Fluctuations in the distribution of Hecke eigenvalues

A thesis
submitted in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

by

Neha Prabhu

ID: 20123212

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

April 2017

Dedicated to

my family

Certificate

Certified that the work incorporated in the thesis entitled "*Fluctuations in the distribution of Hecke eigenvalues*", submitted by *Neha Prabhu* was carried out by the candidate under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date:

Dr. Kaneenika Sinha
Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Date:

Neha Prabhu

Roll Number: 20123212

Abstract

A famous conjecture of Sato and Tate (now a celebrated theorem of Taylor et al) predicts that the normalised p-th Fourier coefficients of a non-CM Hecke eigenform follow the Sato-Tate distribution as we vary the primes p. In 1997, Serre obtained a distribution law for the vertical analogue of the Sato-Tate family, where one fixes a prime p and considers the family of p-th coefficients of Hecke eigenforms. In this thesis, we address a situation in which we vary the primes as well as families of Hecke eigenforms. In the same year, Conrey, Duke and Farmer obtained distribution measures for Fourier coefficients of Hecke eigenforms in these families. Later, in 2006, Nagoshi obtained similar results under weaker conditions. We consider another quantity, namely the number of primes p for which the p-th Fourier coefficient of a Hecke eigenform lies in a fixed interval I. On averaging over families of Hecke eigenforms, we derive an expression for the fluctuations in the distribution of these eigenvalues about the Sato-Tate measure. Further, the fluctuations are shown to follow a Gaussian distribution. In this way, we obtain a conditional Central Limit Theorem for the quantity in question. Similar results are also proved in the setting of Maass forms. This extends a result of Wang (2014), who proved that the Sato-Tate theorem holds on average in the context of Maass forms.

In a separate project, we consider a classical result in number theory: Dirichlet's theorem on the density of primes in an arithmetic progression. We prove a similar result for numbers with exactly k prime factors for $k > 1$. Building upon a proof by E.M. Wright in 1954, we compute the asymptotic density of such numbers where each prime satisfies a congruence condition. As an application, we obtain the density of squarefree $n \leq x$ with k prime factors such that a fixed quadratic equation has exactly 2^k solutions modulo n.

Acknowledgements

It makes me nostalgic to look back at my mathematical journey so far and think about all the people and events that have led to this moment, where I give thanks. I am grateful to a lot of people and I thank the Almighty for the blessing to be able to say so. I would like to start with my school teachers in Dubai, where I spent six years of my schooling. This was where I first decided I wanted to take up Mathematics. I am grateful to Mrs. K. C. Usha, who taught me math in my eleventh and twelfth grade in Chennai and always inspired the class with anecdotes on what it means to work hard. I am immensely thankful to my teachers in the summer math camps of MTTS (Mathematics Training and Talent Search) programmes in the years 2008, 2009 and 2011. These programmes inspired me to pursue a research career in mathematics. I thank Prof. S. Kumaresan, the director of the annual programme for allowing me to participate in the programme and learn mathematics in a way that brought out the beauty of proofs and arguments, seldom taught in the classroom environment of most colleges in India.

IIT Bombay exposed me to a research environment and I thank all my course instructors, especially Prof. B. V. Limaye for teaching me during my masters programme there. This brings me to the place I am currently as I write this, IISER Pune. I am very, very grateful to Dr. Kaneenika Sinha for agreeing to supervise my thesis and for giving me a very interesting problem to work on. Thank you for the independence you gave me to come up with my own ideas and patiently listening to all of them even though many of them eventually didn't work out. I never imagined that a PhD supervisor could be a friend at the same time; thanks for the occassional Friday evening chats that involved sharing past experiences, giving me counsel and of course, sharing a lot of laughs. I also thank Prof. A. Raghuram, who always made time if a student wanted to talk to him. Thank you for your advice and motivating me to aim higher and to never give up. I am very grateful to Prof. Ram Murty who came to India regularly and never refused to give me ample time to discuss my ongoing work or share his valuable insights.

I would like to thank NBHM for the PhD Scholarship throughout the PhD program. In the same spirit, I am grateful to the National Centre for Mathematics for giving me the opportunity to attend various workshops that helped me mathematically and gave me the opportunity to meet experts in the area outside Pune. I thank the organizers of various conferences I attended for their hospitality and exposing me to current research work of world class mathematicians. I would like to thank all the friends I've made over the last decade (too many to individually name!) for supporting me throughout my ups and downs and keeping me in good spirits always. Coming to my family, I am indebted to Steven, my husband and my best friend for his constant support, encouragement, love and patience in the last two years. I thank my parents and sister Nina for their unconditional love, affection and understanding. Thank you for always supporting my decisions. The list of family would be incomplete without the mention of my favourite aunt Vidyakka who taught me to keep faith and who continues to inspire me everyday. There is no doubt I wouldn't have come this far without you.

Statement of Originality

The main results of this thesis which constitute original research are Theorems 1.3.1, 1.3.2, 1.3.3 and 1.3.4.
Propositions 3.5.1, 3.6.1, 4.3.1 and 6.3.1 as well as Theorem 4.3.2 are original subsidiary results that are required to prove the main results.
Theorems 5.2.1 and 5.2.2 are results that follow using the same techniques used in the proof of Theorem 1.3.1 and are stated without proof.

Notation

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the set of natural numbers, integers, rational numbers and real numbers respectively.

- $\mathbb{C}$ denotes the field of complex numbers; for $z \in \mathbb{C}$, $\mathrm{Re}(z)$ will denote its real part, $\mathrm{Im}(z)$ its imaginary part, $|z|$ its absolute value and $\overline{z}$ its complex conjugate.

- $\mathfrak{H}$ denotes the upper-half complex plane.

- For $x \in \mathbb{R}$, the quantity $\pi(x)$ will denote the number of primes not exceeding x.

- For a ring R, $SL_2(R)$ denotes the ring of 2×2 matrices with entries in R of determinant 1. Similarly, $GL_2(R)$ will denote the ring of 2×2 matrices which are invertible. If R is contained in the field of real numbers, then $GL_2^+(R)$ be the subset of $GL_2(R)$ with positive determinant.

- For a finite set S, $|S|$ or $\#S$ will denote the cardinality of S.

- For integers a and b, we write $a|b$ to mean that a is a divisor of b and $\gcd(a, b)$ to denote the greatest common divisor of a and b.

- For a positive integer n, we have the Euler-ϕ function given by

$$\phi(n) = n \prod_{\substack{p|n \\ p:\ \text{prime}}} \left(1 - \frac{1}{p}\right).$$

- For a positive integer n, $\psi(n)$ is given by

$$\psi(n) = n \prod_{\substack{p|n \\ p\text{: prime}}} \left(1 + \frac{1}{p}\right).$$

- Let a and n be a natural numbers. Then a is said to be a *quadratic residue* mod n if there exists a non-zero integer x such that

$$x^2 \equiv a \bmod n.$$

Else it is called a *quadratic non-residue.*

- Let p be an odd prime and a be an integer. The *Legendre symbol* is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p \\ -1 & \text{if } a \text{ is a quadratic non-residue mod } p \\ 0 & \text{if } a \equiv 0 \bmod p. \end{cases} \tag{1}$$

- For real valued functions f and g with $g \neq 0$ we write

$$f \sim g$$

to mean

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

- If g is positive, we write

$$f = \mathrm{O}_R(g)$$

or

$$f \ll_R g$$

to mean that there exists a positive constant $c = c(R)$, depending on some quantity R such that $|f(x)| \leq c(R)|g(x)|$ for all x; if the constant $c(R)$ is absolute, then we simply write

$$f = \mathrm{O}(g)$$

or

$$f \ll g.$$

- We write

$$f = \mathrm{o}(g)$$

to mean that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

Contents

Chapter 1

Introduction

1.1 Preliminaries

In this section, we review some basic properties of modular forms, Maass forms and Hecke operators. These will form the backbone of the problems addressed in this thesis. The interested reader may want to look at [9] and [13] for more details.

1.1.1 Modular forms

The group $SL_2(\mathbb{R})$ acts on the upper half plane

$$\mathfrak{H} := \{x + iy \mid x \in \mathbb{R},\, y > 0\}.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$, the action is given by

$$\gamma z = \frac{az+b}{cz+d}.$$

Definition 1.1.1 *Let $f(z)$ be a meromorphic function on the upper half plane $\mathfrak{H}$ and let k be an even positive integer.*
The function f is called a modular form of weight k for $\Gamma = SL_2(\mathbb{Z})$ if the following conditions are satisfied:
(1)

$$f(\gamma z) = (cz+d)^k f(z) \qquad \textit{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

(2) $f(z)$ is holomorphic at infinity and by (1), we have $f(z+1) = f(z)$ therefore we have the following Fourier series expansion:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q^n, \qquad \textit{where } q = e^{2\pi i z}$$

has $a_n = 0$ for $n < 0$.
*If further we have $a_0 = 0$, that is, the modular form vanishes at infinity, then $f(z)$ is called a **cusp form** of weight k with respect to Γ.*

In general, one could consider finite index subgroups of Γ and look at meromorphic functions that respect the transformation properties as in (1) above with respect to these subgroups. Besides $\Gamma = SL_2(\mathbb{Z})$, some of its subgroups are of special significance. We define them now.
Let N be a positive integer. We define

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N \right\}.$$

This subgroup is called the **principal congruence subgroup of level** N. A subgroup of Γ is called a **congruence subgroup of level** N if it contains $\Gamma(N)$. Given a discrete subgroup Γ' of $GL_2^+(\mathbb{R})$, an element of Γ' is called parabolic if it is conjugate in $GL_2^+(\mathbb{R})$ to a matrix of the form $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$. A **cusp** of Γ' is an element $s \in \mathbb{R} \cup \{\infty\}$ such that s is fixed by a parabolic element of Γ. In the case of a discrete subgroup $\Gamma' \subset SL_2(\mathbb{Z})$, it can be shown (see Proposition 3.5 in [12]) that the set of cusps of Γ' is $\mathbb{Q} \cup \{\infty\}$.
We now define modular forms for congruence subgroups. For

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}),$$

it will be convenient to use the following notation. We denote the value of $f\big|[\gamma]_k$ at z by:

$$f\big|[\gamma]_k(z) = (\det \gamma)^{\frac{k}{2}} (cz+d)^{-k} f(\gamma z).$$

Note that this is an action of $GL_2^+(\mathbb{Z})$ on the space of meromorphic functions on $\mathfrak{H}$. Let $f(z)$ be a meromorphic function in $\mathfrak{H}$ and let $\Gamma' \subset \Gamma$ be a congruence subgroup of level N. Let $k \in \mathbb{Z}$. Let $g = f\big|[\gamma_0]_k$ for some fixed $\gamma_0 \in GL_2^+(\mathbb{Q})$. If f is invariant under Γ', that is, if $f\big|[\gamma]_k = f$ for $\gamma \in \Gamma'$, then it follows that g is invariant under the group $\gamma_0^{-1}\Gamma\gamma_0$. Also, if $\gamma_1 \in \Gamma$ and $\Gamma(N) \subset \Gamma'$ then $\gamma_1^{-1}\Gamma'\gamma_1$ also contains $\Gamma(N)$. Since $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(N)$, we have $g(z+N) = g(z)$ and so $g = f\big|[\gamma_0]_k$ has a Fourier expansion in powers of $q_N = e^{2\pi i z/N}$, that is, we have

$$f\big|[\gamma_0]_k(z) = \sum_{n=-\infty}^{\infty} a_n(f\big|[\gamma_0]_k) q_N^n.$$

We say that f is **holomorphic at the cusps** if there are no negative powers of q_N in the Fourier expansion of $f\big|[\gamma_0]_k$ for any $\gamma_0 \in SL_2(\mathbb{Z})$. We say that f **vanishes at the cusps** if only positive powers occur in the above expansion for all $\gamma_0 \in SL_2(\mathbb{Z})$.

For the purposes of this thesis, we will be concerned with one kind of congruence subgroup which is defined as follows:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}.$$

This is called the Hecke congruence subgroup.

Definition 1.1.2 *A modular form of weight k with respect to $\Gamma_0(N)$ (or "level N") is a function $f : \mathfrak{H} \to \mathbb{C}$ such that*

- *f is holomorphic on $\mathfrak{H}$,*
- *$f\big|[\gamma_0]_k = f$ for all $\gamma_0 \in \Gamma_0(N)$,*
- *f is holomorphic at the cusps.*

If in addition, f vanishes at the cusps, then f is called a cusp form of weight k with respect to $\Gamma_0(N)$. We denote the space for modular forms of weight k and level N by $M(N,k)$ and the space of cusp forms of weight k and level N by $S(N,k)$. Both these spaces are finite dimensional complex vector spaces. Henceforth we will focus our attention on the space $S(N,k)$. The theorems proved will be for level 1, that is, $\Gamma(1) = SL_2(\mathbb{Z})$ and we will make some remarks about the analogous theorems for higher levels.

The space of cusp forms $S(N,k)$ is equipped with an inner product called the **Petersson Inner Product** which is defined below:

Definition 1.1.3 *Let $f, g \in S(N,k)$. The Petersson inner product of f and g is defined to be*

$$\langle f, g \rangle = \int\limits_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}.$$

We remark that in the above definition, we could have taken $f, g \in M(N,k)$ but with at least one of them in $S(N,k)$ in order for the integral to converge. If $N = d_1 d_2$ and $f \in S(d_1, k)$ then it is not hard to see that $f \in S(N,k)$ as well and $g(z) = f(d_2 z) \in S(N,k)$. The subspace of cusp forms spanned by the forms that are obtained from lower levels are called **oldforms**. It is precisely the $\mathbb{C}$-span of

$$\bigcup_{\substack{N'|N \\ N' \neq N}} \bigcup_{d|\frac{N}{N'}} \{f(dz) | f \in S(N',k)\}.$$

The orthogonal complement to the space of oldforms with respect to the Petersson inner product is called the space of **newforms** and we denote it by $S^*(N,k)$.

1.1.2 Hecke Operators for cusp forms

For each weight k and level N, there exists a family of linear operators that preserve the spaces $S(N,k)$ and $M(N,k)$, called the **Hecke operators** and it is the distribution of the eigenvalues of these operators that will be analyzed in a major part of this thesis. We define

$$\Delta^n(N) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{Z}) : a, b, d \in \mathbb{Z},\ 0 \le b \le d-1,\ ad = n, \gcd(a,N) = 1 \right\}.$$

Definition 1.1.4 *Let $f \in M(N,k)$ and n be a positive integer. Then the n-th* ***Hecke operator*** *T_n is defined as*

$$T_n(f) := n^{\frac{k}{2}-1} \sum_{\gamma \in \Delta^n(N)} f\big|[\gamma]_k.$$

Let $f \in M(N,k)$ be non-zero. We say f is a **Hecke eigenform** if, for each n so that $\gcd(n,N) = 1$, there exists a complex number λ_n so that

$$T_n f = \lambda_n f.$$

We record some very useful and important properties of Hecke operators:

- For $m, n \in \mathbb{N}$, if $\gcd(m,n) = 1$ then

$$T_m T_n = T_{mn}.$$

- More generally, if $m, n \in \mathbb{N}$ such that $\gcd(mn, N) = 1$ then

$$T_m T_n = \sum_{d \mid \gcd(m,n)} d^{k-1} T_{\frac{mn}{d^2}}$$

- The Hecke operators T_n for $n \in \mathbb{N}$ commute with each other and are self-adjoint with respect to the Petersson inner product.

- There exists a basis of $S(N,k)$ whose elements are eigenforms for all T_n for which $\gcd(n,N) = 1$. In particular, if $N = 1$ there exists a simultaneous eigenbasis for all $T_n, n \geq 1$. For higher levels, that is, $N > 1$, if we look at the subspace of newforms, then this space has the advantage of having a basis consisting of newforms that are simultaneous eigenforms for all T_n.

- If we were to look at **normalized eigenforms**, i.e., eigenforms so that the first Fourier coefficient $a_f(1) = 1$, then the eigenvalues of T_n coincide with the $n-$th Fourier coefficients $a_f(n)$ for each eigenform f in the eigenbasis consisting of newforms.

1.1.3 Maass forms

The theory of modular forms described earlier is a special case of holomorphic automorphic functions. In general, automorphic functions of weight k for $\Gamma_0(N)$ need not be holomorphic functions of z. We provide the basic definitions in this theory that will be sufficient to understand the statement of the result in this thesis pertaining to Maass forms.

Definition 1.1.5 ***(Moderate growth)*** *A smooth function $f : \mathfrak{H} \to \mathbb{C}$ is said to have moderate growth at a cusp $\mathfrak{a} \in \mathbb{Q} \cup \infty$ if $f(\sigma_{\mathfrak{a}}(x+iy))$ is bounded by a power of y, as $y \to \infty$ for any fixed $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ satisfying $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$. The function f is said to have moderate growth if it has moderate growth at every cusp.*

Definition 1.1.6 ***(Automorphic function of integral weight k with respect to $\Gamma_0(N)$)*** *Let $N, k \in \mathbb{Z}$ and $N \geq 1$. An automorphic function of weight k is a smooth function $f : \mathfrak{H} \to \mathbb{C}$, of moderate growth, which satisfies*

$$f(\gamma z) = \left(\frac{cz+d}{|cz+d|} \right)^k f(z)$$

for all $\gamma \in \Gamma_0(N)$ and $z \in \mathfrak{H}$. We let $\mathcal{A}_k(\Gamma_0(N))$ denote the complex vector space of all automorphic functions of weight k with respect to $\Gamma_0(N)$.

Definition 1.1.7 ***(Laplace operator)*** *For an integer k, we define the weight k Laplace operator*

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky\, \frac{\partial}{\partial x}.$$

The weight k Laplace operator has the property that it maps the space $\mathcal{A}_k(\Gamma_0(N))$ to itself. i.e., for $f \in \mathcal{A}_k(\Gamma_0(N))$,

$$\Delta_k f \in \mathcal{A}_k(\Gamma_0(N)).$$

We now have all the ingredients to define a Maass form (of trivial nebentypus).

Definition 1.1.8 ***(Maass form)*** *Let $N, k \in \mathbb{Z}$ with $N \geq 1$. Let $\nu \in \mathbb{C}$. A Maass form of type ν of weight k with respect to $\Gamma_0(N)$ is a smooth function $f : \mathfrak{H} \to \mathbb{C}$ satisfying the following conditions:*

- $f(\gamma z) = \left(\frac{cz+d}{|cz+d|}\right)^k f(z)$ *for all* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $z \in \mathfrak{H}$
- $\Delta_k f = \nu(1-\nu)f$, *where Δ_k is the Laplace operator given in Definition 1.1.7*
- *f is of moderate growth as in Definition 1.1.5*
- $\iint\limits_{\Gamma_0(N)\backslash\mathfrak{H}} |f(z)|^2 \frac{dxdy}{y^2} < \infty.$

Finally, a Maass form is said to be of level N if it is a Maass form for $\Gamma_0(N)$ and it is not a Maass form for $\Gamma_0(M)$ with $M < N$.

1.1.4 Hecke operators for Maass forms

Let $n \in \mathbb{N}$ and $f \in \mathcal{A}_k(\Gamma_0(N))$. The Hecke operator T_n is defined by

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right), \quad z \in \mathfrak{H}.$$

We describe some important properties. Although some of them are the same as the properties as those of Hecke operators acting on $S(N,k)$, it is worthwhile to see that they hold in the case of Maass forms, because it will be relevant later, when we use these facts to generalize the results in the modular forms setting to Maass forms.

