\epsilon_\Pi}(\Pi)$ such that

$$\frac{L_f(\frac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2))}{P_\infty(\mu, \lambda)\, p^{\epsilon_\Pi}(\Pi)\, L_f(2, \chi_1\chi_2^{-1})} = (\mathrm{vol}(R_f)V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}.$$

This already shows that the left hand side is algebraic. Moreover we can study the action of the Galois group of $\mathbb{Q}$ on the quantities.

5.4 The main identity for the critical values $L_f(m, \Pi \otimes \chi)$

F is totally real

Now recall the fact that the L-value at $s = 1/2$ attached to the pair of representations $(\Pi, \Sigma(\chi_1, \chi_2))$ decomposes as

$$\begin{aligned} L_f(\tfrac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2)) &= L_f(1, \Pi \otimes \chi_1) L_f(0, \Pi \otimes \chi_2) \\ &= L_f(1 + d_1, \Pi \otimes \chi_1^0) L_f(d_2, \Pi \otimes \chi_2^0). \end{aligned}$$

Now we consider the four cases as delineated in Proposition 4.19. Before getting into details, let's comment that, using Equation (3.5), in each case the L-value $L(2, \chi_1\chi_2^{-1})$ in the denominator of the left hand side of the above theorem, which is the same as $L(2+d_1-d_2, \chi_1^0\chi_2^{0-1})$, is in fact a critical value of a classical Dirichlet L-function of a finite order Hecke character of F for the choices of d_j and χ_j^0, $j = 1, 2$.

Case 1. χ is even, that is, $\varepsilon_{\chi_v} = \mathbb{1}$ for all $v \in S_\infty$.

Case 1a. $m \in \{2, 4, \ldots, n_{\mathrm{ev}}\}$. Take $d_1 = m - 1$, $d_2 = -1$, $\chi_1^0 = \chi$, and $\chi_2^0 = \mathbb{1}$. Put $\lambda_v = (1, 1-m)$. Then, Theorem 5.11 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega_r^+(\Pi)\, L_f(2+m, \chi)}$$
$$= (\mathrm{vol}(R_f)V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}, \quad (5.12)$$

where the modified period is defined as

$$\Omega_r^+(\Pi) \;:=\; p^{\epsilon_\Pi}(\Pi) L_f(-1, \Pi)^{-1}. \quad (5.13)$$

Case 1b. $m \in \{1 - n_{\mathrm{ev}}, \dots, -3, -1\}$. Take $d_1 = 1$, $d_2 = m$, $\chi_1^0 = 1\!\!1$, and $\chi_2^0 = \chi$. Put $\lambda_v = (-m, -1)$. In this case Theorem 5.11 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega_l^+(\Pi)\, L_f(3-m, \chi^{-1})}$$
$$= (\mathrm{vol}(R_f)V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}, \quad (5.14)$$

where the modified period is defined as

$$\Omega_l^+(\Pi) \;:=\; p^{\epsilon_\Pi}(\Pi) L_f(2, \Pi)^{-1}. \quad (5.15)$$

Case 2. χ is odd, that is, $\varepsilon_{\chi_v} = \mathrm{sgn}$ for all $v \in S_\infty$. In this case, we fix once and for all, a totally odd quadratic Hecke character ξ of F.

Case 2a. $m \in \{1, 3, \dots, n_{\mathrm{od}}\}$. Take $d_1 = m - 1$, $d_2 = 0$, $\chi_1^0 = \chi$, and $\chi_2^0 = \xi$. Put $\lambda_v = (0, 1-m)$. Then Theorem 5.11 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega_r^-(\Pi)\, L_f(m+1, \chi\xi^{-1})}$$
$$= (\mathrm{vol}(R_f)V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}, \quad (5.16)$$

where the modified period is defined as

$$\Omega_r^-(\Pi) \;:=\; p^{\epsilon_\Pi}(\Pi) L_f(0, \Pi \otimes \xi)^{-1}. \quad (5.17)$$

Case 2b. $m \in \{1 - n_{\mathrm{od}}, \dots, -2, 0\}$. Take $d_1 = 0$, $d_2 = m$, $\chi_1^0 = \xi$, and $\chi_2^0 = \chi$. Put $\lambda_v = (-m, 0)$. Then, Theorem 5.11 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega_l^-(\Pi)\, L_f(2-m, \xi\chi^{-1})}$$
$$= (\mathrm{vol}(R_f) V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta^\circ_{\Pi, \epsilon_\Pi}, \vartheta^\circ_\Sigma \rangle_{C(R_f)}, \quad (5.18)$$

where the modified period is defined as

$$\Omega_l^-(\Pi) \ := \ p^{\epsilon_\Pi}(\Pi) L_f(1, \Pi \otimes \xi)^{-1}. \quad (5.19)$$

F is CM field

Before getting into the details observe that, since there are no real places, signature doesn't appear in the period $p^{\epsilon_\Pi}(\Pi)$ and $\vartheta^\circ_{\Pi,\epsilon_\Pi}$. Hence we denote it as $p(\Pi)$ and ϑ°_Π respectively. Furthermore we prove a lemma regarding the criticality of the L-value $L(2, \chi_1\chi_2^{-1})$ in the denominator of the left hand side of the above theorem, which is the same as $L(2 + \sigma_1 - \sigma_2,\ \chi_1^1\ (\chi_2^1)^{-1})$.

Lemma 5.20 $L(2 + \sigma_1 - \sigma_2,\ \chi_1^1\ (\chi_2^1)^{-1})$ *is indeed a critical value of L-function attached to the unitary algebraic Hecke character* $\chi_1^1\ (\chi_2^1)^{-1}$ *of* F *for the choices of* σ_j *and* χ_j^1, $j = 1, 2$.

Proof. Consider the two cases as delineated in Proposition 4.19.

Case 1. Suppose t is totally positive, $n_2 \leq -2t$ and $n_1 \geq 1$. Take $f_1 = -2$, $\sigma_1 = -1$, $f_2 = 2t$, and $\sigma_2 = m$. Then the L-value

$$L(2 + \sigma_1 - \sigma_2, \chi_1^1(\chi_2^1)^{-1}) = L(1 - m, \chi^\circ),$$

with χ° is such that $\chi^\circ_\infty(z) = \left(\frac{z}{|z|}\right)^{2+2t}$. Now using Section 3.2, $1 - m$ is critical for χ° if

$$-t \leq 1 - m \leq 1 + t,$$

that is, $-t \leq m \leq 1 + t$. We check this condition for the following three sub-cases:

- For $0 \leq n_1 \leq t-2, \quad m \in [2+n_1-t,\ t-n_1-1]$. This gives a series of inequaities

$$-t < 2+n_1-t \leq m \leq t-n_1-1 < t < 1+t.$$

- For $t \leq n_1 \leq 2t-1, \quad m \in [t-n_1,\, n_1+1-t]$. This implies

$$-t < -t+1 < t-n_1 \leq m \leq n_1+1-t \leq t-1 < t+1.$$

- For $n_1 \geq 2t, \quad m \in [1-t,\ t]$, which immediately gives $-t \leq m \leq 1+t$.