- For $m, n \in \mathbb{N}$, the Hecke operators satisfy:

  $$T_m T_n = \sum_{d|\gcd(m,n)} T_{\frac{mn}{d^2}}.$$

  In particular, the Hecke operators commute with one another.

- The Hecke operators T_n commute with Δ_k.

- For $f, g \in \mathcal{A}_k(\Gamma_0(N))$, the Petersson inner product of f and g is defined to be
$$\langle f, g\rangle = \iint\limits_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\overline{g(z)}\frac{dxdy}{y^2}.$$
We write $\mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{H}, k)$ to denote the completion of the space of all functions $f \in \mathcal{A}_k(\Gamma_0(N))$ satisfying the $\mathcal{L}^2$ condition
$$\iint\limits_{\Gamma_0(N)\backslash\mathfrak{H}} |f(z)|^2\frac{dxdy}{y^2} < \infty,$$
with respect to the Petersson inner product.

- If we assume $\gcd(n, N) = 1$, then T_n is a normal operator. The properties so far tell us that we may diagonalize the space $\mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{H}, k)$ with respect to these operators and Δ_k. The Selberg spectral decomposition states that the Hilbert space $\mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{H}, k)$ decomposes into Maass cusp forms, Eisenstein series and residues of Eisenstein series. We will be concerned with the restriction of the Hecke operators and Δ_k to the space of Maass cusp forms. For the precise definition of this space, the reader may consult Chapter 3 of [9].

- We let $\mathcal{C}(\Gamma \setminus \mathfrak{H})$ denote the space of Maass cusp forms with respect to $\Gamma = SL_2(\mathbb{Z})$. We have an orthonormal basis for $\mathcal{C}(\Gamma\backslash\mathfrak{H})$ of simultaneous eigenforms for the Hecke operators T_n and the Laplace operator Δ_k, which we denote by $\{f_j : j \geq 0\}$.

- For an eigenform f_j we have
$$\Delta_k f_j = \left(\frac{1}{4} + t_j^2\right) f_j, \quad T_n f_j = a_j(n) f_j,$$
where $a_j(n)$ are the eigenvalues of T_n.

- For $z = x + iy \in \mathfrak{H}$, each f_j has the Fourier expansion
$$f_j(z) = \sqrt{y}\varrho_j(1)\sum_{n=1}^{\infty} a_j(n)K_{it_j}(2\pi|n|y)e(nx), \tag{1.1}$$

where $a_j(n) \in \mathbb{R}$, $\varrho_j(1) \neq 0$ and K_ν is the K-Bessel function of order ν.

- We order the f_j's so that $0 < t_1 \leq t_2 \leq t_3 \leq \dots$. It is well known that for level 1,

$$\text{(Weyl's law) } r(T) := \#\{j : 0 < t_j \leq T\} = \frac{1}{12}T^2 + \mathrm{O}(T \log T). \quad (1.2)$$

  Weyl's Law was obtained by Selberg [31] as a consequence of the Selberg's Trace Formula, and in particular, it proved that Maass forms exist.

1.2 History and Motivation of the problem

The statistical distribution of eigenvalues of the Hecke operators acting on spaces of modular cusp forms and Maass forms has been well investigated in recent years ([1], [29], [33]). Among the early developments that motivated this study was a famous conjecture, stated independently by M. Sato and J. Tate around 1960. This conjecture predicted a distribution law for the second order terms in the expression for the number of points in a non-CM elliptic curve modulo a prime p as the primes vary. Serre [32] generalised this conjecture in 1968 in the language of modular forms. The modular version of the Sato-Tate conjecture can be understood as follows:
Let k be a positive even integer and N be a positive integer. Let $S(N, k)$ denote the space of modular cusp forms of weight k with respect to $\Gamma_0(N)$. For $n \geq 1$, let T_n denote the n-th Hecke operator acting on $S(N, k)$. We denote the set of all newforms in $S(N, k)$ by $\mathcal{F}_{N,k}$. Any $f(z) \in \mathcal{F}_{N,k}$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n,$$

where $a_f(1) = 1$ and

$$\frac{T_n(f(z))}{n^{\frac{k-1}{2}}} = a_f(n) f(z), \; n \geq 1.$$

A cusp form is said to be a CM form if there is a non-trivial Dirichlet character ϕ such that $a_f(p) = \phi(p)a_f(p)$ for all primes p in a set of primes of density 1. Otherwise, it is called a non-CM form.
Let p be a prime number such that $\gcd(p, N) = 1$. By a theorem of Deligne [7], the eigenvalues $a_f(p)$ lie in the interval $[-2, 2]$. One can study the distribution of the coefficients $a_f(p)$ in different ways:

(A) (Sato-Tate family) Let N and k be fixed and let $f(z)$ be a non-CM newform in $\mathcal{F}_{N,k}$. We consider the sequence $\{a_f(p)\}$ as $p \to \infty$.

(B) (Vertical Sato-Tate family) For a fixed prime p, we consider the families

$$\{a_f(p),\ f \in \mathcal{F}_{N,k}\},\ |\mathcal{F}_{N,k}| \to \infty.$$

(C) (Average Sato-Tate family) We consider the families

$$\{a_f(p),\ p \leq x,\ f \in \mathcal{F}_{N,k}\},\ |\mathcal{F}_{N,k}| \to \infty,\ x \to \infty.$$

Serre's modular version of the Sato-Tate conjecture predicts a distribution law for the sequence defined in (A). More explicitly, let I be a subinterval of $[-2, 2]$ and for a positive real number x and $f \in \mathcal{F}_{N,k}$, let

$$N_I(f, x) := \#\{p \leq x : \gcd(p, N) = 1,\ a_f(p) \in I\}.$$

The Sato-Tate conjecture states that for a fixed non-CM newform $f \in \mathcal{F}_{N,k}$, we have

$$\lim_{x \to \infty} \frac{N_I(f, x)}{\pi(x)} = \int_I \mu_\infty(t) dt,$$

where $\pi(x)$ denotes the number of primes less than or equal to x and

$$\mu_\infty(t) := \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{t^2}{4}} & \text{if } t \in [-2, 2] \\ 0 & \text{otherwise.} \end{cases}$$

The measure $\mu_\infty(t)$ is referred to as the Sato-Tate or semicircle measure in the literature. This conjecture has deep and interesting generalisations and has been a central theme in arithmetic geometry over the last few decades. In 1970, Langlands [16] formulated a general automorphy conjecture which would imply the Sato-Tate conjecture. This conjecture is still open. However, using a very special case of the Langlands funtoriality conjecture, M. R. Murty and V. K. Murty [24] have shown that the general automorphy conjecture follows.

The Sato-Tate conjecture has now been proved in the highly celebrated work of Barnet-Lamb, Geraghty, Harris and Taylor [1]. The methods in [1] to address the Sato-Tate conjecture are different from the approach of Langlands: the authors prove that the L-functions $L_m(s)$ associated to symmetric powers of l-adic representations (l coprime to N) attached to f are *potentially* automorphic.

If these L-functions are automorphic, then one can also obtain error terms in the Sato-Tate distribution. In fact, under the condition that all symmetric power L-functions are automorphic and satisfy the Generalized Riemann Hypothesis, V. K. Murty [23] showed that for a non-CM newform f of weight 2 and square free level N, we have

$$N_I(f,x) = \pi(x) \int_I \mu_\infty(t)dt + \mathrm{O}\left(x^{3/4}\sqrt{\log Nx}\right).$$

Building on Murty's work, Bucur and Kedlaya [5] have obtained, under some analytic assumptions on motivic L-functions, an extension of the effective Sato-Tate error term for arbitrary motives. Recently, Rouse and Thorner [28] have generalised Murty's explicit result for all squarefree N and even $k \geq 2$, further improving the error term by a factor of $\sqrt{\log Nx}$.

In 1984, Sarnak [29] considered a vertical variant of the Sato-Tate conjecture in the case of primitive Maass cusp forms. For a fixed prime p, he obtained a distribution measure for the p-th coefficients of Maass Hecke eigenforms averaged over Laplacian eigenvalues. The Sato-Tate conjecture is still open in the case of primitive Maass forms. One important factor here is that the Ramanujan-Peterrson conjecture, which states that for all primes p, the eigenvalues sasisfy $|a_j(p)| \leq 2$, is open. The best bound known so far is by Kim and Sarnak [19] who proved that for all primes p,

$$|a_j(p)| \leq p^{\theta} + p^{-\theta}$$

where $\theta = 7/64$.

In 1997, Serre [33] considered a similar vertical question for holomorphic Hecke eigenforms. For a fixed prime p, let $|\mathcal{F}_{N,k}| \to \infty$ such that k is a positive even integer and N is coprime to p. Let I be a subinterval of $[-2, 2]$ and

$$N_I(N,k) := \#\{f \in \mathcal{F}_{N,k} : a_f(p) \in I\}.$$

Serre showed that

$$\lim_{|\mathcal{F}_{N,k}| \to \infty} \frac{N_I(N,k)}{|\mathcal{F}_{N,k}|} = \int_I \mu_p(t) dt, \tag{1.3}$$

where

$$\mu_p(t) = \begin{cases} \frac{p+1}{\pi} \frac{(1-t^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2 - t^2} & \text{if } t \in [-2,2] \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$\mu_p(t) = \frac{(p+1)}{(p^{1/2} + p^{-1/2})^2 - t^2} \mu_\infty(t).$$

The measure $\mu_p(t)$ is referred to as the p-adic Plancherel measure in the literature. This theorem was independently proved by Conrey, Duke and Farmer [6] for $N = 1$.

Since averaging over eigenforms provides us with an important tool namely, the Eichler-Selberg trace formula, the quantity $N_I(N,k)$ becomes easier to approach. Error terms in Serre's theorem were obtained by M. R. Murty and Sinha [25]. They prove that for a positive integer N, a prime number p coprime to N and a subinterval I of $[-2, 2]$,

$$N_I(N,k) = |\mathcal{F}_{N,k}| \int_I \mu_p(t) dt + \mathrm{O}\left(\frac{|\mathcal{F}_{N,k}| \log p}{\log kN} \right).$$

In this note, we consider the families described in (C),

$$\{a_f(p),\, p \le x,\, f \in \mathcal{F}_{N,k}\}$$

as $|\mathcal{F}_{N,k}| \to \infty$ and $x \to \infty$. In other words, this is the Sato-Tate family (A) averaged over all newforms in $\mathcal{F}_{N,k}$. In fact, in this direction, the following theorem was proved by Conrey, Duke and Farmer [6]: As $x \to \infty$ and $k = k(x) \to \infty$ with $k > e^x$, for any subinterval I of $[-2, 2]$,

$$\lim_{x\to\infty} \frac{1}{|\mathcal{F}_{1,k}|} \sum_{f\in\mathcal{F}_{1,k}} \frac{N_I(f,x)}{\pi(x)} = \int_I \mu_\infty(t)dt.$$

Nagoshi [27] obtained the same asymptotic under weaker conditions on the growth of k, namely, $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. An effective version of Nagoshi's theorem was proved by Wang [36]. Under the above mentioned conditions, he proves that

$$\frac{1}{|\mathcal{F}_{1,k}|} \sum_{f\in\mathcal{F}_{1,k}} \frac{N_I(f,x)}{\pi(x)} = \int_I \mu_\infty(t)dt + \mathrm{O}\left(\frac{\log x}{\log k} + \frac{\log x \log\log x}{x}\right).$$

The "average" Sato-Tate theorem tells us that for a fixed interval I, the expected value of $N_I(f,x)$ as we vary $f \in \mathcal{F}_{1,k}$,

$$E[N_I(f,x)] := \frac{1}{|\mathcal{F}_{1,k}|} \sum_{f\in\mathcal{F}_{1,k}} N_I(f,x)$$

is asymptotic to

$$\pi(x) \int_I \mu_\infty(t)dt$$

as $x \to \infty$ with $\frac{\log k}{\log x} \to \infty$.

In this thesis, we delve deeper into the nature of the distribution of Hecke eigenvalues by posing the following questions:

- What can be said about the variance of this random variable? In other words, as we vary $f \in \mathcal{F}_{N,k}$, what can be concluded about the fluctuations of $N_I(f,x)$ about the expected value?

- What about higher moments? Is there a distribution that these fluctuations follow?

Both these questions are answered in the context of holomorphic cusp forms in detail.
The case of primitive Maass cusp forms admits a similar analysis to the case of holomorphic modular cusp forms. We therefore make some observations in this case. Using the notation in Section 1.1.4, for an interval $I = [a, b] \subset \mathbb{R}$ and for $1 \leq j \leq r(T)$, we define

$$N_I(j, x) = \#\{p \leq x : a_j(p) \in I\},$$

with $a_j(p)$ defined as in equation (1.1) and ask similar questions regarding the statistics of $N_I(j, x)$.
In the process of studying the Eichler-Selberg Trace formula, a related problem of counting the number of solutions to a given quadratic equation mod N as N varies in a certain subset of positive integers was studied. The last chapter in this thesis records results obtained in this direction.

1.3 Overview of new results.

Theorem 1.3.1 *(Distribution results in the case of holomorphic cusp forms) Let $I = [a, b]$ be a fixed subinterval of $[-2, 2]$. Suppose that $k = k(x)$ satisfies $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Then for any bounded continuous real function h on $\mathbb{R}$ we have*

$$\lim_{x\to\infty} \frac{1}{|\mathcal{F}_{1,k}|} \sum_{f\in\mathcal{F}_{1,k}} h\left(\frac{N_I(f, x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)\left[\mu_\infty(I) - (\mu_\infty(I))^2\right]}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)e^{-\frac{t^2}{2}}dt.$$

In other words, for any real numbers $A < B$,

$$\lim_{x\to\infty} \mathit{Prob}_{\mathcal{F}_{1,k}}\left(A < \frac{N_I(f, x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)\left[\mu_\infty(I) - (\mu_\infty(I))^2\right]}} < B\right) = \frac{1}{\sqrt{2\pi}} \int_{A}^{B} e^{-t^2/2}dt.$$

Theorem 1.3.2 *(Distribution results in the case of primitive Maass forms) Suppose that $T = T(x)$ satisfies $\frac{\log T}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Let $I = [a, b] \subset \mathbb{R}$. Then for any bounded continuous real function h on $\mathbb{R}$ we have*

$$\lim_{x\to\infty} \frac{1}{r(T)} \sum_{j=1}^{r(T)} h\left(\frac{N_I(j,x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)\left[\mu_\infty(I) - (\mu_\infty(I))^2\right]}} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-\frac{t^2}{2}} dt.$$

In other words, for any real numbers $A < B$,

$$\lim_{x\to\infty} \ Prob_{\ 1\le j\le r(T)} \left(A < \frac{N_I(j,x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)\left[\mu_\infty(I) - (\mu_\infty(I))^2\right]}} < B \right) = \frac{1}{\sqrt{2\pi}} \int_{A}^{B} e^{-t^2/2} dt.$$

Theorem 1.3.3 *(A generalization of Dirichlet's density theorem) Let $N, k \in \mathbb{N}$ and consider a k-tuple*

$$\underline{\mathrm{m}}_{[k]} = (m_1, m_2, \ldots, m_k)$$

where each $m_i \in (\mathbb{Z}/N\mathbb{Z})^\times$, the multiplicative group of units in $\mathbb{Z}/N\mathbb{Z}$. The m_i's are not necessarily distinct.
Consider positive integers $n \le x$ with k prime factors, counted with multiplicity. Represent such n as $n = p_1 p_2 \ldots p_k$ with $p_1 \le p_2 \le \ldots \le p_k$. Let $\tau_{k,\underline{\mathrm{m}}_{[k]}}(x)$ denote the number of positive integers $n \le x$ with k prime factors satisfying $p_i \equiv m_i \mod N$ for each $i = 1, \ldots, k$. If the primes are distinct, then n is squarefree. Let $\pi_{k,\underline{\mathrm{m}}_{[k]}}(x)$ denote the number of such squarefree $n \le x$. Then,

$$\pi_{k,\underline{\mathrm{m}}_{[k]}}(x) \sim \tau_{k,\underline{\mathrm{m}}_{[k]}}(x) \sim \frac{1}{\phi(N)^k} \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} \qquad (k \ge 2) \text{ as } x \to \infty.$$

Theorem 1.3.4 *(Density of solutions to quadratic congruences)*
Let $D \in \mathbb{Z} - \{0\}$ *and* $k \in \mathbb{N}$. *Fix a* k*-tuple* $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$ *where each* $\varepsilon_i = \pm 1$ *for each* $i = 1, \ldots, k$. *Then, as* $x \to \infty$,

$$\frac{1}{\pi_k(x)} \# \left\{ n \leq x, n = p_1 p_2 \ldots p_k \text{ with } p_1 < p_2 < \ldots < p_k : \left(\frac{D}{p_i}\right) = \varepsilon_i \text{ for each } i \right\} \sim \frac{1}{2^k},$$

where $\pi_k(x)$ *denotes the number of squarefree numbers less than* x *with* k *prime factors.*

Chapter 2

Beurling-Selberg polynomials

The Beurling-Selberg polynomials are trigonometric polynomials which provide a good approximation to the characteristic functions of intervals on $\mathbb{R}$. The main strength of this technique is that it reduces estimating counting functions to evaluating finite exponential sums. Moreover, the Fourier coefficients can be explicitly calculated, as we shall see in this chapter. Although the exact formula for these coefficients will not be used, the properties satisfied by them allow us to express them as a main term and error term, which will be used repeatedly in the calculations in the thesis problem. The interested reader may wish to read a detailed exposition by Montgomery (see [20], Chapter 1) or consult the paper of Vaaler [35].

2.1 Definitions and properties

For a positive integer M, we define $\Delta_M(x)$ to be Féjer's kernel as follows:

$$\Delta_M(x) = \sum_{|n|<M} \left(1 - \frac{|n|}{M}\right) e(nx) = \frac{1}{M} \left(\frac{\sin \pi M x}{\sin \pi x}\right)^2 .$$

These can be easily seen to be polynomials in $e(x)$.

The M-th order Beurling polynomial is defined as:

$$B_M^*(x) = \frac{1}{M+1} \sum_{k=1}^{M} \left(\frac{k}{M+1} - \frac{1}{2}\right) \Delta_{M+1}\left(x - \frac{k}{M+1}\right) + \frac{1}{2\pi(M+1)} \sin(2\pi(M+1)x)$$

$$-\frac{1}{2\pi}\Delta_{M+1}(x)\sin 2\pi x+\frac{1}{2(M+1)}\Delta_{M+1}(x). \tag{2.1}$$

For an interval $[a,b]$, we define the M-th Selberg polynomial as

$$S_M^+(x)=b-a+B_M^*(x-b)+B_M^*(a-x)$$

and

$$S_M^-(x)=b-a-B_M^*(b-x)-B_M^*(x-a).$$

These polynomials in $e(x)$ are of degree at most M and have the following properties:

1. For a subinterval $I=[a,b]$ of $\left[-\frac{1}{2},\frac{1}{2}\right]$ and $x\in\mathbb{R}$,

$$S_M^-(x)\le\chi_I(x)\le S_M^+(x).$$

2.

$$\int_{-1/2}^{1/2}S_M^{\pm}(x)dx=b-a\pm\frac{1}{M+1}.$$

3. For $0<|m|\le M$,

$$\left|\hat{S}_M^{\pm}(m)-\hat{\chi}_I(m)\right|\le\frac{1}{M+1}, \tag{2.2}$$

where

$$\hat{\chi}_I(m)=\frac{e(-ma)-e(-mb)}{2\pi im}.$$

4. For $n\neq 0$,

$$|\hat{\chi}_I(n)|\le\min\left(b-a,\frac{1}{\pi|n|}\right).$$

Therefore, for non-zero n,

$$|\hat{S}_M^{+}(n)|\le\frac{1}{M+1}+\min\left(b-a,\frac{1}{\pi|n|}\right).$$

2.2 Fourier coefficients

Although the explicit Fourier coefficients will not be required for the proof of results in this thesis, we would like to record it for future reference. The exact formulae for the Fourier coefficients $\hat{S}_M^{\pm}(m)$ can be calculated by first computing the Fourier coefficients $\hat{B}_M^*(n)$ for $-M \leq n \leq M$.