Hence in all the above sub-cases, $1-m$ is critical for χ°.

Case 2b. t is totally negative, $n_1 \geq -2t$ and $n_2 \leq -1$. Take $f_1 = 2t$, $\sigma_1 = m-1$, $f_2 = 2$, and $\sigma_2 = 1$. Then the L-value

$$L(2+\sigma_1-\sigma_2, \chi_1^1(\chi_2^1)^{-1}) = L(m, \chi^\circ),$$

with χ° is such that $\chi_\infty^\circ(z) = \left(\frac{z}{|z|}\right)^{2-2t}$. Again using Section 3.2, m is critical for χ° if

$$t \leq m \leq 1-t.$$

Again we need to check this condition for the following three sub-cases:

- For $t+2 \leq n_2 \leq 0, \quad m \in [2-n_2+t,\ n_2-1-t]$. This gives a series of inequalities

$$t < 2-n_2+t \leq m \leq n_2-1-t < 1-t.$$

- For $2t+1 \leq n_2 \leq t, \quad m \in [n_2-t,\, 1+t-n_2]$. This implies

$$t < t+1 < n_2-t \leq m \leq 1+t-n_2 \leq -t-1 < 1-t.$$

- For $n_2 \leq 2t, \quad m \in [1+t,\ -t]$, which gives

$$t < 1+t \leq m \leq -t < 1-t.$$

In all the above sub-cases, m is critical for χ°.

Hence in each case the L-value $L(2, \chi_1\chi_2^{-1})$ is in fact a critical value. $\square$

Now recall the fact that the L-value at $s = 1/2$ attached to the pair of representations $(\Pi, \Sigma(\chi_1, \chi_2))$ decomposes as

$$\begin{aligned} L_f(\tfrac{1}{2}, \Pi \times \Sigma(\chi_1, \chi_2)) &= L_f(1, \Pi \otimes \chi_1) L_f(0, \Pi \otimes \chi_2) \\ &= L_f(1 + \sigma_1, \Pi \otimes \chi_1^1) L_f(\sigma_2, \Pi \otimes \chi_2^1). \end{aligned}$$

Case 1. t is strictly positive, $n_2 \le -2t$ and $n_1 \ge 1$ for all $v \in S_\infty$.
In this case, we fix once and for all, a unitary Hecke character ϕ of F such that $\phi_\infty(z) = \left(\frac{z}{|z|}\right)^2$. An integer m satisfies Equation (4.17). Take $\chi_1^1 = \phi$, $\sigma_1 = -1$, $\chi_2^1 = \chi$ and $\sigma_2 = m$; put $\lambda_v = (t - m,\ 0;\ 1,\ 1 - t - m)$. Then $f_1 = -2$ and $f_2 = 2t$. Hence Theorem 5.11 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega^+(\Pi)\, L_f(1 - m, \phi\chi^{-1})} = \tag{5.21}$$
$$(\mathrm{vol}(R_f)V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta_\Pi^\circ, \vartheta_\Sigma^\circ \rangle_{C(R_f)},$$

where the modified period is defined as

$$\Omega^+(\Pi) \ := \ p(\Pi) L_f(0, \Pi \otimes \phi)^{-1}. \tag{5.22}$$

Case 2. t is negative, $n_1 \ge -2t$ and $n_2 \le -1$ for all $v \in S_\infty$.
Observe that the inverse ϕ^{-1} of above defined unitary Hecke character of F satisfies $\phi_\infty^{-1}(z) = \left(\frac{z}{|z|}\right)^{-2}$. Also an integer m satisfies Equation (4.18). Take $\chi_1^1 = \chi$, $\sigma_1 = m - 1$, $\chi_2^1 = \phi^{-1}$, and $\sigma_2 = 1$; put $\lambda_v = (0,\ t - m + 1;\ -t - m,\ -1)$. Then $f_1 = 2t$ and $f_2 = 2$. Hence Theorem 5.11 takes the form:

$$\frac{L_f(m, \Pi \otimes \chi)}{P_\infty(\mu, m)\, \Omega^-(\Pi)\, L_f(m, \chi\phi)} \;=$$

$$(\mathrm{vol}(R_f) V_\Sigma) \cdot (L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)) \cdot A_\Sigma \cdot \langle \vartheta_\Pi^\circ, \vartheta_\Sigma^\circ \rangle_{C(R_f)}, \tag{5.23}$$

where the modified period is defined as

$$\Omega^-(\Pi) \;:=\; p(\Pi) L_f(1, \Pi \otimes \phi^{-1})^{-1}. \tag{5.24}$$

Chapter 6

Galois equivariance

6.1 The action of $\mathrm{Aut}(\mathbb{C})$

We study Galois equivariance, i.e., behaviour under the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ of all the quantities in the main identity for each of the four cases in both the theorems. Let's parse the Galois action on the various ingredients involved in the main identities:

• The Poincaré duality pairing $\langle\ ,\ \rangle$ is Galois-equivariant. (See, for example, [30, Proposition 3.14].)

• The Galois action on the class $\vartheta^{\circ}_{\Pi,\epsilon_\Pi}$. Due to our specific choice of finite Whittaker vectors w_{Π_f}, exactly as in [30, Proposition 3.15], we get

$$\sigma(\vartheta^{\circ}_{\Pi,\epsilon_\Pi}) \;=\; \frac{\sigma(\mathcal{G}(\omega_{\Sigma_f}))}{\mathcal{G}(\omega_{{}^\sigma\Sigma_f})}\vartheta^{\circ}_{{}^\sigma\Pi,\epsilon_{{}^\sigma\Pi}} \;=\; \frac{\sigma(\mathcal{G}(\chi_1\chi_2))}{\mathcal{G}({}^\sigma\chi_1{}^\sigma\chi_2)}\vartheta^{\circ}_{{}^\sigma\Pi,\epsilon_{{}^\sigma\Pi}}.$$

Furthermore, for Dirichlet characters χ_1 and χ_2, it's well-known (see [39, Lemma 8]) that

$$\sigma\left(\frac{\mathcal{G}(\chi_1\chi_2)}{\mathcal{G}(\chi_1)\mathcal{G}(\chi_2)}\right) = \frac{\mathcal{G}({}^\sigma\chi_1{}^\sigma\chi_2)}{\mathcal{G}({}^\sigma\chi_1)\mathcal{G}({}^\sigma\chi_2)}.$$

Putting the above two together we get

$$\sigma(\vartheta^{\circ}_{\Pi,\epsilon_\Pi}) \;=\; \frac{\sigma(\mathcal{G}(\chi_1)\mathcal{G}(\chi_2))}{\mathcal{G}({}^\sigma\chi_1)\mathcal{G}({}^\sigma\chi_2)}\,\vartheta^{\circ}_{{}^\sigma\Pi,\epsilon_{{}^\sigma\Pi}}. \tag{6.1}$$

• To understand the Galois action on the class $\vartheta^{\circ}_{\Sigma}$, we begin with the function φ_f which is the finite part of φ_{χ_1,χ_2} as defined in Equation (4.11). Let's denote φ_f also as $\varphi_{\chi_{1f},\chi_{2f}}$. Now, $\sigma \in \mathrm{Aut}(\mathbb{C})$ acts on φ_f by acting on all its local components. The action of $\mathrm{Aut}(\mathbb{C})$ on Σ_v is given by acting on the values of a function in the induced space (see [26, Section 1.1]). For the local components of $\varphi_{\chi_{1f},\chi_{2f}}$ (with notations suitably modified) we get for the spherical vectors:

$$ {}^{\sigma}f_v^{\mathrm{sp}}(\chi_{1v},\chi_{2v}) \;=\; f_v^{\mathrm{sp}}({}^{\sigma}\chi_{1v},{}^{\sigma}\chi_{2v}), $$

and from our choices of new vectors made in Proposition 4.3, we get for $v \in S_{\Sigma} \setminus S_{\chi_2}$:

$$ {}^{\sigma}f_v^{\mathrm{new}}(\chi_{1v},\chi_{2v}) \;=\; f_v^{\mathrm{new}}({}^{\sigma}\chi_{1v},{}^{\sigma}\chi_{2v}), $$

however, for $v \in S_{\chi_2}$–the set of ramified primes for χ_2, we get:

$$ {}^{\sigma}f_v^{\mathrm{new}}(\chi_{1v},\chi_{2v}) \;=\; \frac{\sigma(q_v^{n_2/2})}{q_v^{n_2/2}} f_v^{\mathrm{new}}({}^{\sigma}\chi_{1v},{}^{\sigma}\chi_{2v}). $$

Note that the quantity $\sigma(q_v^{n_2/2})/q_v^{n_2/2}$ is ± 1. Putting these together we get

$$ {}^{\sigma}\varphi_{\chi_{1f},\chi_{2f}} \;=\; \left(\prod_{v \in S_{\chi_2}} \frac{\sigma(q_v^{n_2/2})}{q_v^{n_2/2}}\right) \varphi_{{}^{\sigma}\chi_{1f},{}^{\sigma}\chi_{2f}}. $$

Since the Eisenstein map $\mathcal{F}_{\Sigma}$ in Equation (3.22) is $\mathrm{Aut}(\mathbb{C})$-equivariant, we get

$$ \sigma(\vartheta^{\circ}_{\Sigma}) \;=\; \left(\prod_{v \in S_{\chi_2}} \frac{\sigma(q_v^{n_2/2})}{q_v^{n_2/2}}\right) \vartheta^{\circ}_{{}^{\sigma}\Sigma}. \tag{6.2} $$

• Now we look at Galois action on the quantity A_{Σ}. Recall from Equation (4.7) that $A_{\sigma} = \prod_{v \in S_{\Sigma}} A_v$ and the values of A_v are computed in Proposition

4.6. Now the volume of $\mathcal{O}_v$, by our choice of measures, is rational. We easily deduce that

$$\sigma(A_\Sigma) \;=\; \left(\prod_{v\in S_{\chi_2}} \frac{\sigma(q_v^{-n_2/2})}{q_v^{-n_2/2}}\right)\cdot \frac{\sigma(\mathcal{G}(\chi_2))}{\mathcal{G}({}^\sigma\chi_2)}\, A_{{}^\sigma\Sigma}. \tag{6.3}$$

From Equations (6.2) and (6.3) we get

$$\sigma(A_\Sigma\,\vartheta_\Sigma^\circ) \;=\; \frac{\sigma(\mathcal{G}(\chi_2))}{\mathcal{G}({}^\sigma\chi_2)}\, A_{{}^\sigma\Sigma}\,\vartheta_{{}^\sigma\Sigma}^\circ. \tag{6.4}$$

• The quantities L_Σ and $L_{S_\Sigma}(\frac{1}{2},\Pi\times\Sigma)$ are Galois equivariant as they are finite products of local critical L-values; this follows from [30, Proposition 3.17].

• The volume terms $\mathrm{vol}(R_f)$ and V_Σ are rational numbers by our choice of measures.

• Using Equation (3.6) we have, for a totally even Dirichlet character ϱ of F, and an even positive integer r, it's well-known that we have the following rationality result for the critical value $L_f(r,\varrho)$:

$$\sigma\left(\frac{L_f(r,\varrho)}{(2\pi i)^r\mathcal{G}(\varrho)}\right) \;=\; \frac{L_f(r,{}^\sigma\varrho)}{(2\pi i)^r\mathcal{G}({}^\sigma\varrho)}. \tag{6.5}$$

• Finally recall the Fact 3.8, we have an algebraicity result for the critical value r of $L_f(r,\chi)$ attached to the finite part of an algebraic Hecke character χ over a CM field, which can be rewritten as follows:

$$\left(\frac{L_f(r,\chi)}{(2\pi i)^{rd_0}c^+(\chi)}\right) = \frac{L_f(r,{}^\sigma\chi)}{(2\pi i)^{rd_0}c^+({}^\sigma\chi)}. \tag{6.6}$$

6.2 Proof of Theorem 1.2

Notice that for $\sigma \in \mathrm{Aut}(\mathbb{C})$ applied to the pairing of the cohomology classes, after using Galois-equivariance of the duality pairing, and after using Equations (6.1) and (6.4) we get

$$\sigma\left(\langle \vartheta^{\circ}_{\Pi,\epsilon_\Pi}, A_\Sigma\, \vartheta^{\circ}_{\Sigma}\rangle\right) \;=\; \langle \vartheta^{\circ}_{{}^\sigma\Pi,\epsilon_\Pi}, A_{{}^\sigma\Sigma}\, \vartheta^{\circ}_{{}^\sigma\Sigma}\rangle\, \frac{\sigma(\mathcal{G}(\chi_1)\mathcal{G}(\chi_2)^2)}{\mathcal{G}({}^\sigma\chi_1)\mathcal{G}({}^\sigma\chi_2)^2}. \tag{6.7}$$

Now keeping above Galois equivariant facts in mind, the details of the proof in all the four cases are as follows:

Case 1a. On multiplying and dividing by the factor $(2\pi i)^{2+m}\mathcal{G}(\chi)$, rewrite Equation (5.12) as

$$\frac{L_f(m,\Pi\otimes\chi)}{P^1_\infty(\mu,m)\,\Omega^+_r(\Pi)\mathcal{G}(\chi)}$$
$$= \;(\mathrm{vol}(R_f)V_\Sigma)\cdot\left(L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2},\Pi\times\Sigma)\right)\cdot\left(\frac{L_f(2+m,\chi)}{(2\pi i)^{2+m}\mathcal{G}(\chi)}\right)\cdot\langle \vartheta^{\circ}_{\Pi,\epsilon_\Pi}, A_\Sigma\, \vartheta^{\circ}_{\Sigma}\rangle,$$

where $P^1_\infty(\mu,m) = (2\pi i)^{2+m}P_\infty(\mu,m)$. Now apply $\sigma\in\mathrm{Aut}(\mathbb{C})$ to both sides. The first three parentheses on the right hand side are $\mathrm{Aut}(\mathbb{C})$-equivariant as explained above. In this case we have $\mathcal{G}(\chi_1) = \mathcal{G}(\chi_1^\circ) = \mathcal{G}(\chi)$ and $\mathcal{G}(\chi_2) = \mathcal{G}(\chi_2^\circ) = \mathcal{G}(\mathbb{1}) = 1$. Hence using Equation (6.7) we get

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P^1_\infty(\mu,m)\,\Omega^+_r(\Pi)\mathcal{G}(\chi)}\right)$$
$$= \;(\mathrm{vol}(R_f)V_{{}^\sigma\Sigma})\cdot\left(L_{{}^\sigma\Sigma}\, L_{S_\Sigma}(\tfrac{1}{2},{}^\sigma\Pi\times{}^\sigma\Sigma)\right)\cdot\left(\frac{L_f(2+m,{}^\sigma\chi)}{(2\pi i)^{2+m}\mathcal{G}({}^\sigma\chi)}\right)$$
$$\cdot\,\langle \vartheta^{\circ}_{{}^\sigma\Pi,\epsilon_\Pi}, A_{{}^\sigma\Sigma}\, \vartheta^{\circ}_{{}^\sigma\Sigma}\rangle\frac{\sigma(\mathcal{G}(\chi))}{\mathcal{G}({}^\sigma\chi)},$$

which is the same as

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P^1_\infty(\mu,m)\,\Omega^+_r(\Pi)\mathcal{G}(\chi)}\right) \;=\; \frac{L_f(m,{}^\sigma\Pi\otimes{}^\sigma\chi)}{P^1_\infty({}^\sigma\mu,m)\,\Omega^+_r({}^\sigma\Pi)\mathcal{G}({}^\sigma\chi)}\cdot\frac{\sigma(\mathcal{G}(\chi))}{\mathcal{G}({}^\sigma\chi)},$$

and which in turn may be rewritten as

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P^1_\infty(\mu,m)\,\Omega^+_r(\Pi)\mathcal{G}(\chi)^2}\right) \;=\; \frac{L_f(m,{}^\sigma\Pi\otimes{}^\sigma\chi)}{P^1_\infty({}^\sigma\mu,m)\,\Omega^+_r({}^\sigma\Pi)\mathcal{G}({}^\sigma\chi)^2},$$

proving the Galois-equivariant version. This implies in particular that

$$L_f(m,\Pi\otimes\chi)\;\approx_{\mathbb{Q}(\Pi,\chi)}\; P^1_\infty(\mu,m)\,\Omega^+_r(\Pi)\,\mathcal{G}(\chi)^2,$$

where $\approx_{\mathbb{Q}(\Pi,\chi)}$ is a number field which is the compositum of the rationality fields $\mathbb{Q}(\Pi)$ and $\mathbb{Q}(\chi)$.

Case 1b Rewrite Equation (5.14) as

$$\begin{aligned}
&\frac{L_f(m,\Pi\otimes\chi)}{P^2_\infty(\mu,m)\,\Omega^+_l(\Pi)\mathcal{G}(\chi)}\\
&= \;(\mathrm{vol}(R_f)V_\Sigma)\cdot\big(L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2},\Pi\times\Sigma)\big)\cdot\left(\frac{L_f(2+m,\chi)}{(2\pi i)^{3-m}\mathcal{G}(\chi)}\right)\cdot\langle\vartheta^\circ_{\Pi,\epsilon_\Pi},A_\Sigma\,\vartheta^\circ_\Sigma\rangle,
\end{aligned}$$

where $P^2_\infty(\mu,m)=(2\pi i)^{3-m}P_\infty(\mu,m)$. Now apply $\sigma\in\mathrm{Aut}(\mathbb{C})$ to both sides. As in Case 1a, using the above remarks along with the facts $\mathcal{G}(\chi_1)=\mathcal{G}(\chi_1^\circ)=\mathcal{G}(\mathbb{1})$ and $\mathcal{G}(\chi_2)=\mathcal{G}(\chi_2^\circ)=\mathcal{G}(\chi)$, applying σ to the pairing we get

$$\begin{aligned}
&\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P^2_\infty(\mu,m)\,\Omega^+_l(\Pi)\mathcal{G}(\chi^{-1})}\right)\\
&\quad= \;(\mathrm{vol}(R_f)V_{^\sigma\Sigma})\cdot\big(L_{^\sigma\Sigma}\, L_{S_\Sigma}(\tfrac{1}{2},{}^\sigma\Pi\times{}^\sigma\Sigma)\big)\cdot\left(\frac{L_f(3-m,{}^\sigma\chi^{-1})}{(2\pi i)^{2+m}\mathcal{G}({}^\sigma\chi^{-1})}\right)\\
&\qquad\qquad\qquad\cdot\langle\vartheta^\circ_{^\sigma\Pi,\epsilon_\Pi},A_{^\sigma\Sigma}\,\vartheta^\circ_{^\sigma\Sigma}\rangle\frac{\sigma(\mathcal{G}(\chi)^2)}{\mathcal{G}({}^\sigma\chi)^2},
\end{aligned}$$

which is the same as

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P^2_\infty(\mu,m)\,\Omega^+_l(\Pi)\mathcal{G}(\chi^{-1})}\right) \;=\; \frac{L_f(m,{}^\sigma\Pi\otimes{}^\sigma\chi^{-1})}{P^2_\infty({}^\sigma\mu,m)\,\Omega^+_r({}^\sigma\Pi)\mathcal{G}({}^\sigma\chi^{-1})}\cdot\frac{\sigma(\mathcal{G}(\chi)^2)}{\mathcal{G}({}^\sigma\chi)^2},$$

and which in turn may be rewritten as

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P^2_\infty(\mu,m)\,\Omega^+_l(\Pi)\mathcal{G}(\chi)}\right) \;=\; \frac{L_f(m,{}^\sigma\Pi\otimes{}^\sigma\chi)}{P^1_\infty({}^\sigma\mu,m)\,\Omega^+_l({}^\sigma\Pi)\mathcal{G}({}^\sigma\chi)},$$

proving the Galois-equivariant version. This implies in particular that

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P^1_\infty(\mu, m)\, \Omega^+_r(\Pi)\, \mathcal{G}(\chi).$$