Extracting the coefficient of $e(nx)$ in equation (2.1), we obtain the following:

1. For $|n| > M$,
$$\widehat{B}_M^*(n) = \widehat{B}_M^*(-n) = 0.$$

2. For $-M \leq n \leq M$,
$$\widehat{B}_M^*(n) = \frac{1}{M+1}\left(1 - \frac{|n|}{M+1}\right)\sum_{k=1}^{M}\left(\frac{k}{M+1} - \frac{1}{2}\right) e\left(\frac{-nk}{M+1}\right)$$
$$+\frac{1}{4\pi}\left(\frac{|n-1| - |n+1|}{M+1}\right) + \frac{1}{2(M+1)}\left(1 - \frac{|n|}{M+1}\right).$$

   In particular,
$$\widehat{B}_M^*(0) = \frac{1}{M+1}\sum_{k=1}^{M}\left(\frac{k}{M+1} - \frac{1}{2}\right) + \frac{1}{2(M+1)} = \frac{1}{2(M+1)}.$$

We observe that the Beurling polynomial is periodic of period 1. i.e.,

$$B_M^*(x) = B_M^*(x+n) \qquad \text{for } n \in \mathbb{Z}.$$

Therefore, we have, when $0 < n \leq M$,

$$\widehat{S}_M^+(n) = e(-nb)\widehat{B}_M^*(n) + e(-na)\widehat{B}_M^*(-n),$$

$$\widehat{S}_M^-(n) = -e(-nb)\widehat{B}_M^*(-n) - e(-na)\widehat{B}_M^*(n),$$

and

$$\widehat{S}_M^{\pm}(0) = b - a \pm \frac{1}{M+1}.$$

We record two properties that can be deduced from these explicit formulae:

1. If the interval is symmetric about zero, that is, if it is of the form $[-b, b]$, it is easy to see that $\widehat{S}_M^{\pm}(n) = \widehat{S}_M^{\pm}(-n)$. To be precise, we have

$$\widehat{S}_M^{\pm}(n) = \widehat{S}_M^{\pm}(-n) = \pm e(nb)\widehat{B}_M^*(-n) \pm e(-nb)\widehat{B}_M^*(n).$$

2. For any interval $[a, b] \subset [0, 1]$, the sum

$$\widehat{S}_M^{+}(n) + \widehat{S}_M^{+}(-n)$$

is always real:

Using the expression above for $\widehat{B}_M^*(n)$, we have the following:

1. $$\widehat{B}_M^*(n)e(-nb) = \frac{1}{M+1}\left(1 - \frac{n}{M+1}\right)\sum_{k=1}^{M}\left(\frac{k}{M+1} - \frac{1}{2}\right)e\left(\frac{-nk}{M+1} - nb\right) - \frac{e(-nb)}{2\pi i(M+1)} + \frac{e(-nb)}{2(M+1)}\left(1 - \frac{n}{M+1}\right)$$

2. $$\widehat{B}_M^*(-n)e(nb) = \frac{1}{M+1}\left(1 - \frac{n}{M+1}\right)\sum_{k=1}^{M}\left(\frac{k}{M+1} - \frac{1}{2}\right)e\left(\frac{nk}{M+1} + nb\right) - \frac{e(nb)}{2\pi i(M+1)} + \frac{e(nb)}{2(M+1)}\left(1 - \frac{n}{M+1}\right)$$

3. $$\widehat{B}_M^*(n)e(na) = \frac{1}{M+1}\left(1 - \frac{n}{M+1}\right)\sum_{k=1}^{M}\left(\frac{k}{M+1} - \frac{1}{2}\right)e\left(\frac{-nk}{M+1} + na\right) - \frac{e(na)}{2\pi i(M+1)} + \frac{e(na)}{2(M+1)}\left(1 - \frac{n}{M+1}\right)$$

4. $$\widehat{B}_M^*(-n)e(-na) = \frac{1}{M+1}\left(1 - \frac{n}{M+1}\right)\sum_{k=1}^{M}\left(\frac{k}{M+1} - \frac{1}{2}\right)e\left(\frac{nk}{M+1} - na\right) - \frac{e(-na)}{2\pi i(M+1)} + \frac{e(-na)}{2(M+1)}\left(1 - \frac{n}{M+1}\right)$$

Summing, we have:

$$\widehat{S}_M^+(n) + \widehat{S}_M^+(-n) =$$

$$\frac{1}{M+1}\left(1-\frac{n}{M+1}\right)\sum_{k=1}^{M}\left(\frac{k}{M+1}-\frac{1}{2}\right)\left\{2\cos\left(2\pi\left|na-\frac{nk}{M+1}\right|\right)\right.$$
$$\left.+\,2\cos\left(2\pi\left|nb+\frac{nk}{M+1}\right|\right)\right\}-\frac{1}{\pi(M+1)}(\sin(2\pi na)-\sin(2\pi nb))$$
$$+\frac{1}{M+1}\left(1-\frac{n}{M+1}\right)(\cos(2\pi na)-\cos(2\pi nb)),$$

where a, b are real.
Therefore, $\widehat{S}_M^\pm(n) + \widehat{S}_M^\pm(-n) \in \mathbb{R}$ for all $n \in \mathbb{N}$.

2.3 Preliminary results

In this section we prove some results involving the Fourier coefficients of the Beurling-Selberg polynomials that will be required later.
Henceforth, we will use the following notation: for an interval $I = [a, b] \subseteq [-2, 2]$, we choose a subinterval

$$I_1 = [\alpha, \beta] \subseteq \left[0, \frac{1}{2}\right]$$

so that

$$\theta \in I_1 \Leftrightarrow 2\cos(2\pi\theta) \in I.$$

For $M \geq 1$, let $S_M^\pm(x)$ denote the majorant and minorant Beurling-Selberg polynomials for the interval I_1. We denote, for $0 \leq |m| \leq M$,

$$\hat{\mathcal{S}}^\pm(m) := \left(\hat{S}_M^\pm(m) + \hat{S}_M^\pm(-m)\right) \tag{2.3}$$

By equation (2.2), we have, for $1 \leq |m| \leq M$,

$$\hat{S}_M^\pm(m) = \frac{e(-m\alpha) - e(-m\beta)}{2\pi i m} + \mathrm{O}\left(\frac{1}{M}\right)$$

and

$$\hat{S}_M^\pm(-m) = \frac{e(m\beta) - e(m\alpha)}{2\pi i m} + \mathrm{O}\left(\frac{1}{M}\right).$$

Thus,

$$\hat{\mathcal{S}}^{\pm}(m) = \frac{\sin(2\pi m\beta) - \sin(2\pi m\alpha)}{m\pi} + \mathrm{O}\left(\frac{1}{M}\right). \tag{2.4}$$

Proposition 2.3.1 *For $[\alpha, \beta] \subseteq [0, \frac{1}{2}]$ and an integer $M \geq 1$, we have*

$$2\sum_{m=1}^{M} \hat{\mathcal{S}}^{\pm}(m)^2 = 2(\beta-\alpha) - 4(\beta-\alpha)^2 + \mathrm{O}\left(\frac{\log M}{M}\right) \tag{2.5}$$

and for $M \geq 3$,

$$2\sum_{m=1}^{M-2} \hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2) = -\frac{1}{\pi^2}(\sin(2\pi\beta) - \sin(2\pi\alpha))^2 + \frac{1}{2\pi}(1-4(\beta-\alpha))(\sin(4\pi\beta) - \sin(4\pi\alpha)) + \mathrm{O}\left(\frac{\log M}{M}\right). \tag{2.6}$$

Proof. We start by the following observation:

$$\sum_{m=1}^{M} \left|\frac{\sin(2\pi m\beta) - \sin(2\pi m\alpha)}{m\pi}\right| = \mathrm{O}(\log\, M).$$

We have

$$\begin{aligned}
&2\sum_{m=1}^{M} \hat{\mathcal{S}}^{\pm}(m)^2 \\
&= 2\sum_{m=1}^{M}\left(\frac{\sin(2\pi m\beta) - \sin(2\pi m\alpha)}{m\pi} + \mathrm{O}\left(\frac{1}{M}\right)\right)^2 \\
&= \frac{2}{\pi^2}\sum_{m=1}^{M}\left(\frac{\sin^2(2\pi m\beta)}{m^2} + \frac{\sin^2(2\pi m\alpha)}{m^2} - 2\frac{\sin(2\pi m\beta)\sin(2\pi m\alpha)}{m^2}\right) + \mathrm{O}\left(\frac{\log M}{M}\right) \\
&= \frac{2}{\pi^2}\left(\sum_{m=1}^{M}\frac{\sin^2(2\pi m\beta)}{m^2} + \sum_{m=1}^{M}\frac{\sin^2(2\pi m\alpha)}{m^2} - \sum_{m=1}^{M}\frac{\cos(2\pi m(\beta-\alpha))}{m^2}\right. \\
&\qquad\left. + \sum_{m=1}^{M}\frac{\cos(2\pi m(\beta+\alpha))}{m^2}\right) + \mathrm{O}\left(\frac{\log M}{M}\right)
\end{aligned}$$

$$= \frac{2}{\pi^2}\Bigg(\sum_{m=1}^{\infty}\frac{\sin^2(2\pi m\beta)}{m^2} + \sum_{m=1}^{\infty}\frac{\sin^2(2\pi m\alpha)}{m^2} - \sum_{m=1}^{\infty}\frac{\cos(2\pi m(\beta-\alpha))}{m^2}$$

$$+ \sum_{m=1}^{\infty}\frac{\cos(2\pi m(\beta+\alpha))}{m^2}\Bigg) + \mathrm{O}\left(\frac{\log M}{M}\right).$$

We now use the trigonometric sum (see [2, p. 360] or Equation (520) of [11]),

$$\sum_{m=1}^{\infty}\frac{\sin^2(m\theta)}{m^2} = \frac{1}{2}\theta(\pi-\theta), \quad 0 \le \theta \le \pi.$$

We also have (see [2, p. 370] or Equation (573) of [11]),

$$\sum_{m=1}^{\infty}\frac{\cos(2\pi m\theta)}{(m\pi)^2} = \theta^2 - \theta + \frac{1}{6}, \quad 0 < \theta < 1.$$

Therefore, we have

$$2\sum_{m=1}^{M}\hat{\mathcal{S}}^{\pm}(m)^2 = 2\beta(1-2\beta) + 2\alpha(1-2\alpha) - 2\left((\beta-\alpha)^2 - (\beta-\alpha) + \frac{1}{6}\right)$$

$$+ 2\left((\beta+\alpha)^2 - (\beta+\alpha) + \frac{1}{6}\right) + \mathrm{O}\left(\frac{\log M}{M}\right)$$

$$= 2\beta(1-2\beta) + 2\alpha(1-2\alpha) + 8\alpha\beta - 4\alpha + \mathrm{O}\left(\frac{\log M}{M}\right)$$

$$= 2(\beta-\alpha) - 4(\beta-\alpha)^2 + \mathrm{O}\left(\frac{\log M}{M}\right).$$

This proves equation (2.5).

In order to prove equation (2.6), we observe,

$$\begin{aligned}
&2\sum_{m=1}^{M-2}\hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2)\\
&= 2\sum_{m=1}^{M-2}\left(\frac{\sin(2\pi m\beta)-\sin(2\pi m\alpha)}{m\pi}\right)\left(\frac{\sin(2\pi(m+2)\beta)-\sin(2\pi(m+2)\alpha)}{(m+2)\pi}\right)\\
&\quad + \mathrm{O}\left(\frac{\log M}{M}\right)\\
&= \frac{2}{\pi^2}\sum_{m=1}^{M-2}\frac{\sin(2\pi m\beta)\sin(2\pi(m+2)\beta)}{m(m+2)} - \frac{2}{\pi^2}\sum_{m=1}^{M-2}\frac{\sin(2\pi m\beta)\sin(2\pi(m+2)\alpha)}{m(m+2)}\\
&\quad - \frac{2}{\pi^2}\sum_{m=1}^{M-2}\frac{\sin(2\pi m\alpha)\sin(2\pi(m+2)\beta)}{m(m+2)} + \frac{2}{\pi^2}\sum_{m=1}^{M-2}\frac{\sin(2\pi m\alpha)\sin(2\pi(m+2)\alpha)}{m(m+2)}\\
&\quad + \mathrm{O}\left(\frac{\log M}{M}\right).
\end{aligned}$$

We write:

$$2\sin(2\pi m\beta)\sin(2\pi(m+2)\beta) = \cos(4\pi\beta) - \cos(4\pi(m+1)\beta),$$

$$2\sin(2\pi m\alpha)\sin(2\pi(m+2)\alpha) = \cos(4\pi\alpha) - \cos(4\pi(m+1)\alpha).$$

Next, we combine the right hand sides of

$$2\sin(2\pi m\beta)\sin(2\pi(m+2)\alpha) = \cos(2\pi m(\beta-\alpha)-4\pi\alpha)-\cos(2\pi m(\beta+\alpha)+4\pi\alpha)$$

and

$$2\sin(2\pi m\alpha)\sin(2\pi(m+2)\beta) = \cos(2\pi m(\alpha-\beta)-4\pi\beta)-\cos(2\pi m(\alpha+\beta)+4\pi\beta)$$

to get

$$\cos(2\pi m(\beta-\alpha)-4\pi\alpha)+\cos(2\pi m(\alpha-\beta)-4\pi\beta) = 2\cos(2\pi(\alpha+\beta))\cos(2\pi(m+1)(\beta-\alpha))$$

and

$$\cos(2\pi m(\beta+\alpha)+4\pi\alpha)+\cos(2\pi m(\alpha+\beta)+4\pi\beta) = 2\cos(2\pi(\beta-\alpha))\cos(2\pi(m+1)(\alpha+\beta)).$$

Therefore we have,

$$2\sum_{m=1}^{M-2}\hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2)=$$

$$\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi\beta)}{m(m+2)}-\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi(m+1)\beta)}{m(m+2)}+\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi\alpha)}{m(m+2)}$$
$$-\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi(m+1)\alpha)}{m(m+2)}-\frac{2\cos(2\pi(\alpha+\beta))}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(2\pi(m+1)(\beta-\alpha))}{m(m+2)}$$
$$+\frac{2\cos(2\pi(\beta-\alpha))}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(2\pi(m+1)(\beta+\alpha))}{m(m+2)}+\mathrm{O}\left(\frac{\log M}{M}\right).$$

We use the following trigonometric sum (see [2, p. 368] or Equation (605) of [11]):

$$\sum_{m=1}^{\infty}\frac{\cos((m+1)\theta)}{m(m+2)}=\frac{1}{2}+\frac{\cos(\theta)}{4}-\frac{\pi-\theta}{2}\sin(\theta),\quad 0<\theta<2\pi.$$

Using the above equation and the following identity:

$$\sum_{m=1}^{\infty}\frac{1}{m(m+2)}=\frac{3}{4},$$

we deduce the following:

(A) $\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi\beta)}{m(m+2)}=\frac{3}{4\pi^2}\cos(4\pi\beta)+\mathrm{O}\left(\frac{1}{M}\right).$

(B)

$$-\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi(m+1)\beta)}{m(m+2)}=-\frac{1}{\pi^2}\left(\frac{1}{2}+\frac{\cos(4\pi\beta)}{4}-\frac{(\pi-4\pi\beta)}{2}\sin(4\pi\beta)\right)$$
$$+\mathrm{O}\left(\frac{1}{M}\right).$$

(C) $\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi\alpha)}{m(m+2)}=\frac{3}{4\pi^2}\cos(4\pi\alpha)+\mathrm{O}\left(\frac{1}{M}\right).$

(D)

$$-\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi(m+1)\alpha)}{m(m+2)} = -\frac{1}{\pi^2}\left(\frac{1}{2}+\frac{\cos(4\pi\alpha)}{4}-\frac{(\pi-4\pi\alpha)}{2}\sin(4\pi\alpha)\right) + \mathrm{O}\left(\frac{1}{M}\right).$$

(E) $$-\frac{2\cos(2\pi(\alpha+\beta))}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(2\pi(m+1)(\beta-\alpha))}{m(m+2)}$$
$$= -\frac{2\cos(2\pi(\alpha+\beta))}{\pi^2}\left(\frac{1}{2}+\frac{\cos(2\pi(\beta-\alpha))}{4}-\frac{(\pi-2\pi(\beta-\alpha))}{2}\sin(2\pi(\beta-\alpha))\right) + \mathrm{O}\left(\frac{1}{M}\right).$$

(F) $$\frac{2\cos(2\pi(\beta-\alpha))}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(2\pi(m+1)(\beta+\alpha))}{m(m+2)}$$
$$= \frac{2\cos(2\pi(\beta-\alpha))}{\pi^2}\left(\frac{1}{2}+\frac{\cos(2\pi(\alpha+\beta))}{4}-\frac{(\pi-2\pi(\alpha+\beta))}{2}\sin(2\pi(\alpha+\beta))\right) + \mathrm{O}\left(\frac{1}{M}\right).$$

From equations (A)-(D) above, we obtain

$$\frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi\beta)}{m(m+2)} - \frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi(m+1)\beta)}{m(m+2)} + \frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi\alpha)}{m(m+2)} - \frac{1}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(4\pi(m+1)\alpha)}{m(m+2)}$$
$$= \frac{\cos(4\pi\beta)-1}{2\pi^2} + \frac{\cos(4\pi\alpha)-1}{2\pi^2} + \frac{1}{2\pi}\left((1-4\beta)\sin(4\pi\beta)+(1-4\alpha)\sin(4\pi\alpha)\right) + \mathrm{O}\left(\frac{1}{M}\right).$$

From equations (E) and (F) above, we get

$$-\frac{2\cos(2\pi(\alpha+\beta))}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(2\pi(m+1)(\beta-\alpha))}{m(m+2)} + \frac{2\cos(2\pi(\beta-\alpha))}{\pi^2}\sum_{m=1}^{M-2}\frac{\cos(2\pi(m+1)(\beta+\alpha))}{m(m+2)}$$

$$
\begin{aligned}
&= \frac{1}{\pi^2}\left(\cos(2\pi(\beta-\alpha)) - \cos(2\pi(\alpha+\beta))\right) \\
&\quad + \frac{1}{\pi}(1-2(\beta-\alpha))\cos(2\pi(\alpha+\beta))\sin(2\pi(\beta-\alpha)) \\
&\quad - \frac{1}{\pi}(1-2(\alpha+\beta))\cos(2\pi(\beta-\alpha))\sin(2\pi(\beta+\alpha)) + \mathrm{O}\left(\frac{1}{M}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{\pi^2}(\sin(2\pi\beta)\sin(2\pi\alpha)) + \frac{(1-2\beta)}{\pi}\cos(2\pi(\alpha+\beta))\sin(2\pi(\beta-\alpha)) \\
&\quad - \frac{(1-2\beta)}{\pi}\cos(2\pi(\beta-\alpha))\sin(2\pi(\beta+\alpha)) + \frac{2\alpha}{\pi}\cos(2\pi(\alpha+\beta))\sin(2\pi(\beta-\alpha)) \\
&\quad + \frac{2\alpha}{\pi}\cos(2\pi(\beta-\alpha))\sin(2\pi(\beta+\alpha)) + \mathrm{O}\left(\frac{1}{M}\right)
\end{aligned}
$$

$$
= \frac{2}{\pi^2}(\sin(2\pi\beta)\sin(2\pi\alpha)) - \frac{(1-2\beta)}{\pi}\sin(4\pi\alpha) + \frac{2\alpha}{\pi}\sin(4\pi\beta) + \mathrm{O}\left(\frac{1}{M}\right).
$$

Therefore, putting all of it together we have

$$
\begin{aligned}
&2\sum_{m=1}^{M-2}\hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2) \\
&= \frac{\cos(4\pi\beta)-1}{2\pi^2} + \frac{\cos(4\pi\alpha)-1}{2\pi^2} + \frac{1}{2\pi}\left((1-4\beta)\sin(4\pi\beta) + (1-4\alpha)\sin(4\pi\alpha)\right) \\
&+ \frac{2}{\pi^2}(\sin(2\pi\beta)\sin(2\pi\alpha)) - \frac{(1-2\beta)}{\pi}\sin(4\pi\alpha) + \frac{2\alpha}{\pi}\sin(4\pi\beta) + \mathrm{O}\left(\frac{\log M}{M}\right).
\end{aligned}
$$

Simplifying, we get equation (2.6). This proves the proposition.