Case 2a Rewrite Equation (5.16) as

$$\begin{aligned}
&\frac{L_f(m, \Pi \otimes \chi)}{P^1_\infty(\mu, m)\, \Omega^-_r(\Pi)\mathcal{G}(\chi)\mathcal{G}(\xi)^{-1}} \\
&\quad = \ (\mathrm{vol}(R_f)V_\Sigma) \cdot \big(L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)\big) \cdot \left(\frac{L_f(m+1, \chi\xi^{-1})}{(2\pi i)^{m+1}\mathcal{G}(\chi)\mathcal{G}(\xi)^{-1}}\right) \\
&\qquad\qquad \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, A_\Sigma\, \vartheta^\circ_\Sigma \rangle,
\end{aligned}$$

where $P^3_\infty(\mu, m) = (2\pi i)^{m+1} P_\infty(\mu, m)$. Clearly the first three parentheses on the right hand side are $\mathrm{Aut}(\mathbb{C})$-equivariant as explained earlier. Use Equation (6.7) and substitute the folloing: $\mathcal{G}(\chi_1) = \mathcal{G}(\chi_1^\circ) = \mathcal{G}(\chi)$ and $\mathcal{G}(\chi_2) = \mathcal{G}(\chi_2^\circ) = \mathcal{G}(\xi)$. For σ applied to the pairing of the cohomology classes we get

$$\begin{aligned}
&\sigma\left(\frac{L_f(m, \Pi \otimes \chi)}{P^3_\infty(\mu, m)\, \Omega^-_r(\Pi)\mathcal{G}(\chi)\mathcal{G}(\xi)^{-1}}\right) \\
&\quad = \ (\mathrm{vol}(R_f)V_{^\sigma\Sigma}) \cdot \big(L_{^\sigma\Sigma}\, L_{S_\Sigma}(\tfrac{1}{2}, {}^\sigma\Pi \times {}^\sigma\Sigma)\big) \cdot \left(\frac{L_f(m+1, {}^\sigma\chi {}^\sigma\xi^{-1})}{(2\pi i)^{m+1}\mathcal{G}({}^\sigma\chi)\mathcal{G}({}^\sigma\xi)^{-1}}\right) \cdot \\
&\qquad\qquad \cdot \langle \vartheta^\circ_{^\sigma\Pi,\epsilon_\Pi}, A_{^\sigma\Sigma}\, \vartheta^\circ_{^\sigma\Sigma} \rangle \frac{\sigma(\mathcal{G}(\chi)\mathcal{G}(\xi)^2}{\mathcal{G}({}^\sigma\chi)\mathcal{G}({}^\sigma\xi)^2},
\end{aligned}$$

which is the same as

$$\sigma\left(\frac{L_f(m, \Pi \otimes \chi)}{P^3_\infty(\mu, m)\, \Omega^-_r(\Pi)\mathcal{G}(\chi)}\right) \ = \ \frac{L_f(m, {}^\sigma\Pi \otimes {}^\sigma\chi)}{P^3_\infty({}^\sigma\mu, m)\, \Omega^-_r({}^\sigma\Pi)\mathcal{G}({}^\sigma\chi)} \cdot \frac{\sigma(\mathcal{G}(\chi)\mathcal{G}(\xi)}{\mathcal{G}({}^\sigma\chi)\mathcal{G}({}^\sigma\xi)},$$

and which proves the theorem, since

$$L_f(m, \Pi \otimes \chi) \ \approx_{\mathbb{Q}(\Pi,\chi)} \ P^3_\infty(\mu, m)\, \Omega^-_r(\Pi)\, \mathcal{G}(\chi)^2\mathcal{G}(\xi).$$

Case 2b. Rewrite Equation (5.12) as

$$\frac{L_f(m, \Pi \otimes \chi)}{P^4_\infty(\mu, m)\,\Omega^-_l(\Pi)\mathcal{G}(\xi)\mathcal{G}(\chi)^{-1}}$$

$$= (\mathrm{vol}(R_f)V_\Sigma) \cdot \left(L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)\right) \cdot \left(\frac{L_f(2-m, \xi\chi^{-1})}{(2\pi i)^{2-m}\mathcal{G}(\xi)\mathcal{G}(\chi) - 1}\right) \cdot \langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, A_\Sigma\, \vartheta^\circ_\Sigma\rangle,$$

where $P^4_\infty(\mu, m) = (2\pi i)^{2-m} P_\infty(\mu, m)$. Now apply $\sigma \in \mathrm{Aut}(\mathbb{C})$ to both sides. The first three parentheses on the right hand side are $\mathrm{Aut}(\mathbb{C})$-equivariant as explained earlier. Applying σ to the pairing of the cohomology classes, and using $\mathcal{G}(\chi_1) = \mathcal{G}(\chi_1^\circ) = \mathcal{G}(\xi)$ and $\mathcal{G}(\chi_2) = \mathcal{G}(\chi_2^\circ) = \mathcal{G}(\chi)$ we get

$$\sigma\left(\langle \vartheta^\circ_{\Pi,\epsilon_\Pi}, A_\Sigma\, \vartheta^\circ_\Sigma\rangle\right) = \langle \vartheta^\circ_{{}^\sigma\Pi,\epsilon_\Pi}, A_{{}^\sigma\Sigma}\, \vartheta^\circ_{{}^\sigma\Sigma}\rangle\, \frac{\sigma(\mathcal{G}(\xi)\mathcal{G}(\chi)^2)}{\mathcal{G}({}^\sigma\xi)\mathcal{G}({}^\sigma\chi)^2}.$$

Hence

$$\sigma\left(\frac{L_f(m, \Pi \otimes \chi)}{P^4_\infty(\mu, m)\,\Omega^-_l(\Pi)\mathcal{G}(\xi)\mathcal{G}(\chi)^{-1}}\right)$$

$$= (\mathrm{vol}(R_f)V_{{}^\sigma\Sigma}) \cdot \left(L_{{}^\sigma\Sigma}\, L_{S_\Sigma}(\tfrac{1}{2}, {}^\sigma\Pi \times {}^\sigma\Sigma)\right) \cdot \left(\frac{L_f(2-m, {}^\sigma\chi)}{(2\pi i)^{2-m}\mathcal{G}({}^\sigma\xi)\mathcal{G}({}^\sigma\chi)^{-1}}\right) \cdot \cdot \langle \vartheta^\circ_{{}^\sigma\Pi,\epsilon_\Pi}, A_{{}^\sigma\Sigma}\, \vartheta^\circ_{{}^\sigma\Sigma}\rangle \frac{\sigma(\mathcal{G}(\xi)\mathcal{G}(\chi)^2)}{\mathcal{G}({}^\sigma\xi)\mathcal{G}({}^\sigma\chi)^2}.$$

Since ξ is quadratic, $\mathcal{G}(\xi)^2 = \mathcal{G}(\xi^2) = \mathcal{G}(\mathbb{1}) = 1$. We get

$$\sigma\left(\frac{L_f(m, \Pi \otimes \chi)}{P^4_\infty(\mu, m)\,\Omega^-_l(\Pi)\mathcal{G}(\chi)}\right) = \frac{L_f(m, {}^\sigma\Pi \otimes {}^\sigma\chi)}{P^4_\infty({}^\sigma\mu, m)\,\Omega^-_l({}^\sigma\Pi)\mathcal{G}({}^\sigma\chi)},$$

proving the Galois-equivariant version. This implies in particular that

$$L_f(m, \Pi \otimes \chi) \approx_{\mathbb{Q}(\Pi,\chi)} P^4_\infty(\mu, m)\,\Omega^-_l(\Pi)\,\mathcal{G}(\chi).$$

This concludes the proof of Theorem 1.2.

6.3 Proof of Theorem 1.3

The line of proof is exactly same for both the cases. The details are as follows:

Case 1. On multiplying and dividing by $(2\pi i)^{(1-m)d_0} c^+(\phi\chi^{-1})$, Equation (5.21) may be rewritten as

$$\frac{L_f(m, \Pi \otimes \chi)}{P^+_\infty(\mu, m)\, \Omega^+(\Pi) c^+(\phi\chi^{-1})}$$
$$= (\mathrm{vol}(R_f) V_\Sigma) \cdot \left(L_\Sigma\, L_{S_\Sigma}(\tfrac{1}{2}, \Pi \times \Sigma)\right) \cdot \left(\frac{L_f(1-m, \phi\chi^{-1})}{(2\pi i)^{(1-m)d_0} c^+(\phi\chi^{-1})}\right) \cdot \langle \vartheta^\circ_\Pi, A_\Sigma\, \vartheta^\circ_\Sigma\rangle,$$

where $P^+_\infty(\mu, m) = (2\pi i)^{(1-m)d_0} P_\infty(\mu, m)$. Now apply $\sigma \in \mathrm{Aut}(\mathbb{C})$ to both sides. Using above facts, the first three parentheses on the right hand side of the equation are $\mathrm{Aut}(\mathbb{C})$-equivariant. Applying σ to the pairing of the cohomology classes, after using Galois-equivariance of the duality pairing, as well as using Equations (6.1) and (6.4) we get

$$\sigma\left(\langle \vartheta^\circ_\Pi, A_\Sigma\, \vartheta^\circ_\Sigma\rangle\right) = \langle \vartheta^\circ_{{}^\sigma\Pi}, A_{{}^\sigma\Sigma}\, \vartheta^\circ_{{}^\sigma\Sigma}\rangle \frac{\sigma(\mathcal{G}(\chi_1)\mathcal{G}(\chi_2)^2)}{\mathcal{G}({}^\sigma\chi_1)\mathcal{G}({}^\sigma\chi_2)^2}. \tag{6.8}$$

In this case we have $\mathcal{G}(\chi_1) = \mathcal{G}(\chi_1^1) = \mathcal{G}(\phi)$ and $\mathcal{G}(\chi_2) = \mathcal{G}(\chi_2^1) = \mathcal{G}(\chi)$. Hence using the above substitutions we get

$$\sigma\left(\frac{L_f(m, \Pi \otimes \chi)}{P^+_\infty(\mu, m)\, \Omega^+(\Pi) c^+(\phi\chi^{-1})}\right)$$
$$= (\mathrm{vol}(R_f) V_{{}^\sigma\Sigma}) \cdot \left(L_{{}^\sigma\Sigma}\, L_{S_\Sigma}(\tfrac{1}{2}, {}^\sigma\Pi \times {}^\sigma\Sigma)\right) \cdot \left(\frac{L_f(1-m, {}^\sigma\phi^\sigma\chi^{-1})}{(2\pi i)^{(1-m)d_0} c^+({}^\sigma\phi^\sigma\chi^{-1})}\right) \cdot \langle \vartheta^\circ_{{}^\sigma\Pi}, A_{{}^\sigma\Sigma}\, \vartheta^\circ_{{}^\sigma\Sigma}\rangle \frac{\sigma(\mathcal{G}(\phi)\mathcal{G}(\chi)^2)}{\mathcal{G}({}^\sigma\phi)\mathcal{G}({}^\sigma\chi)^2},$$

which is the same as

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P_\infty^+(\mu,m)\,\Omega^+(\Pi)c^+(\phi\chi^{-1})}\right)$$
$$= \frac{L_f(m,{}^\sigma\Pi\otimes{}^\sigma\chi)}{P_\infty^+({}^\sigma\mu,m)\,\Omega^+({}^\sigma\Pi)c^+({}^\sigma\phi^\sigma\chi^{-1})}\cdot\frac{\sigma(\mathcal{G}(\phi)\mathcal{G}(\chi)^2)}{\mathcal{G}({}^\sigma\phi)\mathcal{G}({}^\sigma\chi)^2}.$$

Hence the Galois-equivariant version of Theorem 1.3 in **Case 1** is as follows:

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P_\infty^+(\mu,m)\,\Omega^+(\Pi)\,c^+(\phi\chi^{-1})\mathcal{G}(\chi)^2\mathcal{G}(\phi)}\right)$$
$$= \frac{L_f(m,{}^\sigma\Pi\otimes{}^\sigma\chi)}{P_\infty^+({}^\sigma\mu,m)\,\Omega^+({}^\sigma\Pi)c^+({}^\sigma\phi^\sigma\chi^{-1})\mathcal{G}({}^\sigma\chi)^2\mathcal{G}({}^\sigma\phi)}.$$

This implies in particular that

$$L_f(m,\Pi\otimes\chi)\ \approx_{\mathbb{Q}(\Pi,\chi,\phi)}\ P_\infty^+(\mu,m)\,\Omega^+(\Pi)\,c^+(\phi\chi^{-1})\,\mathcal{G}(\chi)^2\ \mathcal{G}(\phi),$$

where $\approx_{\mathbb{Q}(\Pi,\chi,\phi)}$ is a number field which is the compositum of the rationality fields $\mathbb{Q}(\Pi)$, $\mathbb{Q}(\chi)$ and $\mathbb{Q}(\phi)$.