□

We record the following bound, which is not optimal, but good enough for our purposes. For $M \geq 3$ and $1 \leq m \leq M$, let

$$
\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m) := \begin{cases} \hat{\mathcal{S}}^{\pm}(m) - \hat{\mathcal{S}}^{\pm}(m+2), & \text{if } 1 \leq m \leq M-2 \\ \hat{\mathcal{S}}^{\pm}(m), & \text{if } m = M-1,\, M. \end{cases}
$$

where $\hat{\mathcal{S}}^{\pm}(m)$ is as defined in equation (2.3).

Lemma 2.3.2 *Let $I = [\alpha, \beta]$ be a fixed interval and $\underline{m}_r = (m_1, \ldots, m_r)$ be an r-tuple of positive integers where each $1 \leq m_i \leq M$. Let $\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_r) =$*

$\hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(m_1)\cdots\hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(m_r)$.

$$\sum_{\underline{m}_i}^{(3)} |\hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(\underline{m}_r)| = \mathrm{O}(\log M)^r.$$

Here, the implied constant depends on r.

Proof. From equation (2.2), we observe that for any $1 \leq m \leq M$,

$$|\hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(m)| \leq \frac{2}{\pi|m|} + \frac{2}{M+1},$$

the implied constant being absolute. Thus,

$$\begin{aligned}
\sum_{\underline{m}_r}^{(3)} |\hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(\underline{m}_r)| &= \mathrm{O}\left(\sum_{\underline{m}_r}\prod_{j=1}^{r}\left(\frac{1}{\pi m_j} + \frac{1}{M+1}\right)\right) \\
&= \mathrm{O}\left(\sum_{k=0}^{r}\frac{1}{\pi^k}\frac{1}{(M+1)^{r-k}}\sum_{m_{j_1},m_{j_2},\ldots,m_{j_k}}\frac{1}{m_{j_1}m_{j_2}\ldots m_{j_k}}\right) \\
&= \mathrm{O}\left(\sum_{k=0}^{r}\binom{r}{k}\frac{1}{\pi^k}\frac{1}{(M+1)^{r-k}}M^{r-k}(\log M)^k\right) \\
&= \mathrm{O}_r(\log M)^r.
\end{aligned}$$

□

Chapter 3

The first moment

3.1 Groundwork

Unless otherwise mentioned, henceforth, the level N will be assumed to be equal to 1. We denote $\mathcal{F}_k$ to be the set of normalized eigenforms in $S(1,k)$ and s_k its dimension.
For an interval $I = [a,b] \subseteq [-2,2]$, we define

$$N_I(f,x) := \# \{ p \leq x : \; a_f(p) \in I \} .$$

By a deep result of Deligne [7] that settled the Ramanujan-Petersson conjecture for modular forms, we know that the eigenvalues $a_f(p) \in [-2,2]$. Therefore, we may write

$$a_f(p) = 2\cos\theta_f(p), \text{ with } \theta_f(p) \in [0,\pi].$$

In order to ease the calculations later that help with simplifying exponential sums, we introduce some symmetry and consider the families

$$\left\{ \pm \frac{\theta_f(p)}{2\pi}, \; f \in \mathcal{F}_k \right\}.$$

As before, we choose a subinterval

$$I_1 = [\alpha, \beta] \subseteq \left[0, \frac{1}{2}\right]$$

so that

$$\frac{\theta_f(p)}{2\pi} \in I_1 \iff 2\cos\theta_f(p) \in I.$$

Let $I_2 = (\alpha, \beta]$. We do so in order to avoid counting zero, if it occurs as an endpoint, twice. Note that the approximating functions for the characteristic function of an interval $(a, b]$ or $[a, b)$ or (a, b) are the same as those of $[a, b]$, because these functions only depend on the length of the interval and the end points. Now we go back to our quantity of interest and write

$$N_I(f, x) = \sum_{p \le x} \left[\chi_{I_1}\left(\frac{\theta_f(p)}{2\pi}\right) + \chi_{I_2}\left(-\frac{\theta_f(p)}{2\pi}\right)\right],$$

since

$$\chi_{I_2}\left(-\frac{\theta_f(p)}{2\pi}\right) = 0.$$

Following the notation and properties of the Beurling-Selberg polynomials from the previous section, we have

$$\sum_{p \le x} \left[S_M^-\left(\frac{\theta_f(p)}{2\pi}\right) + S_M^-\left(-\frac{\theta_f(p)}{2\pi}\right)\right] \le N_I(f, x) \le \sum_{p \le x} \left[S_M^+\left(\frac{\theta_f(p)}{2\pi}\right) + S_M^+\left(-\frac{\theta_f(p)}{2\pi}\right)\right] \tag{3.1}$$

We now focus our attention on the quantity

$$X_f(x) := N_I(f, x) - \pi(x) \int_I \mu_\infty(t) dt$$

and explore the moments

$$\lim_{x \to \infty} \frac{1}{s_k} \sum_{f \in \mathcal{F}_k} \left(N_I(f, x) - \pi(x) \int_I \mu_\infty(t) dt \right)^r$$

as $k = k(x)$ satisfies $\frac{\log k}{\sqrt{x} \log x} \to \infty$. The strategy is to use equation (3.1) to approximate the above expression by certain trigonometric polynomials and evaluate the moments of these polynomials.

3.2 Computing the expected value

Using the Fourier expansion, we have

$$S_M^{\pm}(x) = \sum_{m=1}^{M} \hat{S}_M^{\pm}(m) e(mx).$$

This is where introducing $-\theta_f(p)$ for each $\theta_f(p)$ eases the calculation. Evaluating the function $S_M^{\pm}$ at x and $-x$ and adding, we have

$$\begin{aligned} S_M^{\pm}(x) + S_M^{\pm}(-x) &= \sum_{|m|\le M} \Big(\hat{S}_M^{\pm}(m)e(mx) + \hat{S}_M^{\pm}(m)e(-mx)\Big) \\ &= \sum_{|m|\le M} \hat{S}_M^{\pm}(m) 2\cos(|m|\,\theta_f(p)) \\ &= 2\hat{S}_M^{\pm}(0) + \sum_{m=1}^{M} \Big(\hat{S}_M^{\pm}(m) + \hat{S}_M^{\pm}(-m)\Big) 2\cos(m\,\theta_f(p)). \end{aligned}$$

This allows us to write

$$\begin{aligned} N_I(f,x) &\le \sum_{p\le x} \left[S_M^{+}\left(\frac{\theta_f(p)}{2\pi}\right) + S_M^{+}\left(-\frac{\theta_f(p)}{2\pi}\right)\right] \\ &= \sum_{p\le x}\sum_{|m|\le M} \hat{S}_M^{+}(m) 2\cos(m\,\theta_f(p)) \\ &= \sum_{p\le x} 2\hat{S}_M^{+}(0) + \sum_{m=1}^{M} (\hat{S}_M^{+}(m) + \hat{S}_M^{+}(-m)) \sum_{p\le x} 2\cos(m\,\theta_f(p)) \\ &= \pi(x)\left(2(\beta-\alpha) + \frac{2}{M+1}\right) + \sum_{m=1}^{M} (\hat{S}_M^{+}(m) + \hat{S}_M^{+}(-m)) \sum_{p\le x} 2\cos(m\,\theta_f(p)) \\ &= \pi(x)\left(2(\beta-\alpha) + \frac{2}{M+1}\right) + \sum_{m=1}^{M} \hat{\mathcal{S}}^{+}(m) \sum_{p\le x} 2\cos(m\,\theta_f(p)), \end{aligned} \tag{3.2}$$

using the notation defined in equation (2.4). By a similar argument, we derive

$$N_I(f,x) \ge \pi(x)\left(2(\beta-\alpha) - \frac{2}{M+1}\right) + \sum_{m=1}^{M} \hat{\mathcal{S}}^{-}(m) \sum_{p\le x} 2\cos(m\,\theta_f(p)). \tag{3.3}$$

We know from before that for $m = 1$, $2\cos(m\,\theta_f(p)) = a_f(p)$. A classical expression for $m > 1$, see Serre [33], is described below.

Lemma 3.2.1 *For a prime p and $f \in \mathcal{F}_k$, let $\theta_f(p)$ be the unique angle in $[0, \pi]$ such that $a_f(p) = 2\cos\theta_f(p)$. For $m \geq 0$,*

$$a_f(p^m) = X_m(a_f(p)),$$

where the m-th Chebyshev polynomial is defined as follows:

$$X_m(x) = \frac{\sin(m+1)\theta}{\sin\theta},\ x = 2\cos\theta.$$

We observe that for $m \geq 2$,

$$2\cos m\theta = X_m(2\cos\theta) - X_{m-2}(2\cos\theta).$$

Thus, we have the following corollary to the above lemma.

Corollary 3.2.2 *With the same notation as in Lemma 3.2.1, for $m \in \mathbb{Z}$, $m \neq 0$,*

$$2\cos(m\,\theta_f(p)) = \begin{cases} a_f(p) & \text{if } |m| = 1 \\ a_f(p^{|m|}) - a_f(p^{|m|-2}) & \text{if } |m| \geq 2. \end{cases}$$

Let us denote

$$\begin{aligned} S^{\pm}(M,f)(x) := &\sum_{m=1}^{2} (\hat{S}^{\pm}_M(m) + \hat{S}^{\pm}_M(-m)) \sum_{p \leq x} a_f(p^m) \\ &+ \sum_{m=3}^{M} (\hat{S}^{\pm}_M(m) + \hat{S}^{\pm}_M(-m)) \sum_{p \leq x} \left(a_f(p^m) - a_f(p^{m-2})\right) \\ = &\sum_{m=1}^{2} \hat{\mathcal{S}}^{\pm}(m) \sum_{p \leq x} a_f(p^m) + \sum_{m=3}^{M} \hat{\mathcal{S}}^{\pm}(m) \sum_{p \leq x} \left(a_f(p^m) - a_f(p^{m-2})\right). \end{aligned}$$

By combining equations (3.2), (3.3) and Corollary 3.2.2, we get

$$N_I(f,x) - \pi(x)\left[2(\beta - \alpha) - \hat{\mathcal{S}}^{+}(2)\right] \leq S^{+}(M,f)(x) + 2\frac{\pi(x)}{M+1} \tag{3.4}$$

and

$$-2\frac{\pi(x)}{M+1} + S^{-}(M,f)(x) \leq N_I(f,x) - \pi(x)\left[2(\beta-\alpha) - \hat{\mathcal{S}}^{-}(2)\right]. \qquad (3.5)$$

We are now ready to calculate the first moment of $N_I(f,x)$. Henceforth, for any function $\phi:\, S_k \to \mathbb{C}$, we denote the average

$$\langle \phi(f) \rangle := \frac{1}{s_k}\sum_{f\in\mathcal{F}_k} \phi(f).$$

In order to derive the moments $\langle (X_f(x))^r \rangle$, we explore the moments of $S^{\pm}(M,f)(x)$. In this direction, we state a proposition, which shows that the Sato-Tate conjecture is true on average as $x \to \infty$. The usefulness of averaging over the set of eigenforms lies in the fact that it enables us to use the trace formula. We first give a brief description of this formula.

3.3 Eichler-Selberg Trace Formula and some estimates

Let n be a positive integer coprime to N. The Eichler-Selberg trace formula describes the trace of T_n acting on $S(N,k)$. Following the presentation of this formula in ([12], p. 370), for every integer $n \geq 1$, Tr $T_n(N,k) = \sum_{i=1}^{4} A_i(n,N,k)$, where A_i's are as follows :

$$A_1(n,N,k) = \frac{k-1}{12}\psi(N)\begin{cases} n^{(k/2-1)} & \text{if } n \text{ is a square,}\\ 0 & \text{otherwise.}\end{cases}$$

$$A_2(n,N,k) = -\frac{1}{2}\sum_{t\in\mathbb{Z},\, t^2<4n} \frac{\varrho^{k-1}-\overline{\varrho}^{k-1}}{\varrho-\overline{\varrho}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right)\mu(t,f,n).$$

$$A_3(n,N,k) = -{\sum}'_{\substack{d|n,\\ 0<d\leq\sqrt{n}}} d^{k-1}F(N)_d,$$

where $F(N)_d$ is a multiplicative function of N defined as

$$F(N)_d = \sum_{\substack{c|N\\ \gcd\,(c,\frac{N}{c})|\frac{n}{d}-d}} \phi\left(\gcd\left(c,\frac{N}{c}\right)\right).$$

$$A_4(n,N,k) = \begin{cases} \sum_{t|n,\, t>0} t & \text{if } k=2, \\ 0 & \text{otherwise.} \end{cases}$$

In the above terms,

- $\psi(N) = N \prod_{p|N} \left(1+\frac{1}{p}\right)$,
- ϱ and $\overline{\varrho}$ are the complex zeroes of the polynomial $x^2 - tx + n$.
- The inner sum in the second term runs over all positive divisors f of t^2-4n such that $(t^2-4n)/f^2 \in \mathbb{Z}$ is congruent to 0 or $1 \bmod 4$.
- $h_w(\Delta)$ is the class number of the imaginary quadratic order of discriminant Δ divided by 2 (resp. 3) if the discriminant is -4 (resp. -3).
- We have
$$\mu(t,f,n) = \frac{\psi(N)}{\psi\left(\frac{N}{N_f}\right)} M(t,n,NN_f),$$
where $N_f = \gcd(N,f)$ and $M(t,n,NN_f)$ denotes the number of elements of $(\mathbb{Z}/N\mathbb{Z})^*$ which lift to solutions of $x^2-tx+n \equiv 0 \mod NN_f$.
- The dash on top of the summation in the third term of $Tr\, T_n(N,k)$ indicates that if there is a contribution from the term $d=\sqrt{n}$, it should be multiplied by 1/2.

We now state some results involving estimates of the trace formula that will be used in the calculation of the first moment and later in the chapter on higher moments.

The following lemma is a direct consequence of the multiplicative relations satisfied by the Hecke operators T_n, stated in the introduction. Nevertheless, we state it here since this is the exact form in which it will be used in later chapters.

Lemma 3.3.1 *Let $f \in \mathcal{F}_k$. For primes p_1, p_2 and non-negative integers i, j,*

$$a_f(p_1^i)a_f(p_2^j) = \begin{cases} a_f(p_1^i p_2^j) & \text{if } p_1 \neq p_2 \\ \sum_{l=0}^{\min(i,j)} a_f(p_1^{i+j-2l}) & \text{if } p_1 = p_2. \end{cases}$$

Proposition 3.3.2 *Let k be a positive even integer and n be a positive integer. We have*

$$\sum_{f \in \mathcal{F}_k} a_f(n) = \begin{cases} \frac{k-1}{12}\left(\frac{1}{\sqrt{n}}\right) + \mathrm{O}\left(n^c\right) & \text{if } n \text{ is a square} \\ \mathrm{O}\left(n^c\right) & \text{otherwise,} \end{cases}$$

where the constant c is absolute and $0 < c < 1$. The implied constant in the error term is also absolute.

Proof. This proposition follows from the Eichler-Selberg trace formula for Hecke operators T_n, $n \geq 1$ acting on S_k. The Eichler-Selberg trace formula (see [25, Sections 7, 8] and [33, Section 4]) states that for every integer $n \geq 1$,

$$\sum_{f \in \mathcal{F}_k} a_f(n) = \sum_{i=1}^{4} B_i(n),$$

where $B_i(n)$'s are as follows:

$$B_1(n) = \begin{cases} \frac{k-1}{12}\left(\frac{1}{\sqrt{n}}\right) & \text{if } n \text{ is a square} \\ 0 & \text{otherwise.} \end{cases}$$

$$B_2(n) = -\frac{1}{2}\frac{1}{n^{(k-1)/2}} \sum_{t \in \mathbb{Z},\, t^2 < 4n} \frac{\varrho^{k-1} - \overline{\varrho}^{k-1}}{\varrho - \overline{\varrho}} H(4n - t^2).$$

Here, ϱ and $\overline{\varrho}$ denote the zeroes of the polynomial $x^2 - tx + n$ and for a positive integer l, $H(l)$ denotes the Hurwitz class number.

$$B_3(n) = -\frac{1}{n^{(k-1)/2}} \sum_{\substack{d|n \\ 0 \leq d \leq \sqrt{n}}}^{(b)} d^{k-1}.$$

The notation (b) on top of the summation denotes that if there is a contribution from $d = \sqrt{n}$, it should be multiplied with $1/2$. Finally,

$$B_4(n) = \begin{cases} \frac{1}{n^{(k-1)/2}} \sum_{d|n} d & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

To estimate $B_2(n)$, we observe that $|\varrho| = \sqrt{n}$. Thus,

$$\left| \frac{\varrho^{k-1} - \overline{\varrho}^{k-1}}{\varrho - \overline{\varrho}} \right| \leq \frac{2n^{(k-1)/2}}{\sqrt{4n - t^2}}.$$

Following a classical estimate of Hurwitz, we have

$$H(4n - t^2) \ll \sqrt{4n - t^2} \log^2(n),$$

the implied constant being absolute. Thus,

$$|B_2(n)| \ll \sqrt{n} \log^2 n.$$

One can immediately observe that

$$|B_3(n)| \ll \sum_{\substack{d|n \\ d \leq \sqrt{n}}} 1$$

and

$$|B_4(n)| \ll \sqrt{n} \sum_{d|n} 1.$$

Combining the above estimates, we prove Proposition 3.3.2. □

In particular, $n = 1$ in the above proposition gives us

$$s_k = \frac{k-1}{12} + \mathrm{O}(1). \tag{3.6}$$

We also record the following important estimate:

$$\sum_{p \leq x} \frac{1}{p} = \mathrm{O}(\log \log x).$$

Remark 3.3.3 *In the calculations that follow, we will encounter sums of the form*

$$\sum_{p \leq x} \frac{1}{p^n}, \quad n > 1,$$

which is of course $\mathrm{O}(1)$*. However, since the power of* p *will be guaranteed to be positive and we obtain no improvement with respect to error terms on separating the case where* $n = 1$ *and* $n > 1$*, we combine the estimates and use:*

$$\sum_{p \leq x} \frac{1}{p^n} = \mathrm{O}(\log \log x), \quad n \geq 1. \tag{3.7}$$

We are now ready to state a lemma which will be repeatedly used in the coming chapters.