Case 2. The proof is similar to Case 1. So let's briefly present the details. In this case multiply and divide by $(2\pi i)^{(m)d_0}c^+(\chi\phi)$, Equation (5.23) may be rewritten as

$$\frac{L_f(m,\Pi\otimes\chi)}{P_\infty^-(\mu,m)\,\Omega^-(\Pi)\,c^+(\chi\phi)}$$
$$= (\mathrm{vol}(R_f)V_\Sigma)\cdot\left(L_\Sigma\,L_{S_\Sigma}(\tfrac{1}{2},\Pi\times\Sigma)\right)\cdot\left(\frac{L_f(m,\chi\phi)}{(2\pi i)^{(m)d_0}c^+(\chi\phi)}\right)$$
$$\cdot\langle\vartheta_\Pi^\circ,A_\Sigma\,\vartheta_\Sigma^\circ\rangle,$$

where $P_\infty^-(\mu,m)=(2\pi i)^{(m)d_0}P_\infty(\mu,m)$.

Now apply $\sigma\in\mathrm{Aut}(\mathbb{C})$ to both sides. The first three parentheses on the right hand side are $\mathrm{Aut}(\mathbb{C})$-equivariant as explained above. Further we have $\mathcal{G}(\chi_1) = \mathcal{G}(\chi_1^1) = \mathcal{G}(\chi)$ and $\mathcal{G}(\chi_2) = \mathcal{G}(\chi_2^1) = \mathcal{G}(\phi^{-1})$. Using these facts along with Equation (6.8) for Case 2 we get

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P_\infty^-(\mu,m)\,\Omega^-(\Pi)\,c^+(\chi\phi)}\right)$$
$$= (\mathrm{vol}(R_f)V_{{}^\sigma\Sigma})\cdot\left(L_{{}^\sigma\Sigma}\,L_{S_\Sigma}(\tfrac{1}{2},{}^\sigma\Pi\times{}^\sigma\Sigma)\right)\cdot\left(\frac{L_f(m,{}^\sigma\chi{}^\sigma\phi)}{(2\pi i)^{(m)d_0}c^+({}^\sigma\chi{}^\sigma\phi)}\right)$$
$$\cdot\,\langle\vartheta^\circ_{{}^\sigma\Pi},A_{{}^\sigma\Sigma}\,\vartheta^\circ_{{}^\sigma\Sigma}\rangle\frac{\sigma(\mathcal{G}(\chi)\mathcal{G}(\phi^{-1})^2)}{\mathcal{G}({}^\sigma\chi)\mathcal{G}({}^\sigma\phi^{-1})^2},$$

which is the same as

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P_\infty^-(\mu,m)\,\Omega^-(\Pi)\,c^+(\chi\phi)}\right)$$
$$= \frac{L_f(m,{}^\sigma\Pi\otimes{}^\sigma\chi)}{P_\infty^-({}^\sigma\mu,m)\,\Omega^-({}^\sigma\Pi)c^+({}^\sigma\chi{}^\sigma\phi)}\cdot\frac{\sigma(\mathcal{G}(\chi)\mathcal{G}(\phi)^{-2})}{\mathcal{G}({}^\sigma\chi)\mathcal{G}({}^\sigma\phi)^{-2}},$$

which further may be rewritten as

$$\sigma\left(\frac{L_f(m,\Pi\otimes\chi)}{P_\infty^-(\mu,m)\,\Omega^-(\Pi)\,c^+(\chi\phi)\mathcal{G}(\chi)\mathcal{G}(\phi)^{-2}}\right)$$
$$= \frac{L_f(m,{}^\sigma\Pi\otimes{}^\sigma\chi)}{P_\infty^-({}^\sigma\mu,m)\,\Omega^-({}^\sigma\Pi)\,c^+({}^\sigma\chi{}^\sigma\phi)\,\mathcal{G}({}^\sigma\chi)\mathcal{G}({}^\sigma\phi)^{-2}},$$

proving the Galois-equivariant version of Theorem 1.3 in **Case 2**. This implies in particular that

$$L_f(m,\Pi\otimes\chi)\ \approx_{\mathbb{Q}(\Pi,\chi,\phi)}\ P_\infty^-(\mu,m)\,\Omega^-(\Pi)\,c^+(\chi\phi)\,\mathcal{G}(\chi)\ \mathcal{G}(\phi)^{-2}.$$

This concludes the proof of Theorem 1.3.

6.4 Application: Symmetric square L-function

The purpose of this section is to put on record that Theorems 1.2 and 1.3 gives as an application, a rationality result for all the critical values of the symmetric square L-functions. We will consider real and imaginary case seperately:

F: totally real

Let φ be a primitive holomorphic cuspidal Hilbert modular form over F of weight $(k_1, \ldots, k_d)$, where $d = d_F$. Suppose that all the k_j have the same parity, and that φ is not of CM-type. We want to apply Theorem. 1.2 to get a rationality result for all the critical values of the symmetric square L-function $L(s, \mathrm{Sym}^2\varphi, \chi)$ attached to φ, twisted by a finite order Dirichlet character χ.

We will work with the L-function $L(s, \mathrm{Sym}^2\varphi, \chi)$ in the automorphic context. Let $\Pi(\varphi)$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ attached to φ. By Gelbart–Jacquet [13] we know the existence of an isobaric automorphic representation $\mathrm{Sym}^2(\Pi(\varphi))$ of $\mathrm{GL}_3(\mathbb{A}_F)$, defined as $\mathrm{Sym}^2(\Pi(\varphi)) = \otimes'_v \mathrm{Sym}^2(\Pi_v(\varphi))$, where $\mathrm{Sym}^2(\Pi_v(\varphi))$ is an irreducible admissible representation of $\mathrm{GL}_3(F_v)$ obtained via the local Langlands symmetric-square transfer of $\Pi_v(\varphi)$. If φ is not of CM-type, i.e., $\Pi(\varphi)$ is not a dihedral representation then it's well-known that $\mathrm{Sym}^2(\Pi(\varphi))$ is cuspidal. If $L(s, \mathrm{Sym}^2(\Pi(\varphi)) \otimes \chi)$ denotes the standard degree-3 L-function of $\mathrm{Sym}^2(\Pi(\varphi))$ twisted by χ then we have

$$L(s, \mathrm{Sym}^2\varphi, \chi) = L(s - k_0 + 1, \mathrm{Sym}^2(\Pi(\varphi)) \otimes \chi), \tag{6.9}$$

where $k_0 = \min(k_j)$. This may be seen as in the verification of [34, Theorem 1.4, (1)].