Lemma 3.3.4 *Suppose $k = k(x)$ runs over positive even integers such that $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. Then, for any positive integer m and and positive real number a, we have*

$$\lim_{x\to\infty} \frac{1}{(\pi(x))^a s_k} \sum_{p\le x} \sum_{f\in\mathcal{F}_k} a_f(p^m) = 0.$$

More generally, for non-negative integers m_1, m_2, not both zero,

$$\lim_{x\to\infty} \frac{1}{(\pi(x))^a s_k} \sum_{p_1\ne p_2\le x} \sum_{f\in\mathcal{F}_k} a_f(p_1^{m_1} p_2^{m_2}) = 0.$$

Proof. From Proposition 3.3.2, equations (3.6) and (3.7), one deduces, for $m \ge 1$, the following:

$$\frac{1}{s_k} \sum_{p\le x} \sum_{f\in\mathcal{F}_k} a_f(p^m) = \begin{cases} \mathrm{O}(\log\log x) + \mathrm{O}\left(\frac{x^{2m}\pi(x)}{s_k}\right) & \text{if } m \ge 2,\ m \text{ is even} \\ \mathrm{O}\left(\frac{x^{2m}\pi(x)}{s_k}\right) & \text{if } m \ge 1,\ m \text{ is odd.} \end{cases} \tag{3.8}$$

Similarly, we have

$$\frac{1}{s_k} \sum_{p_1\ne p_2\le x} \sum_{f\in\mathcal{F}_k} a_f(p_1^{m_1} p_2^{m_2})$$

$$= \begin{cases} \mathrm{O}(\log\log x)^2 + \mathrm{O}\left(\frac{x^{2(m_1+m_2)}\pi(x)}{s_k}\right) & \text{if } m_1, m_2 \text{ are both even} \\ \mathrm{O}\left(\frac{x^{2(m_1+m_2)}\pi(x)}{s_k}\right) & \text{otherwise.} \end{cases} \tag{3.9}$$

Since $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$,

$$\lim_{x\to\infty} \frac{x^r}{k} = 0$$

for any real power $r > 0$. Moreover,

$$\log\log x = \mathrm{o}(\pi(x))^a,$$

for any $a > 0$. The lemma follows immediately. □

3.4 The average Sato-Tate theorem

In our attempt to compute the first moment we shall be proving the following theorem, which we call the average Sato-Tate Theorem.

Theorem 3.4.1 *Let $k = k(x)$ be a positive even integer. Then, for any interval $I = [a, b] \subseteq [-2, 2]$,*

$$\langle N_I(f, x) \rangle = \pi(x) \int_a^b \mu_\infty(t) dt + \mathrm{O}\left(\frac{\pi(x) \log x}{\log k} + \log \log x \right).$$

Thus, if $k = k(x)$ runs over positive even integers such that $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$, then

$$\lim_{x \to \infty} \langle N_I(f, x) \rangle = \pi(x) \int_a^b \mu_\infty(t) dt.$$

Remark 3.4.2 *Proposition 3.4.1 is essentially due to Y. Wang [36, Theorem 1.1]. He proves an analogous result for primitive Maass forms and indicates that a similar technique works for the average Sato-Tate family. We provide a brief proof of this proposition as a first step in evaluating moments of the polynomials $S^{\pm}(M, f)(x)$.*

Proof. We have, by equation (2.4),

$$(\hat{S}_M^{\pm}(2) + \hat{S}_M^{\pm}(-2)) = \frac{\sin 4\pi\beta - \sin 4\pi\alpha}{2\pi} + \mathrm{O}\left(\frac{1}{M}\right).$$

Combining the above with equations (3.4) and (3.5), we can find constants C and D such that

$$\begin{aligned} S^-(M, f)(x) + C\left(\frac{\pi(x)}{M+1}\right) &\leq N_I(f, x) - \pi(x)\left((2\beta - 2\alpha) - \frac{\sin 4\pi\beta - \sin 4\pi\alpha}{2\pi}\right) \\ &\leq S^+(M, f)(x) + D\left(\frac{\pi(x)}{M+1}\right). \end{aligned} \tag{3.10}$$

We observe, for $[\alpha,\beta] \in [0,1/2]$ as chosen before,

$$\begin{aligned}(2\beta-2\alpha)-\frac{\sin 4\pi\beta-\sin 4\pi\alpha}{2\pi} &= 2\int_\alpha^\beta (1-\cos 4\pi\theta)d\theta \\ &= 4\int_\alpha^\beta \sin^2 2\pi\theta d\theta \\ &= \int_a^b \mu_\infty(t)dt.\end{aligned}$$

Thus, for every positive integer M,

$$S^-(M,f)(x)+C\left(\frac{\pi(x)}{M+1}\right) \le N_I(f,x)-\pi(x)\int_a^b \mu_\infty(t)dt \le S^+(M,f)(x)+D\left(\frac{\pi(x)}{M+1}\right). \tag{3.11}$$

By equation (3.8),

$$\begin{aligned}&\left\langle S^\pm(M,f)(x)\right\rangle \\ &= \mathrm{O}\left(\sum_{\substack{m=1\\ m \text{ even}}}^M |\hat{S}_M^\pm(m)| \sum_{p\le x}\begin{cases}\frac{1}{p}, & \text{if } m=2\\ \frac{1}{p^{\frac{m}{2}-1}}, & \text{if } m\ge 4\end{cases}\right)+\mathrm{O}\left(\sum_{m=1}^M |\hat{S}_M^\pm(m)|\frac{x^{2m}\pi(x)}{k}\right).\end{aligned}$$

Since

$$|\hat{S}_M^\pm(m)| \le \frac{1}{M+1}+\min\left\{\beta-\alpha,\frac{1}{\pi|m|}\right\},$$

we get, for every positive integer M,

$$\begin{aligned}\left\langle S^\pm(M,f)(x)\right\rangle &= \mathrm{O}\left(\sum_{p\le x}\left(\frac{1}{p}+\sum_{m=2}^\infty \frac{1}{p^m}\right)+\sum_{m=1}^M |\hat{S}_M^\pm(m)|\frac{x^{2m}\pi(x)}{k}\right) \\ &= \mathrm{O}\left(\log\log x+\frac{x^{2M}\pi(x)}{k}+\frac{\pi(x)}{k}\sum_{m=1}^M \frac{x^{2m}}{m}\right).\end{aligned}$$

That is, for every positive integer M, by equation (3.11), we have

$$\left\langle N_I(f,x)-\pi(x)\int_a^b \mu_\infty(t)dt\right\rangle = \mathrm{O}\left(\log\log x+\frac{x^{2M}\pi(x)}{k}+\frac{\pi(x)}{M+1}\right).$$

We now choose

$$M=\left\lfloor \frac{d\log k}{2\log x}\right\rfloor$$

for some $0<d<1$. This proves the proposition. □

3.5 Second moment

In this section, we will compute

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \left\langle (S^{\pm}(M,f)(x))^2 \right\rangle$$

for a suitable choice of $M = M(x)$ which grows as a function of x.

Proposition 3.5.1 *Let $[\alpha, \beta]$ be a fixed interval in $[0, 1/2]$. Suppose $k = k(x)$ runs over positive even integers such that $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Let $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$. Then,*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \left\langle (S^{\pm}(M,f)(x))^2 \right\rangle$$

$$= 2\sum_{m=1}^{M} \hat{\mathcal{S}}^{\pm}(m)^2 - 2\sum_{m=1}^{M-2} \hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2) - \hat{\mathcal{S}}^{\pm}(1)^2 - \hat{\mathcal{S}}^{\pm}(2)^2.$$

Proof. Recall, for $M \geq 3$ and $1 \leq m \leq M$, let

$$\hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(m) := \begin{cases} \hat{\mathcal{S}}^{\pm}(m) - \hat{\mathcal{S}}^{\pm}(m+2), & \text{if } 1 \leq m \leq M-2 \\ \hat{\mathcal{S}}^{\pm}(m), & \text{if } m = M-1,\ M. \end{cases}$$

We start by observing that

$$\begin{aligned}
&S^{\pm}(M,f)(x))^2 \\
&= \left(\sum_{m=1}^{2} \hat{\mathcal{S}}^{\pm}(m) \sum_{p\leq x} a_f(p^m) + \sum_{m=3}^{M} \hat{\mathcal{S}}^{\pm}(m) \sum_{p\leq x} \left(a_f(p^m) - a_f(p^{m-2}) \right) \right)^2 \\
&= \Bigg(\sum_{m=1}^{M-2} (\hat{\mathcal{S}}^{\pm}(m) - \hat{\mathcal{S}}^{\pm}(m+2)) \sum_{p\leq x} a_f(p^m) + \hat{\mathcal{S}}^{\pm}(M-1) \sum_{p\leq x} a_f(p^{M-1}) \\
&\qquad + \hat{\mathcal{S}}^{\pm}(M) \sum_{p\leq x} a_f(p^M) \Bigg)^2 \\
&= \left(\sum_{m=1}^{M} \hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(m) \sum_{p\leq x} a_f(p^m) \right)^2 \\
&= \sum_{m_1,m_2=1}^{M} \hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(m_1)\,\hat{\mathcal{U}}^{\pm}_{\mathrm{M}}(m_2) \sum_{p_1,p_2\leq x} a_f(p^{m_1}) a_f(p_2^{m}).
\end{aligned} \tag{3.12}$$

Applying Lemma 3.3.1 and Proposition 3.3.2 along with equations (3.6) and (3.7), we deduce that if $p_1 \neq p_2$ then

$$\frac{1}{s_k}\sum_{f\in\mathcal{F}_k}\sum_{p_1\neq p_2\leq x} a_f(p_1^{m_1})a_f(p_2^{m_2}) = \sum_{p_1\neq p_2\leq x}\left[\frac{\delta(m_1)\delta(m_2)}{(p_1)^{m_1/2}(p_2)^{m_2/2}} + \mathrm{O}\left(\frac{p_1^{cm_1}p_2^{cm_2}}{s_k}\right)\right]$$
$$= \mathrm{O}\left(\frac{\pi(x)^2 x^{(m_1+m_2)c}}{s_k}\right) + \mathrm{O}(\log\log x)^2,$$

where $\delta(n) = 1$ if n is even and is zero otherwise.
If $p_1 = p_2$, then for any m_1, $m_2 \geq 1$,

$$\frac{1}{s_k}\sum_{f\in\mathcal{F}_k}\sum_{p_1\leq x} a_f(p_1^{m_1})a_f(p_1^{m_2}) = \frac{1}{s_k}\sum_{f\in\mathcal{F}_k}\sum_{p_1\leq x}\sum_{i=0}^{\min\{m_1,m_2\}} a_f(p_1^{m_1+m_2-2i})$$
$$= \sum_{p\leq x}\sum_{i=0}^{\min\{m_1,m_2\}} \frac{\delta(m_1+m_2)}{p^{(m_1+m_2-2i)/2}} + \mathrm{O}\left(\frac{p^{(m_1+m_2-2i)c}}{s_k}\right).$$

The sum

$$\sum_{p\leq x}\sum_{i=0}^{\min\{m_1,m_2\}} \frac{\delta(m_1+m_2)}{p^{(m_1+m_2-2i)/2}}$$

contributes $\pi(x)$ when $m_1 = m_2 = i = m$. Otherwise, each term in this sum contributes at most $\mathrm{O}(\log\log x)$. Moreover,

$$\sum_{p\leq x}\sum_{i=0}^{\min\{m_1,m_2\}}\left(\frac{p^{(m_1+m_2-2i)c}}{s_k}\right) = \mathrm{O}\left(\frac{\pi(x)x^{(m_1+m_2)c}}{k}\right).$$

Combining the above information with equation (3.12) and bounds for $\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m)$ from Lemma 2.3.2, we deduce

$$\frac{1}{\pi(x)}\left\langle (S^{\pm}(M,f)(x))^2\right\rangle$$
$$= \sum_{m=1}^{M}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m)^2 + \mathrm{O}\left(\sum_{m_1,m_2=1}^{M}|\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m_1)\,\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m_2)|\left(\frac{(\log\log x)^2}{\pi(x)} + \frac{\pi(x)x^{2Mc}}{k}\right)\right)$$
$$= \left(\sum_{m=1}^{M-2}\left(\hat{\mathcal{S}}^{\pm}(m) - \hat{\mathcal{S}}^{\pm}(m+2)\right)^2 + \hat{\mathcal{S}}^{\pm}(M-1)^2 + \hat{\mathcal{S}}^{\pm}(M)^2\right)$$
$$+ \mathrm{O}\left(\frac{(\log M)^2(\log\log x)^2}{\pi(x)} + \frac{(\log M)^2\pi(x)x^{2Mc}}{k}\right).$$

We now choose

$$M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor.$$

If $k = k(x)$ runs over positive even integers such that $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$, then

$$\lim_{x\to\infty} \frac{(\log M)^2 \pi(x) x^{2Mc}}{k} = 0.$$

Thus,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \left\langle (S^{\pm}(M,f)(x))^2 \right\rangle$$

$$= \sum_{m=1}^{M-2} (\hat{\mathcal{S}}^{\pm}(m) - \hat{\mathcal{S}}^{\pm}(m+2))^2 + \hat{\mathcal{S}}^{\pm}(M-1)^2 + \hat{\mathcal{S}}^{\pm}(M)^2$$

$$= \sum_{m=1}^{M-2} (\hat{\mathcal{S}}^{\pm}(m)^2 + \hat{\mathcal{S}}^{\pm}(m+2)^2 - 2\,\hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2)) + \hat{\mathcal{S}}^{\pm}(M-1)^2 + \hat{\mathcal{S}}^{\pm}(M)^2$$

$$= 2\sum_{m=1}^{M} \hat{\mathcal{S}}^{\pm}(m)^2 - 2\sum_{m=1}^{M-2} \hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2) - \hat{\mathcal{S}}^{\pm}(1)^2 - \hat{\mathcal{S}}^{\pm}(2)^2, \tag{3.13}$$

proving the proposition. □

Remark 3.5.2 *In fact, under the above mentioned growth condition on k, for $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$ and for any $n \geq 1$ and any constant a*

$$\lim_{x\to\infty} \frac{(\log M)^n x^{aM}}{k} = 0.$$

This will prove useful to us when we calculate the asymptotics of higher moments of $S^{\pm}(M,f)(x)$ in later sections. The reason behind our choice of M will become clear in Section 5.1.

3.6 Asymptotic formula for the variance

We compute an asymptotic formula for

$$V_M^{\pm} := 2\sum_{m=1}^{M} \hat{\mathcal{S}}^{\pm}(m)^2 - 2\sum_{m=1}^{M-2} \hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2) - \hat{\mathcal{S}}^{\pm}(1)^2 - \hat{\mathcal{S}}^{\pm}(2)^2.$$

Proposition 3.6.1 *For an interval $I = [a, b] \subset [-2, 2]$ and a positive integer $M \geq 1$, we have*

$$V_M^{\pm} = \mu_\infty(I) - \mu_\infty(I)^2 + \mathrm{O}\left(\frac{\log M}{M}\right).$$

Proof. We have, by equation (2.4),

$$-\hat{\mathcal{S}}^{\pm}(1)^2 - \hat{\mathcal{S}}^{\pm}(2)^2 = -\frac{1}{\pi^2}(\sin(2\pi\beta) - \sin(2\pi\alpha))^2 - \frac{1}{4\pi^2}(\sin(4\pi\beta) - \sin(4\pi\alpha))^2 + \mathrm{O}\left(\frac{1}{M}\right). \tag{3.14}$$

Combining equations (2.5) and (2.6), we have, after suitable rearrangement,

$$2\sum_{m=1}^{M} \hat{\mathcal{S}}^{\pm}(m)^2 - 2\sum_{m=1}^{M-2} \hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2) =$$

$$\begin{aligned} &2(\beta-\alpha) - 4(\beta-\alpha)^2 + \frac{1}{\pi^2}(\sin(2\pi\beta) - \sin(2\pi\alpha))^2 \\ &- \frac{1}{2\pi}(1 - 4(\beta-\alpha))(\sin(4\pi\beta) - \sin(4\pi\alpha)) + \mathrm{O}\left(\frac{\log M}{M}\right). \end{aligned} \tag{3.15}$$

Finally, putting equations (3.14) and (3.15) together, we arrive at the following:

$$\begin{aligned} V_M^{\pm} &= 2(\beta-\alpha) - \frac{(\sin(4\pi\beta) - \sin(4\pi\alpha)}{2\pi} - 4(\beta-\alpha)^2 + \frac{2(\beta-\alpha)}{\pi}(\sin(4\pi\beta) - \sin(4\pi\alpha)) \\ &\quad - \frac{1}{4\pi^2}(\sin(4\pi\beta) - \sin(4\pi\alpha))^2 + \mathrm{O}\left(\frac{\log M}{M}\right) \\ &= 2(\beta-\alpha) - \frac{(\sin(4\pi\beta) - \sin(4\pi\alpha)}{2\pi} - \left(2(\beta-\alpha) - \frac{(\sin(4\pi\beta) - \sin(4\pi\alpha)}{2\pi}\right)^2. \end{aligned}$$

Since

$$2(\beta-\alpha) - \frac{(\sin(4\pi\beta) - \sin(4\pi\alpha)}{2\pi} = \mu_\infty(I),$$

we conclude,

$$V_M^{\pm} = \mu_\infty(I) - \mu_\infty(I)^2 + \mathrm{O}\left(\frac{\log M}{M}\right). \tag{3.16}$$

□

Chapter 4

Higher moments

In the earlier chapters the first and second moments were computed. We now calculate higher moments, that is, we set

$$T_M^{\pm}(x) := \frac{S^{\pm}(M,f)(x)}{\sqrt{\pi(x)}}$$

and evaluate the moments

$$\frac{1}{s_k}\sum_{f\in\mathcal{F}_k}\left(T_M^{\pm}(x)\right)^n \tag{4.1}$$

for positive integers $n \geq 3$.

4.1 Overview of the strategy

By definition, we have

$$\left(T_M^{\pm}(x)\right)^n =$$

$$\frac{1}{\pi(x)^{\frac{n}{2}}}\left[\sum_{p\leq x}\left(\hat{\mathcal{S}}^{\pm}(1)a_f(p)+\hat{\mathcal{S}}^{\pm}(2)a_f(p^2)+\sum_{m=3}^{M}\hat{\mathcal{S}}^{\pm}(m)(a_f(p^m)-a_f(p^{m-2}))\right)\right]^n.$$

For a prime p, let

$$Y_M^{\pm}(p)=\sum_{m=1}^{M}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m)a_f(p^m),$$

where we denote, for $M \geq 1$ and $1 \leq m \leq M$,

$$\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m) := \begin{cases} \hat{\mathcal{S}}^{\pm}(m) - \hat{\mathcal{S}}^{\pm}(m+2), & \text{if } 1 \leq M-2 \\ \hat{\mathcal{S}}^{\pm}(m), & \text{if } m = M-1,\ M. \end{cases}$$

Therefore,

$$\left(T_M^{\pm}(x)\right)^n = \frac{1}{\pi(x)^{\frac{n}{2}}} \left(\sum_{p \leq x} Y_M^{\pm}(p) \right)^n .$$

Using the multinomial formula, we may write the above equation as follows.

$$\left(T_M^{\pm}(x)\right)^n$$

$$= \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{u=1}^{n} \sum_{(r_1, r_2, \ldots, r_u)}^{(1)} \frac{n!}{r_1! r_2! \cdots r_u!} \frac{1}{u!} \sum_{(p_1, p_2, \ldots, p_u)}^{(2)} Y_M^{\pm}(p_1)^{r_1} Y_M^{\pm}(p_2)^{r_2} \cdots Y_M^{\pm}(p_u)^{r_u}, \tag{4.2}$$

where

- The sum $\sum\limits_{(r_1, r_2, \ldots, r_u)}^{(1)}$ is taken over tuples of positive integers $r_1, r_2, \ldots, r_u$ so that

  $r_1 + r_2 + \cdots + r_u = n$, that is, a partition of n into u positive parts.

- The sum $\sum\limits_{(p_1, p_2, \ldots, p_u)}^{(2)}$ is over u-tuples of distinct primes not exceeding x.

We now focus our attention on the terms in the innermost sum, namely the $Y_M^{\pm}(p_i)^{r_i}$. The product $Y_M^{\pm}(p_1)^{r_1} Y_M^{\pm}(p_2)^{r_2} \cdots Y_M^{\pm}(p_u)^{r_u}$ involves many terms and so we simplify the notation as much as we can. Following the same goal as in Proposition 3.5.1, that is, the calculation of the second moment, observe that what we really need is to identify which terms survive when we use the trace formula and let $x \to \infty$ in equation (4.1). This is a little harder than the analogous task in the second moment case, because now we have many more possibilities coming from the various partitions of n. In other words, once we fix the number of parts u of a partition of n, we need to see which terms survive for a given triple $(u, (r_1, \ldots, r_u), (p_1, \ldots, p_u))$. Given such a triple, we first need to extract the coefficients of $a_f(1)$ from each $Y_M^{\pm}(p_i)^{r_i}$ and show that the remaining terms converge to zero in the limit. Then we add up all the surviving terms coming from each triple. Recall that this is exactly the idea of the proof of Proposition 3.5.1, where we saw that the non-zero limits come from terms where $p_1 = p_2$ and $m_1 = m_2$.

Thanks to our growth condition on the weight k and the estimates coming from using the trace formula, as we shall see, the following happens: For a given n, there is at most one u and corresponding to it, exactly one partition $(r_1, \ldots, r_u)$ that will give us something non-trivial! We need to be cautious while summing, because there are about $\mathrm{O}(M^n)$ terms in the numerator and M grows as a function of x. As we shall see, the terms are weighted nicely enough to allow for more than just trivial estimation. The final sum is a quantity that is a power of $\log M$ in the numerator and a positive power of $\pi(x)$ in the denominator for all but one partition. With our choice of M, such a sum is asymptotically zero. Since the number of partitions of n doesn't depend on x, the limit is legitimate.

We now simplify the notation and proceed to understand how this happens. To this end, by repeated use of Lemma 3.3.1, we may write, for each $1 \leq i \leq u$,

$$\begin{aligned} Y_M^{\pm}(p_i)^{r_i} &= \sum_{\underline{m}_i}^{(3)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i) \left[D_{r_i,\underline{m}_i}(0) + \sum_{\substack{t \in \mathcal{I}(\underline{m}_i) \\ t \geq 1}} D_{r_i,\underline{m}_i}(t) a_f(p_i^t) \right] \\ &= C_M^{\pm}(i) + \sum_{\underline{m}_i}^{(3)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i) \sum_{\substack{t \in \mathcal{I}(\underline{m}_i) \\ t \geq 1}} D_{r_i,\underline{m}_i}(t) a_f(p_i^t), \end{aligned} \tag{4.3}$$

where

1. $\underline{m}_i$ denotes an r_i-tuple $(m_{j_1}, \ldots, m_{j_{r_i}})$.

2. $\sum\limits_{\underline{m}_i}^{(3)}$ denotes that the sum is taken over r_i-tuples $\underline{m}_i$ where $1 \leq m_{j_l} \leq M$

for each $1 \leq l \leq r_i$.

3. The term $\hat{\mathcal{U}}^{\pm}(\underline{m}_i)$ denotes the product $\hat{\mathcal{U}}^{\pm}(m_{j_1}) \cdots \hat{\mathcal{U}}^{\pm}(m_{j_{r_i}})$.

4. For each r_i-tuple $\underline{m}_i$, $\mathcal{I}(\underline{m}_i)$ denotes the set of non negative integers t that occur in the power of p_i on using Lemma 3.3.1 and for each $t \in \mathcal{I}(\underline{m}_i)$, $D_{r_i,\underline{m}_i}(t)$ denotes the coefficient of $a_f(p_i^t)$ so obtained.
For example, if $r_i = 4$ and $\underline{m}_i = (1, 3, 2, 4)$ then

$$a_f(p_i)a_f(p_i^3)a_f(p_i^2)a_f(p_i^4) = a_f(p_i^{10}) + 3a_f(p_i^8) + 5a_f(p_i^6) + 6a_f(p_i^4) + 5a_f(p_i^2) + 2.$$

Therfore,in this case, $\mathcal{I}(\underline{m}_i) = \mathcal{I}(1, 3, 2, 4) = \{0, 2, 4, 6, 8, 10\}$ and

$$D_{r_i,\underline{m}_i}(t) = D_{4,(1,3,2,4)}(t) = \begin{cases} 2 & \text{if } t = 0 \\ 5 & \text{if } t = 2 \\ 6 & \text{if } t = 4 \\ 5 & \text{if } t = 6 \\ 3 & \text{if } t = 8 \\ 1 & \text{if } t = 10 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{I}(\underline{m}_i)$ is a finite set for each $\underline{m}_i$. We observe that $D_{r_i,\underline{m}_i}(t)$ is independent of the prime p_i.

5. $C_M^{\pm}(i)$ is the sum of the coefficients of $a_f(1) = 1$, coming from the the expansion using Lemma 3.3.1. That is,

$$C_M^{\pm}(i) = \sum_{\underline{m}_i}^{(3)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i) D_{r_i,\underline{m}_i}(0).$$

Observe that $C_M^{\pm}(i)$ is independent of the prime p_i and is in fact a polynomial expression in $\hat{\mathcal{S}}^{\pm}(m)$, $1 \leq m \leq M$.

4.2 Preliminary lemmas

We now prove the following proposition:

Proposition 4.2.1 *Let $1 \le i \le u$ and $\underline{m}_i$ be an r_i-tuple as specified above. Then, for $t \in \mathcal{I}(\underline{m}_i)$,*

$$D_{r_i,\underline{m}_i}(t) = \begin{cases} 0, & \text{if } r_i = 1,\ t = 0 \\ 1, & \text{if } r_i = 1,\ t \ge 1 \\ \mathrm{O}(1), & \text{if } r_i = 2,\ t \ge 0 \\ \mathrm{O}\left(M^{r_i-2}\right), & \text{if } r_i \ge 3,\ t \ge 1 \\ \mathrm{O}\left(M^{r_i-3}\right), & \text{if } r_i \ge 3,\ t = 0. \end{cases}$$

Proof. While focusing on an r_i-tuple $\underline{m}_i$, we may also denote $D_{r_i,\underline{m}_i}(t)$ as $D_{r_i}(t)$ for brevity.

The cases $r_i = 1, 2$ are clear. In fact, for $r_i = 2$, we have

$$Y_M(p)^2 = \sum_{m_1,\,m_2=1}^{M} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m_1)\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m_2) \sum_{i=0}^{\min\{m_1,\,m_2\}} a_f(p^{m_1+m_2-2i})$$

$$= \sum_{m_1,\,m_2=1}^{M} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m_1)\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m_2) \sum_{t\in\mathcal{I}(m_1,m_2)} a_f(p^t),$$

so the coefficient of $a_f(p^t) = 1$ if $t \in \mathcal{I}(m_1, m_2)$ and zero otherwise.

Remark 4.2.2 *In particular, if $t = 0$,*

$$D_{2,(m_1,m_2)}(0) = \begin{cases} 1 & \text{if } m_1 = m_2 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that if $m_1 + m_2$ is even, then

$$\mathcal{I}(m_1, m_2) \subseteq \{0,\ 2,\ \ldots, m_1 + m_2\}.$$

On the other hand, if $m_1 + m_2$ is odd, then

$$\mathcal{I}(m_1, m_2) \subseteq \{1,\ 3,\ \ldots, m_1 + m_2\}.$$

In either case,

$$|\mathcal{I}(m_1, m_2)| \leq \left(\frac{m_1 + m_2}{2}\right) + 1 \leq M + 1.$$

We now address the case $r_i = 3$. Let $l \in \mathcal{I}(m_1, m_2, m_3)$. The product

$$a_f(p^{m_1})a_f(p^{m_2})a_f(p^{m_3})$$

equals

$$a_f(p^{m_3}) \sum_{i=0}^{\min\{m_1, m_2\}} a_f(p^{m_1+m_2-2i}).$$

We observe that in the above product, $a_f(p^l)$ can occur at most in all possible expansions

$$a_f(p^{m_3})a_f(p^j),\ j \in \mathcal{I}(m_1, m_2).$$

Since $D_2(t) = 1$ for all $t \in \mathcal{I}(m_1, m_2)$ and $|\mathcal{I}(m_1, m_2)| \leq M + 1$, we deduce

$$D_3(l) \leq M + 1 = \mathrm{O}(M).$$

This proves $D_3(r_i) = \mathrm{O}(M^{r_i-2})$ for $r_i = 3$.

We now proceed by induction. Assume that for some $k \geq 3$, $D_k(l) = \mathrm{O}(M^{k-2})$. We observe that for each k-tuple $\underline{m}_i$,

$$\begin{aligned} |\mathcal{I}(\underline{m}_i)| &\leq \left\lfloor \frac{m_1 + m_2 + \cdots + m_k}{2} \right\rfloor + 1 \\ &\leq \left\lfloor \frac{kM}{2} \right\rfloor + 1 = \mathrm{O}_k(M). \end{aligned} \tag{4.4}$$

Now, in the expansion

$$\begin{aligned} &(a_f(p^{m_1})a_f(p^{m_2}) \cdots a_f(p^{m_k}))a_f(p^{m_{k+1}}) \\ &= a_f(p^{m_{k+1}}) \sum_{t \in \mathcal{I}(m_1, m_2, \ldots, m_k)} D_k(t) a_f(p^t), \end{aligned}$$

any $a_f(p^l)$ can occur at most in all possible expansions

$$a_f(p^{m_{k+1}})a_f(p^j),\ j \in \mathcal{I}(m_1, m_2, \ldots, m_k).$$

By the induction hypothesis,

$$D_k(l) = O_k(M^{k-2}).$$

Thus, by equation (4.4), we have

$$D_{k+1}(l) \leq |\mathcal{I}(\underline{m}_i)||D_k(l)| = \mathrm{O}_k(M^{k-1}). \tag{4.5}$$

Therefore, by induction, we have proved that if $r_i \geq 3$, $t \geq 0$,

$$D_{r_i}(t) = \mathrm{O}\left(M^{r_i-2}\right).$$

Note that the implied constant depends on r_i. We now use these estimates to get a better estimate for $D_{r_i}(0)$ for $r_i \geq 3$. We prove

$$D_{r_i}(0) = \mathrm{O}\left(M^{r_i-3}\right),\ r_i \geq 3.$$

(It is not difficult to show that for $r_i = 2$, $D_{r_i}(0) \leq 1$.)
For $r_i = 3$, looking again at the expansion

$$a_f(p^{m_1})a_f(p^{m_2})a_f(p^{m_3}) = a_f(p^{m_3}) \sum_{j \in \mathcal{I}(m_1,m_2)} D_2(j)a_f(p^j)$$

$$= \sum_{j \in \mathcal{I}(m_1,m_2)} \sum_{i=0}^{\min\{j,m_3\}} D_2(j)a_f(p^{m_3+j-2i}),$$

we observe that $m_3 + j - 2i = 0$ for some i if and only if $j = m_3$. Thus,

$$D_3(0) \leq D_2(m_3) = \mathrm{O}(1).$$

In general, for $r_i \geq 3$,

$$a_f(p^{m_1}) \cdots a_f(p^{m_{r_i-1}})a_f(p^{m_{r_i}}) = a_f(p^{m_{r_i}}) \sum_{j \in \mathcal{I}(m_1,\ldots,m_{r_i-1})} D_{r_i-1}(j)a_f(p^j)$$

$$= \sum_{j \in \mathcal{I}(m_1,\ldots,m_{r_i-1})} \sum_{i=0}^{\min\{j,m_{r_i}\}} D_{r_i-1}(j)a_f(p^{m_{r_i}+j-2i}).$$

As before, $m_{r_i} + j - 2i = 0$ if and only if $i = j = m_{r_i}$. Therefore,

$$D_{r_i}(0) \leq D_{r_i-1}(m_{r_i}) = \mathrm{O}(M^{r_i-1-2}) = \mathrm{O}(M^{r_i-3}).$$

Here, the implied constant depends on r_i. This proves the proposition. □

We record the following Lemma.

Lemma 4.2.3 *For $r_i = 2$, $C_M^{\pm}(i) = V_M^{\pm}$, where $V_M^{\pm}$ is as defined in equation (3.16).*

Proof. Observe that for $r_i = 2$, from Remark 4.2.2, it follows that

$$C_M^{\pm}(2) = \sum_{m=1}^{M} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(m)^2,$$

which is exactly the right hand side of equation (3.13), denoted by $V_M^{\pm}$. The claim follows. □

4.3 Gaussian distribution of $T_M^{\pm}(x)$

Equipped with the results proved in the previous section, we now proceed to estimate the product

$$Y_M^{\pm}(p_1)^{r_1} \cdots Y_M^{\pm}(p_i)^{r_u}.$$

Using the notation from equation (4.3) and taking a product of $Y_M^{\pm}(p_i)^{r_i}$ over $i = 1, \ldots, u$, we have

$$\sum_{(p_1,p_2,\ldots,p_u)}^{(2)} Y_M^{\pm}(p_1)^{r_1} \cdots Y_M^{\pm}(p_i)^{r_u} = \tag{4.6}$$

$$\sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \sum_{(\underline{m}_1,\ldots,\underline{m}_u)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1,\ldots,\underline{m}_u) \sum_{(t_1,\ldots,t_u)}^{(4)} D_{\underline{r},\underline{m}}(\underline{t}) a_f(p_1^{t_1} \cdots p_u^{t_u}).$$

where

1. $\sum\limits_{(t_1,\ldots,t_u)}^{(4)}$ denotes that the sum is taken over u-tuples $\underline{t} = (t_1, \ldots, t_u)$, where each $t_i \geq 0$ and $t_i \in \mathcal{I}(\underline{m}_i)$.

2. We abbreviate the notation by setting

$$\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1, \ldots, \underline{m}_u) := \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1) \cdots \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_u)$$

and for a given tuple $\underline{m} = (\underline{m}_1, \ldots, \underline{m}_u)$,

$$D_{\underline{r},\underline{m}}(\underline{t}) := D_{r_1,\underline{m}_1}(t_1) D_{r_2,\underline{m}_2}(t_2) \cdots D_{r_u,\underline{m}_u}(t_u).$$

We now prove the following proposition:

Proposition 4.3.1 *Suppose $k = k(x)$ runs over positive even integers such that $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Let $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$. For each partition $(r_1, r_2, \ldots, r_u)$ of n,*

$$\lim_{x \to \infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \frac{1}{s_k} \sum_{f \in \mathcal{F}_k} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} Y_M^{\pm}(p_1)^{r_1} Y_M^{\pm}(p_2)^{r_2} \cdots Y_M^{\pm}(p_u)^{r_u}$$

$$= \begin{cases} (V_M^{\pm})^u & \text{if } (r_1, r_2, \ldots, r_u) = (2, \ldots, 2) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From equation (4.6), we have, for each partition $(r_1, \ldots, r_u)$ of n,

$$\sum_{(p_1,p_2,\ldots,p_u)}^{(2)} Y_M^{\pm}(p_1)^{r_1} \cdots Y_M^{\pm}(p_i)^{r_u} =$$

$$\sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \sum_{(\underline{m}_1,\ldots,\underline{m}_u)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1, \ldots, \underline{m}_u) \sum_{(t_1,\ldots,t_u)}^{(4)} D_{\underline{r},\underline{m}}(\underline{t}) a_f(p_1^{t_1} \cdots p_u^{t_u}).$$

For each tuple $(\underline{m}_1, \ldots, \underline{m}_u)$, on applying Proposition 3.3.2, we have

$$\frac{1}{\pi(x)^{\frac{n}{2}}} \left(\frac{1}{s_k} \sum_{f \in \mathcal{F}_k} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \sum_{(t_1,\ldots,t_u)}^{(4)} D_{\underline{r},\underline{m}}(\underline{t}) a_f(p_1^{t_1} \cdots p_u^{t_u}) \right) =$$

$$\frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \sum_{(t_1,\ldots,t_u)}^{(4)} D_{\underline{r},\underline{m}}(\underline{t}) \left(\frac{\delta(t_1, \ldots, t_u)}{(p_1^{t_1} \cdots p_u^{t_u})^{\frac{1}{2}}} + \mathrm{O}\left(\frac{(p_1^{t_1} \cdots p_u^{t_u})^c}{k} \right) \right),$$

where $\delta(t_1,\ldots,t_u)=1$ if $2|t_i$ for every $t_i>0$ and $\delta(t_1,\ldots,t_u)=0$ otherwise. Observe that for each $1\le i\le u$, t_i is even if and only if the sum of the components of the corresponding $\underline{m}_i$ is even.
Therefore, the sum

$$\frac{1}{s_k}\sum_{f\in\mathcal{F}_k}\frac{1}{\pi(x)^{\frac{n}{2}}}\sum_{(p_1,p_2,\ldots,p_u)}^{(2)} Y_M^{\pm}(p_1)^{r_1}\cdots Y_M^{\pm}(p_i)^{r_u}$$
$$=\frac{1}{\pi(x)^{\frac{n}{2}}}\sum_{(p_1,p_2,\ldots,p_u)}^{(2)}\sum_{(\underline{m}_1,\ldots,\underline{m}_u)}^{(\star)}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1,\ldots,\underline{m}_u)\sum_{(t_1,\ldots,t_u)}^{(4)} D_{\underline{r},\underline{m}}(\underline{t})\frac{1}{(p_1^{t_1}\cdots p_u^{t_u})^{\frac{1}{2}}}$$
$$+\,\mathrm{O}\left(\frac{1}{\pi(x)^{\frac{n}{2}}}\sum_{(p_1,p_2,\ldots,p_u)}^{(2)}\sum_{(\underline{m}_1,\ldots,\underline{m}_u)}|\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1,\ldots,\underline{m}_u)|\sum_{(t_1,\ldots,t_u)}^{(4)} D_{\underline{r},\underline{m}}(\underline{t})\frac{(p_1^{t_1}\cdots p_u^{t_u})^c}{k}\right), \tag{4.7}$$

where
$\sum\limits_{(\underline{m}_1,\ldots,\underline{m}_u)}^{(\star)}$ denotes that the sum is over those tuples such that $\delta(t_1,\ldots,t_u)=1$.