For convenience, let's suppose that all the $k_j \geq 2$ are even. Then $\Pi(\varphi)$ is cohomological to the weight $\mu = (\mu_j)$ where $\mu_j = ((k_j - 2)/2, -(k_j - 2)/2)$.

Following [29], we may verify that $\mathrm{Sym}^2(\Pi(\varphi)) \in \mathrm{Coh}(G_3; \mathrm{Sym}^2(\mu))$ for the weight $\mathrm{Sym}^2(\mu) = (\mathrm{Sym}^2(\mu_j))$, where $\mathrm{Sym}^2(\mu_j) = (k_j - 2, 0, 2 - k_j)$.

In the notation of Theorem 1.2, we get $n = \min(n_j) = k_0 - 2$. For the sake of simplicity, let's consider the case when χ is totally even. Then, the critical set for $L(s, \mathrm{Sym}^2(\Pi(\varphi)) \otimes \chi)$ is given by

$$\{3 - k_0, \ldots, -1;\ 2, 4, \ldots, k_0 - 2\}.$$

From Equation (6.9) we get that the critical set for $L(s, \mathrm{Sym}^2\varphi, \chi)$ is the set

$$\{2, 4, \ldots, k_0 - 2;\ k_0 + 1, k_0 + 3, \ldots, 2k_0 - 3\}.$$

This is to be interpreted as an empty set if $k_0 = 2$. If m is critical for $L(s, \mathrm{Sym}^2\varphi, \chi)$, and is on the right of the center of symmetry, i.e., $k_0 + 1 \leq m \leq 2k_0 - 3$ and m is odd, then **Case 1a.** of Theorem 1.2 takes the form:

$$L(m, \mathrm{Sym}^2\varphi, \chi) \approx_{\mathbb{Q}(\varphi,\chi)} P^1_\infty(\mathrm{Sym}^2(\mu), m - k_0 + 1)\, \Omega_r^+(\mathrm{Sym}^2(\Pi(\varphi))\, \mathcal{G}(\chi)^2. \tag{6.10}$$

It's an easy exercise to see that $\mathbb{Q}(\mathrm{Sym}^2(\Pi(\varphi)), \chi) \subset \mathbb{Q}(\Pi(\varphi), \chi) = \mathbb{Q}(\varphi, \chi)$. (See also statements (4) and (5) of [34, Theorem 1.4].) Moreover, we may state this result in a Galois-equivariant manner.

A comparison of Equation (6.10) with the main result of Sturm [40] for the critical values of the symmetric-square L-functions for an elliptic modular form would lead us to speculate that our global period $\Omega_r^+(\mathrm{Sym}^2(\Pi(\varphi))$ is related in some explicit way to the Petersson inner product $\langle \varphi, \varphi \rangle$.

F: CM field

Let π be a cohomological cuspidal automorphic representation of GL_2/F with purity weight w, that is, $\pi \in \mathrm{Coh}(G_2, \mu)$ with $\mu \in X_0^+(T_2)$. Then for each $v \in S_c$, $\mu_v = (\mu^\iota, \mu^{\bar\iota})$, where $\mu^\iota = (a, b)$; $\mu^{\bar\iota} = (a^*, b^*)$ such that $a \geq b$; $a^* \geq b^*$ and $a + b^* = b + a^* = \mathsf{w}$. By Gelbart–Jacquet [13] we know the existence of an isobaric automorphic representation $\mathrm{Sym}^2(\pi)$ of $\mathrm{GL}_3(\mathbb{A}_F)$, defined as

$\mathrm{Sym}^2(\pi) = \otimes'_v \mathrm{Sym}^2(\pi_v)$, where $\mathrm{Sym}^2(\pi_v)$ is an irreducible admissible representation of $\mathrm{GL}_3(F_v)$ obtained via the local Langlands symmetric-square transfer of π_v. Furthermore, following [29, Theorem 3.2],

$$\mathrm{Sym}^2(\pi) \in \mathrm{Coh}(G_3, \mathrm{Sym}^2(\mu)), \tag{6.11}$$

with $\mathrm{Sym}^2(\mu) = (\mathrm{Sym}^2(\mu)^\iota, \mathrm{Sym}^2(\mu)^{\bar{\iota}})_{\iota \in \varepsilon_F}$ such that

$$\mathrm{Sym}^2(\mu)^\iota = (2a,\ a+b,\ 2b) \text{ and } \mathrm{Sym}^2(\mu)^{\bar{\iota}} = (2a^*,\ a^*+b^*,\ 2b^*). \tag{6.12}$$

Clearly the $\mathrm{Sym}^2(\mu) \in X_0^+(T_3)$.

6.4.1 Proof of Theorem 1.6

By thinking of the symmetric square L-function on $\mathrm{GL}_3 \times \mathrm{GL}_1$ as the standard L-function of the symmetric-square–twisted by an algebraic Hecke character χ, and after using Equations (6.11) and (6.12), we get

$$\mathrm{Sym}^2(\pi) \in \mathrm{Coh}(G_3, \mathrm{Sym}^2(\mu)),$$

such that for each $v \in S_\infty$, $\mathrm{Sym}^2(\mu_v) = (2a, 0, -2a;\ 2a, 0, -2a)$. In terms of notations of Theorem 1.3, we get $n_1 = 2a$ and $n_2 = -2a$. Under the assumption that $a \geq t$, the critical set for $L(s, \mathrm{Sym}^2(\pi) \otimes \chi)$ is given by $\{1 - t \leq m \leq t\}$. Then **Case 1** of Theorem 1.3 takes the form:

$$L_f(m, \mathrm{Sym}^2(\pi) \otimes \chi)$$

$$\approx_{\mathbb{Q}(\mathrm{Sym}^2(\pi), \chi, \phi)} P_\infty^+(\mathrm{Sym}^2(\mu)), m)\, \Omega^+(\mathrm{Sym}^2(\pi))\, c^+(\phi\chi^{-1})\, \mathcal{G}(\chi)^2\, \mathcal{G}(\phi),$$

where $P_\infty^+(\mathrm{Sym}^2(\mu)), m) = (2\pi i)^{1-m} p_\infty(\mathrm{Sym}^2(\mu), m)$ and $\Omega^+(\mathrm{Sym}^2(\pi)) = p(\mathrm{Sym}^2(\pi)) L(0, \mathrm{Sym}^2(\pi) \otimes \phi)^{-1}$. Again its an easy exercise to see that

$$\mathbb{Q}(\mathrm{Sym}^2(\pi)).\chi, \phi) \subset \mathbb{Q}(\pi, \chi, \phi).$$

This proves the theorem.

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