Henceforth, to ease the notation, we will supress writing $t_i\in\mathcal{I}(\underline{m}_i)$ and assume that when we write t_i under the summation sign, we mean t_i such that $t_i\in\mathcal{I}(\underline{m}_i)$.
Now we consider the first term on the right hand side of (4.7) , which is

$$\frac{1}{\pi(x)^{\frac{n}{2}}}\sum_{(p_1,p_2,\ldots,p_u)}^{(2)}\sum_{(\underline{m}_1,\ldots,\underline{m}_u)}^{(\star)}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1,\ldots,\underline{m}_u)\sum_{(t_1,\ldots,t_u)}^{(4)} D_{\underline{r},\underline{m}}(\underline{t})\frac{1}{(p_1^{t_1}\ldots p_u^{t_u})^{\frac{1}{2}}}$$
$$=\frac{1}{\pi(x)^{\frac{n}{2}}}\sum_{(p_1,p_2,\ldots,p_u)}^{(2)}\left(\sum_{\underline{m}_1}^{(\star)}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1)\sum_{t_1\ge 0}\frac{D_{r_1,\underline{m}_1}(t_1)}{p_1^{t_1/2}}\right)\cdots\left(\sum_{\underline{m}_u}^{(\star)}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_u)\sum_{t_u\ge 0}\frac{D_{r_u,\underline{m}_u}(t_u)}{p_u^{t_u/2}}\right).$$

We write each

$$\left(\sum_{\underline{m}_i}^{(\star)}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)\sum_{t_i\ge 0}\frac{D_{r_i,\underline{m}_i}(t_i)}{p_i^{t_i/2}}\right)$$

as

$$\sum_{\underline{m}_i}^{(\star)}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)D_{r_i,\underline{m}_i}(0)+\sum_{\underline{m}_i}^{(\star)}\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)\sum_{t_i\ge 2}\frac{D_{r_i,\underline{m}_i}(t_i)}{p_i^{t_i/2}}.$$

Therefore, denoting

$$C_M^{\pm}(i) = \sum_{\underline{m}_i}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i) D_{r_i,\underline{m}_i}(0),$$

we have, for a partition $(r_1, r_2, \dots r_u)$ of n,

$$\frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \left(\sum_{\underline{m}_1}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1) \sum_{t_1 \geq 0} \frac{D_{r_1,\underline{m}_1}(t_1)}{p_1^{t_1/2}} \right) \cdots \left(\sum_{\underline{m}_u}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_u) \sum_{t_u \geq 0} \frac{D_{r_u,\underline{m}_u}(t_u)}{p_u^{t_u/2}} \right)$$
$$= \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \left(\prod_{i=1}^{u} C_M^{\pm}(i) \right)$$
$$+ \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \sum_{(\varepsilon_1,\dots,\varepsilon_u)} \prod_{i=1}^{u} \left(C_M^{\pm}(i) \right)^{1-\varepsilon_i} \left(\sum_{\underline{m}_i}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i) \sum_{t_i \geq 0} \frac{D_{r_i,\underline{m}_i}(t_i)}{p_i^{t_i/2}} \right)^{\varepsilon_i}. \tag{4.8}$$

Here, in the second term on the right hand side, $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_u)$ runs over all u-tuples such that for each $i = 1, \dots, u$, $\varepsilon_i \in \{0, 1\}$ and at least one ε_i is non-zero. The tuple $(0, \dots, 0)$ is accounted for by the first term. We also follow the convention that if $C_M^{\pm}(i) = 0$, then ε_i is fixed to be 1 and $C_M^{\pm}(i)^{1-\varepsilon_i} = 1$.

Let

$$\tilde{D}_{\underline{m}_i}(r_i) := \max\{D_{r_i,\underline{m}_i}(t_i) : \ t_i \in \mathcal{I}_{\underline{m}_i}\}.$$

Then, we have

$$\sum_{\underline{m}_i}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i) \sum_{t_i \geq 2} \frac{D_{r_i,\underline{m}_i}(t_i)}{p_i^{t_i/2}} \ll \sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i}.$$

From this, we derive,

$$\frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \left(\sum_{\underline{m}_1}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1) \sum_{t_1 \geq 0} \frac{D_{r_1,\underline{m}_1}(t_1)}{p_1^{t_1/2}} \right) \cdots \left(\sum_{\underline{m}_u}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_u) \sum_{t_u \geq 0} \frac{D_{r_u,\underline{m}_u}(t_u)}{p_u^{t_u/2}} \right)$$

$$= \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \left(\prod_{i=1}^{u} C_M^{\pm}(i) \right)$$

$$+ \mathrm{O}\left(\frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \sum_{(\varepsilon_1,\dots,\varepsilon_u)} \prod_{i=1}^{u} \left|C_M^{\pm}(i)\right|^{1-\varepsilon_i} \left(\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i} \right)^{\varepsilon_i} \right). \tag{4.9}$$

Our goal is to show that the error term goes to zero as $x \to \infty$. To this end, for each tuple $(\varepsilon_1, \dots, \varepsilon_u)$, observe that we may write

$$\sum_{(p_1,p_2,\dots,p_u)}^{(2)} \prod_{i=1}^{u} \left|C_M^{\pm}(i)\right|^{1-\varepsilon_i} \left(\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i} \right)^{\varepsilon_i}$$

as

$$\sum_{(p_1,p_2,\dots,p_u)}^{(2)} \left(\prod_{\substack{i=1 \\ \varepsilon_i=0}}^{u} \left|C_M^{\pm}(i)\right| \right) \prod_{\substack{i=1 \\ \varepsilon_i=1}}^{u} \left(\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i} \right).$$

For $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_u)$, we define

$$\alpha(\underline{\varepsilon}) := \alpha(\varepsilon_1, \dots, \varepsilon_u) := \#\{1 \leq i \leq u :\, \varepsilon_i = 0\}.$$

We observe that if $r_i = 1$, then $C_M^{\pm}(i) = 0$. In general, for $r_i \geq 2$, we have

$$|C_M^{\pm}(i)| \ll \sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)||D_{r_i,\underline{m}_i}(0)|.$$

If $r_i = 2$, then for each $\underline{m}_i$,

$$D_{r_i,\underline{m}_i}(0) = \mathrm{O}(1).$$

Thus, by Lemma 2.3.2,

$$|C_M^{\pm}(i)| \ll \sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \ll (\log M)^{r_i}.$$

On the other hand, if $r_i \geq 3$, then, by Proposition 4.2.1, for each $\underline{m}_i$,

$$D_{r_i,\underline{m}_i}(0) = \mathrm{O}(M^{r_i-3}).$$

Once again, by Lemma 2.3.2,

$$\begin{aligned}|C_M^{\pm}(i)| &\ll \sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)||D_{r_i,\underline{m}_i}(0)| \\ &\ll \begin{cases} (\log M)^{r_i} & \text{if } r_i = 1,2 \\ M^{r_i-3}(\log M)^{r_i} & \text{if } r_i \geq 3. \end{cases}\end{aligned} \tag{4.10}$$

Similarly, by another application of Proposition 4.2.1 and Lemma 2.3.2 , we have

$$\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i} \ll \begin{cases} \dfrac{(\log M)}{p_i} & \text{if } r_i = 1 \\ \\ \dfrac{M^{r_i-2}(\log M)^{r_i}}{p_i} & \text{if } r_i \geq 2. \end{cases} \tag{4.11}$$

The partition $(r_1, r_2, \ldots, r_u)$ can be of two types as described below.

Case 1: The partition $(r_1, \ldots, r_u)$ satisfies the condition $r_i > 1$ for $i = 1, \ldots, u$. Observe that this means $u \leq \frac{n}{2}$.

In this case, by equations (4.10) and (4.11), for each tuple $(\varepsilon_1, \ldots, \varepsilon_u)$, we have

$$\begin{aligned}&\sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{\substack{i=1 \\ \varepsilon_i=0}}^{u} \left|C_M^{\pm}(i)\right| \right) \prod_{\substack{i=1 \\ \varepsilon_i=1}}^{u} \left(\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i} \right) \\ &\ll \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{\substack{i=1 \\ \varepsilon_i=0}}^{u} M^{r_i-2}(\log M)^{r_i} \right) \left(\prod_{\substack{i=1 \\ \varepsilon_i=1}}^{u} M^{r_i-2} \frac{(\log M)^{r_i}}{p_i} \right) \\ &\ll M^{n-2\alpha(\underline{\varepsilon})-2(u-\alpha(\underline{\varepsilon}))}(\log M)^n \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \frac{1}{\prod\limits_{\substack{i=1 \\ \varepsilon_i=1}}^{u} p_i} \\ &\ll M^{n-2\alpha(\underline{\varepsilon})-2(u-\alpha(\underline{\varepsilon}))}(\log M)^n \pi(x)^{\alpha(\underline{\varepsilon})}(\log\log x)^{u-\alpha(\underline{\varepsilon})} \\ &\ll M^{n-2u}\pi(x)^{\alpha(\underline{\varepsilon})}(\log\log x)^{u-\alpha(\underline{\varepsilon})}(\log M)^n.\end{aligned} \tag{4.12}$$

We now choose $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$. The above error term is

$$\ll \pi(x)^{\frac{n}{2}-u} \pi(x)^{u-1} (\log\log x)^u (\log x)^n,$$

since $\alpha(\underline{\varepsilon}) \le u-1$. Thus, for each tuple $(\varepsilon_1, \dots, \varepsilon_u)$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)^{n/2}} \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \left(\prod_{\substack{i=1\\ \varepsilon_i=0}}^{u} |C_M^{\pm}(i)| \right) \prod_{\substack{i=1\\ \varepsilon_i=1}}^{u} \left(\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i} \right)$$

$$\ll \lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \pi(x)^{\frac{n}{2}-1} (\log\log x)^u (\log x)^n = 0.$$

Since the number of tuples $(\varepsilon_1, \dots, \varepsilon_u)$ depends only on u, we conclude that

$$\lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \sum_{(\varepsilon_1,\dots,\varepsilon_u)} \prod_{i=1}^{u} \left(C_M^{\pm}(i) \right)^{1-\varepsilon_i} \left(\sum_{\underline{m}_i}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i) \sum_{t_i \ge 0} \frac{D_{r_i,\underline{m}_i}(t_i)}{p_i^{t_i/2}} \right)^{\varepsilon_i} = 0.$$

Case 2: The partition $(r_1, \dots, r_u)$ has at least one component r_i equal to 1. Let l be the number of 1's in the partition. Without loss of generality, we may assume that the last l parts are equal to one while $r_1, \dots, r_{u-l}$ are at least 2. By our convention, since $C_M^{\pm}(i) = 0$ if $r_i = 1$, we have $\varepsilon_i = 1$ for $u-l+1 \le i \le u$. Also, if $r_i = 1$, $\tilde{D}_{\underline{m}_i}(r_i) = 1$. For $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_u) = (\varepsilon_1, \dots, \varepsilon_{u-l}, 1, \dots, 1)$, let

$$\alpha_l(\underline{\varepsilon}) = \#\{1 \le i \le u-l : \varepsilon_i = 0\}.$$

Therefore, if the partition in consideration has l components equal to 1, we have, for each $\underline{\varepsilon}_l = (\varepsilon_1, \dots, \varepsilon_{u-l}, 1, \dots, 1)$,

$$\sum_{(p_1,p_2,\dots,p_u)}^{(2)} \prod_{i=1}^{u} \left| C_M^{\pm}(i) \right|^{1-\varepsilon_i} \left(\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i} \right)^{\varepsilon_i}$$

$$= \sum_{(p_1,p_2,\dots,p_u)}^{(2)} \prod_{i=1}^{u-l} \left[\left| C_M^{\pm}(i) \right|^{1-\varepsilon_i} \left(\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{\tilde{D}_{\underline{m}_i}(r_i)}{p_i} \right)^{\varepsilon_i} \right] \prod_{i=u-l+1}^{u} \left(\sum_{\underline{m}_i}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i)| \frac{1}{p_i} \right).$$

Again, using equations (4.10) and (4.11), for each tuple $\underline{\varepsilon_l} = (\varepsilon_1, \ldots, \varepsilon_{u-l}, 1, \ldots, 1)$ we have

$$\sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{\substack{i=1 \\ \varepsilon_i=0}}^{u-l} |C_M^{\pm}(i)| \right) \prod_{\substack{i=1 \\ \varepsilon_i=1}}^{u-l} \left(\sum_{\underline{m_i}}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m_i})| \frac{\tilde{D}_{\underline{m_i}}(r_i)}{p_i} \right) \prod_{i=u-l+1}^{u} \left(\sum_{\underline{m_i}}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m_i})| \frac{\tilde{D}_{\underline{m_i}}(1)}{p_i} \right)$$

$$\ll \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{\substack{i=1 \\ \varepsilon_i=0}}^{u-l} M^{r_i-2} (\log M)^{r_i} \right) \left(\prod_{\substack{i=1 \\ \varepsilon_i=1}}^{u-l} M^{r_i-2} \frac{(\log M)^{r_i}}{p_i} \right) \frac{(\log M)^l}{p_{u-l+1} \cdots p_u}$$

$$\ll M^{n-l-2\alpha_l(\underline{\varepsilon})-2(u-l-\alpha_l(\underline{\varepsilon}))} (\log M)^n \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \frac{1}{\prod\limits_{\substack{i=1 \\ \varepsilon_i=1}}^{u} p_i}$$

$$\ll M^{n-2\alpha_l(\underline{\varepsilon})-2(u-l-\alpha_l(\underline{\varepsilon}))} (\log M)^n \pi(x)^{\alpha(\underline{\varepsilon})} (\log\log x)^{u-\alpha_l(\underline{\varepsilon})}$$

$$\ll M^{n-l-2(u-l)} \pi(x)^{\alpha_l(\underline{\varepsilon})} (\log\log x)^{u-\alpha_l(\underline{\varepsilon})} (\log M)^n. \tag{4.13}$$

Substituting our chosen value for M and using the bound $\alpha_l(\underline{\varepsilon}) \leq u-l$, the above error term is

$$\ll \pi(x)^{\frac{n}{2}-\frac{l}{2}} (\log\log x)^u (\log x)^n.$$

Therefore,

$$\lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \sum_{\underline{\varepsilon_l}} \prod_{i=1}^{u} |C_M^{\pm}(i)|^{1-\varepsilon_i} \left(\sum_{\underline{m_i}}^{(\star)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m_i})| \frac{\tilde{D}_{\underline{m_i}}(r_i)}{p_i} \right)^{\varepsilon_i}$$

$$\ll_u \lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \pi(x)^{-\frac{1}{2}} (\log\log x)^u (\log x)^n = 0$$

noting that $l \geq 1$.

From the analysis in Cases 1 and 2, we deduce that for any partition $(r_1,\, r_2, \ldots, r_u)$ of n,

$$\lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \sum_{(\varepsilon_1,\ldots,\varepsilon_u)} \prod_{i=1}^{u} \left(C_M^{\pm}(i)\right)^{1-\varepsilon_i} \left(\sum_{\underline{m}_i}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_i) \sum_{t_i\geq 0} \frac{D_{r_i,\underline{m}_i}(t_i)}{p_i^{t_i/2}} \right)^{\varepsilon_i} = 0, \tag{4.14}$$

where we are summing over all tuples $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_u)$ with at least one ε_i is non-zero.
From equations (4.8) and (4.14), we deduce that for a partition $(r_1,\, r_2, \ldots, r_u)$ of n,

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\sum_{\underline{m}_1}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1) \sum_{t_1\geq 0} \frac{D_{r_1,\underline{m}_1}(t_1)}{p_1^{t_1/2}} \right) \cdots \left(\sum_{\underline{m}_u}^{(\star)} \hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_u) \sum_{t_u\geq 0} \frac{D_{r_u,\underline{m}_u}(t_u)}{p_u^{t_u/2}} \right) \\ &= \lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{i=1}^{u} C_M^{\pm}(i) \right). \end{aligned} \tag{4.15}$$

We now study the term

$$\frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{i=1}^{u} C_M^{\pm}(i) \right)$$

as $x \to \infty$.
We know, from Lemma 4.2.3, that for $r_i = 2$,

$$C_M^{\pm}(i) = V_M^{\pm} = 2\sum_{m=1}^{M} \hat{\mathcal{S}}^{\pm}(m)^2 - 2\sum_{m=1}^{M-2} \hat{\mathcal{S}}^{\pm}(m)\,\hat{\mathcal{S}}^{\pm}(m+2) - \hat{\mathcal{S}}^{\pm}(1)^2 - \hat{\mathcal{S}}^{\pm}(2)^2.$$

Plugging in our choice of $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$ into the asymptotic formula for $V_M^{\pm}$ calculated in Proposition 3.6.1, we may write

$$V_M^{\pm} = \mu_\infty(I) - \mu_\infty(I)^2 + \mathrm{O}\left(\frac{\log \sqrt{x}}{\sqrt{\pi(x)}} \right).$$

Once again, the partitions $(r_1,\, r_2, \ldots, r_u)$ are of three different types as described in the three cases below.

Case 1: If $(r_1, r_2, \ldots, r_u) = (2, 2, \ldots, 2)$, then $u = n/2$ and

$$\frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{i=1}^{u} C_M^{\pm}(i)\right) = \left(\prod_{i=1}^{u} C_M^{\pm}(i)\right)$$

$$= (V_M^{\pm})^{n/2}$$

$$= (\mu_\infty(I) - \mu_\infty(I)^2)^{\frac{n}{2}} + \mathrm{O}_n\left(\frac{\log\sqrt{x}}{\sqrt{\pi(x)}}\right)^{\frac{n}{2}}.$$

Case 2: If $r_i = 1$ for some r_i in the given partition, then the corresponding $C_M^{\pm}(i)$ is 0. Thus,

$$\frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{i=1}^{u} C_M^{\pm}(i)\right) = 0.$$

Case 3: Each $r_i \geq 2$ with at least one $r_i \geq 3$. Without loss of generality, for some $1 \leq l \leq u$, suppose we have $r_1, r_2, \ldots, r_l \geq 3$ and $r_{l+1} = \cdots = r_u = 2$.

Thus, $(r_1 + \cdots + r_l) + 2(u - l) = n$. By equation (4.10),

$$\left|\prod_{i=1}^{u} C_M^{\pm}(i)\right| \ll M^{\sum_{i=1}^{l}(r_i-3)}(\log M)^n$$

$$\ll M^{n-2(u-l)-3l}(\log M)^n = M^{n-l-2u}(\log M)^n.$$

Choosing $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$,

$$\frac{1}{\pi(x)^{n/2}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left|\prod_{i=1}^{u} C_M^{\pm}(i)\right|$$

$$\ll \frac{1}{\pi(x)^{n/2}} (\pi(x))^{\frac{n}{2}-\frac{l}{2}-u+u}(\log x)^n.$$

Since $l \geq 1$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{i=1}^{u} C_M^{\pm}(i)\right) = 0.$$

From the above three cases, we deduce that for $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$,

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \left(\prod_{i=1}^{u} C_M^{\pm}(i) \right) \\ &= \begin{cases} (\mu_\infty(I) - \mu_\infty(I)^2)^{n/2} & \text{if } (r_1, r_2, \ldots, r_u) = (2, \ldots, 2) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.16}$$

We now look at the error term on the right hand side of equation (4.7),

$$\mathrm{O}\left(\frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \sum_{(\underline{m}_1,\ldots,\underline{m}_u)} |\hat{\mathcal{U}}_{\mathrm{M}}^{\pm}(\underline{m}_1, \ldots, \underline{m}_u)| \sum_{(t_1,\ldots,t_u)}^{(4)} D_{\underline{r},\underline{m}}(\underline{t}) \frac{(p_1^{t_1} \cdots p_u^{t_u})^c}{k} \right).$$

We observe that for each i,

$$\sum_{\substack{t_i \geq 0 \\ t_i \in \mathcal{I}(\underline{m}_i)}} (p_i^c)^{t_i} \ll p_i^{c(r_i M+1)},$$

since the t_i can at most be $r_i M$. Thus, by Proposition 4.2.1 and Lemma 2.3.2, the above error term from equation (4.7) becomes

$$\begin{aligned} &= \mathrm{O}\left(\frac{1}{\pi(x)^{\frac{n}{2}}} \pi(x)^u (\log M)^n M^{n-2u} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} \frac{p_1^{c(r_1M+1)} p_2^{c(r_2M+1)} \cdots p_u^{c(r_uM+1)}}{k} \right) \\ &= \mathrm{O}\left(\frac{1}{\pi(x)^{\frac{n}{2}}} \pi(x)^u (\log M)^n M^{n-2u} \pi(x)^u \frac{x^{cn(M+1)}}{k} \right), \end{aligned}$$

since $r_1 + \cdots + r_u = n$. For $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$, this is

$$\mathrm{O}\left(\frac{1}{\pi(x)^{\frac{n}{2}}} \pi(x)^u (\log x)^n (\pi(x))^{n/2-u} \pi(x)^u \frac{x^{cn\sqrt{x}}}{k} \right).$$

If

$$\frac{\log k}{\sqrt{x}\log x} \to \infty \text{ as } x \to \infty,$$

then

$$\lim_{x\to\infty} \frac{x^{A\sqrt{x}}}{k} = 0 \text{ for any constant } A > 0.$$

In particular, given $n \geq 1$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \pi(x)^u (\log x)^n (\pi(x))^{n/2-u} \pi(x)^u \frac{x^{cn\sqrt{x}}}{k} = 0.$$

Combining this information with (4.15) and (4.16), we prove Proposition 4.3.1. □

Using Proposition 4.3.1 in equation (4.2), we deduce, under the same assumptions on M and k as above,

$$\lim_{x\to\infty} \frac{1}{s_k} \sum_{f\in\mathcal{F}_k} \left(T_M^{\pm}(x)\right)^n$$

$$= \lim_{x\to\infty} \frac{1}{\pi(x)^{\frac{n}{2}}} \sum_{u=1}^{n} \sum_{(r_1,r_2,\ldots,r_u)}^{(1)} \frac{n!}{r_1!r_2!\ldots r_u!} \frac{1}{u!} \sum_{(p_1,p_2,\ldots,p_u)}^{(2)} Y_M^{\pm}(p_1)^{r_1} Y_M^{\pm}(p_2)^{r_2} \ldots Y_M^{\pm}(p_u)^{r_u}$$

$$= \sum_{u=1}^{n} \sum_{(r_1,r_2,\ldots,r_u)}^{(1)} \frac{n!}{r_1!r_2!\ldots r_u!} \frac{1}{u!} \begin{cases} (\mu_\infty(I) - \mu_\infty(I)^2)^u & \text{if } (r_1,r_2,\ldots,r_u) = (2,\,2,\,2,\ldots,2) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{n!}{(\frac{n}{2})!2^{\frac{n}{2}}} (\mu_\infty(I) - \mu_\infty(I)^2)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

We have proved the following theorem:

Theorem 4.3.2 *Let $I = [a,b]$ be a fixed interval in $[-2,2]$. Let $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$ and suppose $k = k(x)$ runs over positive even integers such that $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Then,*

$$\lim_{x\to\infty} \left\langle \left(\frac{\mathrm{S}^{\pm}(M,f)(x)}{\sqrt{\pi(x)(\mu_\infty(I) - \mu_\infty(I)^2)}} \right)^n \right\rangle = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{n!}{(n/2)!2^{n/2}} & \text{if } n \text{ is even.} \end{cases}$$

Chapter 5

Proofs of the main theorems

In this chapter we bring together the results proved in the chapters so far to prove theorems 1.3.1 and 1.3.2.

5.1 Proof of Theorem 1.3.1

We recall (3.10) below:

$$S^{-}(M,f)(x)+C\left(\frac{\pi(x)}{M+1}\right) \leq N_I(f,x)-\pi(x)\mu_\infty(I) \leq S^{+}(M,f)(x)+D\left(\frac{\pi(x)}{M+1}\right). \tag{5.1}$$

The above tells us that

$$\langle N_I(f,x) - \pi(x)\mu_\infty(I) - S^{\pm}(M,f)(x)\rangle = \mathrm{O}\left(\frac{\pi(x)}{M+1}\right).$$

Thus, we have

$$\left\langle \left| \frac{N_I(f,x) - \pi(x)\mu_\infty(I) - S^{\pm}(M,f)(x)}{\sqrt{\pi(x)(\mu_\infty(I) - \mu_\infty(I)^2)}} \right| \right\rangle$$

$$= \mathrm{O}\left(\frac{\sqrt{\pi(x)}}{(M+1)\sqrt{(\mu_\infty(I) - \mu_\infty(I)^2)}}\right).$$

Remark 5.1.1 *Observe that to prove Theorem 4.3.2, it would have sufficed to take M to be $\lfloor\sqrt{\pi(x)}\rfloor$. However, to conclude the convergence in mean,* *looking at the error term in the above equation, we deduce that our choice of M should be a function of x that grows faster than $\sqrt{\pi(x)}$. This explains the reason behind our choice of M.*

Therefore, choosing $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$, we get

Proposition 5.1.2 *Let $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$. Then, as $x \to \infty$,*

$$\left\langle \left| \frac{N_I(f,x) - \pi(x)\mu_\infty(I) - S^{\pm}(M,f)(x)}{\sqrt{\pi(x)(\mu_\infty(I) - \mu_\infty(I)^2)}} \right| \right\rangle \to 0.$$

The above proposition tells us that the random variable

$$\frac{N_I(f,x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)(\mu_\infty(I) - \mu_\infty(I)^2)}}$$

converges in mean to

$$\frac{S^{\pm}(M,f)(x)}{\sqrt{\pi(x)(\mu_\infty(I) - \mu_\infty(I)^2)}}$$

as $x \to \infty$. Observe that Theorem 4.3.2 implies the following:
For $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$ and $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$,

$$\left\langle \left(\frac{S^{\pm}(M,f)(x)}{\sqrt{\pi(x)(\mu_\infty(I) - \mu_\infty(I)^2)}} \right)^n \right\rangle \to \begin{cases} 0 & \text{if n is odd} \\ \frac{n!}{(n/2)!2^{n/2}} & \text{if } n \text{ is even.} \end{cases}$$

Recall, for $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n e^{-\frac{t^2}{2}} dt = \begin{cases} 0 & \text{if n is odd} \\ \frac{n!}{(n/2)!2^{n/2}} & \text{if } n \text{ is even.} \end{cases}$$

Since the Gaussian distribution is uniquely characterized by its moments, this tells us that

$$\frac{S^{\pm}(M,f)(x)}{\sqrt{\pi(x)(\mu_\infty(I) - \mu_\infty(I)^2)}}$$

follows a Gaussian distribution. Finally, using Proposition 5.1.2 and the fact that convergence in mean implies convergence in distribution, this proves that the quantity we're interested in also follows a Gaussian distribution under the specified growth conditions on the weight k. More explicitly, we have proved the following theorem.

Theorem 1.3.1 Let $I = [a, b]$ be a subinterval of $[-2, 2]$. Suppose that $k = k(x)$ satisfies $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Then for any bounded continuous real function h on $\mathbb{R}$ we have

$$\lim_{x\to\infty} \frac{1}{|\mathcal{F}_k|} \sum_{f\in\mathcal{F}_k} h\left(\frac{N_I(f,x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)\left[\mu_\infty(I) - (\mu_\infty(I))^2\right]}} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-\frac{t^2}{2}} dt.$$

In other words, for any real numbers $A < B$,

$$\lim_{x\to\infty} \text{Prob}_{\mathcal{F}_k} \left(A < \frac{N_I(f,x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)\left[\mu_\infty(I) - (\mu_\infty(I))^2\right]}} < B \right) = \frac{1}{\sqrt{2\pi}} \int_A^B e^{-t^2/2} dt.$$

5.2 Variants

Before we present the proof of the analogous theorem for Maass forms, we state some results that are variants of the theorem proved in the case of modular forms. These can be proved using the same techniques used in proving Theorem 1.3.1.

5.2.1 Harmonic averaging

One could consider a weighted variant of Theorem 1.3.1. Instead of uniformly averaging over cusp forms in $\mathcal{F}_k$, we consider the case of harmonic averaging. That is, for $f \in \mathcal{F}_k$, we denote

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle},$$

where $\langle f, g\rangle$ denotes the Petersson inner product of f, $g \in S_k$. We define

$$h_k := \sum_{f\in\mathcal{F}_k} \omega_f.$$

For a function $\phi : S_k \to \mathbb{C}$, we denote its harmonic average as follows:

$$\langle \phi(f) \rangle_{h_k} := \frac{1}{h_k} \sum_{f \in \mathcal{F}_k} \omega_f \phi(f).$$

Using the Peterrson trace formula, one can prove the following analogue of Theorem 1.3.1 with harmonic weights attached to the quantities in consideration.

Theorem 5.2.1 *Let $I = [a, b]$ be a fixed interval in $[-2, 2]$. Suppose that $k = k(x)$ satisfies $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Then for any bounded, continuous, real-valued function g on $\mathbb{R}$, we have*

$$\lim_{x\to\infty} \frac{1}{h_k} \sum_{f \in \mathcal{F}_k} g\left(\frac{\omega_f N_I(f,x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)\left[\mu_\infty(I) - (\mu_\infty(I))^2\right]}} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-\frac{t^2}{2}} dt.$$

5.2.2 Remarks on higher levels.

Now that we have a proof for the case of full level, it is natural to ask if the theorem extends to higher levels. For higher levels, one could consider the subspace $S^*(N, k)$ of newforms sitting inside $S(N, k)$. We consider the subspace of newforms because this subspace has a basis consisting of simultaneous eigenforms for all the Hecke operators T_p, including primes p dividing N. Of course, one would have to consider the estimates of the trace formula for T_n acting on $S^*(N, k)$ and a formula for the dimension $s^*(N, k)$ in this case. We state them here.

Let $f_0(N)$ be the multiplicative function satisfying

$$f_0(p) = 1 - \frac{1}{p},$$

$$f_0(p^2) = 1 - \frac{1}{p} - \frac{1}{p^2}$$

and

$$f_0(p^m) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right) \text{ for } m \geq 3.$$

As before, let $F_{N,k}$ denote the basis of $S^*(N,k)$ consisting of normalized Hecke eigenforms. Then we have

1. $$\sum_{f \in \mathcal{F}_{N,k}} a_f(n) = \begin{cases} k\dfrac{Nf_0(N)}{12} + \mathrm{O}_N\left(n^c\right) & \text{if } n \text{ is a square} \\ \\ \mathrm{O}_N\left(n^c\right) & \text{otherwise,} \end{cases}$$

   for some positive integer c.

2. $$s^*(N,k) = k\left(\frac{Nf_0(N)}{12}\right) + \mathrm{O}_N\left(1\right).$$

These can be found in [26] and [18]. It is not hard to see that Lemma 3.3.4 holds in this case as well, if one replaces s_k with $s^*(N,k)$. Proceeding analogously as presented in the earlier chapters, the following theorem holds:

Theorem 5.2.2 *Let $I = [a,b]$ be a subinterval of $[-2,2]$. Suppose that $k = k(x)$ satisfies $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Then for any real numbers $A < B$,*

$$\lim_{x\to\infty} \text{Prob}_{\mathcal{F}_{N,k}}\left(A < \frac{N_I(f,x) - \pi(x)\mu_\infty(I)}{\sqrt{\pi(x)\left[\mu_\infty(I) - (\mu_\infty(I))^2\right]}} < B\right) = \frac{1}{\sqrt{2\pi}}\int_A^B e^{-t^2/2}dt.$$

5.3 Proof of Theorem 1.3.2

We conclude this chapter with a proof of Theorem 1.3.2. The ideas in the proof are similar to those for the case of holomorphic newforms. However, one of the main tools used in the study of Maass cusp forms is the unweighted Kuznetsov trace formula as opposed to the Eichler Selberg trace formula in previous sections. Additionally, the Ramanujan-Peterrson Conjecture, which is the assertion that for all primes p,

$$|a_j(p)| \leq 2$$

is still open. This causes a subtle change in the growth conditions under which Theorems 1.3.1 and 1.3.2 hold true, since we have to account for the eigenvalues lying outside the interval $[-2,2]$.

Following the notation of subsection 1.1.3, for an interval $I=[a,b]\subset\mathbb{R}$ and for $1\leq j\leq r(T)$, let us define

$$N_I(j,x)=\#\{p\leq x:\ a_j(p)\in I\}.$$

The Sato-Tate conjecture for a primitive Maass form f_j (unproved as yet) is the prediction that for an interval $I=[a,b]\subset[-2,2]$,

$$\lim_{x\to\infty}\frac{N_I(j,x)}{\pi(x)}=\int_I\mu_\infty(t)dt.$$

In [29], Sarnak considered a vertical analogue of the Sato-Tate conjecture and showed that if we fix a prime p and let $j\to\infty$, the sequence $\{a_j(p)\}$ is equidistributed in $[-2,2]$ with respect to the measure $\mu_p(t)$ defined in equation (1.3). An effective version of Sarnak's theorem was derived by Lau and Wang [17]. Adapting the techniques of [17], Wang [36] also proves an average version of the Sato-Tate conjecture for primitive Maass forms along with effective error terms. He shows the following:

Theorem 5.3.1 *Suppose* $T=T(x)$ *satisfies* $\frac{\log T}{\log x}\to\infty$ *as* $x\to\infty$*. For any* $I=[a,b]\subset\mathbb{R}$,

$$\frac{1}{r(T)}\sum_{j=1}^{r(T)}N_I(j,x)=\pi(x)\mu_\infty(I)+\mathrm{O}\left(\frac{\pi(x)\log x}{\log T}\right)+\mathrm{O}(\log\log x).$$

In order to prove the above theorem, the author uses a modified version of the Beurling-Selberg polynomials to approximate the counting function $N_I(j,x)$ and then applies an unweighted Kuznetsov trace formula to evaluate the exponential sums that appear in this case. This unweighted trace formula had been previously stated and proved by Lau and Wang [17, Lemma 3.3] in order to derive effective versions of Sarnak's theorem. We state it below.

Proposition 5.3.2 *Let $\kappa = 11/155$, $\eta = 43/620$ and m, n be positive integers. For arbitrarily small $\epsilon > 0$ we have,*

$$\sum_{t_j \le T} a_j(m) a_j(n) = \frac{1}{12} T^2 \delta_{mn=\square} \frac{\sigma(\gcd(m,n))}{\sqrt{mn}} + \mathrm{O}_\varepsilon(T^{2-\kappa+\varepsilon}(mn)^{\eta+\varepsilon}),$$

where $\sigma(l) = \sum_{d|l} d$ and $\delta_{l=\square} = 1$ if l is a square and $\delta_{l=\square} = 0$ otherwise.

The following proposition, analogous to Proposition 3.3.2, follows immediately from their unweighted trace formula.

Proposition 5.3.3 *Let $\kappa = 11/155$, $\eta = 43/620$ and m be a positive integer. For $\epsilon > 0$,*

$$\sum_{t_j \le T} a_j(p^m) = \begin{cases} \frac{T^2}{12} \frac{1}{p^{m/2}} + \mathrm{O}_\epsilon\left(T^{2-\kappa+\epsilon} p^{m\eta+\epsilon}\right) & \text{if } m \text{ is even} \\ \mathrm{O}_\epsilon\left(T^{2-\kappa+\epsilon} p^{m\eta+\epsilon}\right) & \text{if } m \text{ is odd.} \end{cases}$$

The techniques presented in the work of Lau and Wang [17] and Wang [36] take adequate care to treat the exceptional eigenvalues, that is, those eigenvalues $a_j(p) \notin [-2, 2]$. For a prime p, we may write for each $1 \le j \le r(T)$, the eigenvalue

$$a_j(p) = 2\cos(\theta_j(p)),$$

where $\theta_j(p) \in [0, \pi]$ if $a_j(p) \in [-2, 2]$ and $\theta_j(p) = i\vartheta_j(p)$ or $\pi + i\vartheta_j(p)$, with $\vartheta_j(p) \in \mathbb{R}$ if $|a_j(p)| > 2$. In all cases, $a_j(p) \in \mathbb{R}$. As in the case of holomorphic Hecke eigenforms, we have, for $m \ge 1$,

$$a_j(p^m) = \frac{\sin(m+1)\theta_j(p)}{\sin\theta_j(p)} = X_m(2\cos(\theta_j(p))),$$

and therefore, for $m \ge 2$,

$$\begin{aligned} 2\cos(m\theta_j(p)) &= X_m(2\cos(\theta_j(p))) - X_{m-2}(2\cos(\theta_j(p))) \\ &= a_j(p^m) - a_j(p^{m-2}). \end{aligned}$$

From Weyl's law (equation (1.2)) and Proposition 5.3.3, we deduce the following analogue of Lemma 3.3.4:

Lemma 5.3.4 *Suppose $T = T(x)$ satisfies $\frac{\log T}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$. Then, for any $m \geq 1$ and $a > 0$,*

$$\lim_{x\to\infty} \frac{1}{(\pi(x))^a r(T)} \sum_{p\leq x} \sum_{1\leq j\leq r(T)} a_j(p^m) = 0.$$

More generally, for non-negative integers m_1, m_2, not both zero,

$$\lim_{x\to\infty} \frac{1}{(\pi(x))^a r(T)} \sum_{p_1\neq p_2\leq x} \sum_{1\leq j\leq r(T)} a_j(p_1^{m_1} p_2^{m_2}) = 0.$$

We have, for any $I \subset \mathbb{R}$,

$$N_I(j,x) = N_{I'}(j,x) + E_j(x),$$

where $I' \subseteq [-2,2]$ and $E_j(x) = \#\{p \leq x : \, |a_j(p)| > 2\}$. Lau and Wang [17, Lemma 4.3] show that for a fixed prime p and sufficiently large T,

$$\frac{1}{r(T)} \#\{1 \leq j \leq r(T) : \, |a_j(p)| > 2\} = \mathrm{O}\left(\frac{\log p}{\log T}\right)^2,$$

the implied constant being absolute.
So we have

$$\frac{1}{r(T)} \sum_{j=1}^{r(T)} E_j(x) = O\left(\frac{\pi(x)(\log x)^2}{(\log T)^2}\right).$$

As in Section 4, considering the Beurling-Selberg polynomials for I' and denoting

$$\begin{aligned} S^{\pm}(M,j)(x) := &\sum_{m=1}^{2} (\hat{S}_M^{\pm}(m) + \hat{S}_M^{\pm}(-m)) \sum_{p\leq x} a_j(p^m) \\ &+ \sum_{m=3}^{M} (\hat{S}_M^{\pm}(m) + \hat{S}_M^{\pm}(-m)) \sum_{p\leq x} \left(a_j(p^m) - a_j(p^{m-2})\right), \end{aligned}$$

we get

$$N_{I'}(j,x) - \pi(x)\left[(2\beta - 2\alpha) - (\hat{S}_M^{+}(2) + \hat{S}_M^{+}(-2))\right] \leq S^{+}(M,j)(x) + 2\frac{\pi(x)}{M+1}.$$

and

$$-2\frac{\pi(x)}{M+1}+S^{-}(M,j)(x) \leq N_{I'}(j,x)-\pi(x)\left[(2\beta-2\alpha)-(\hat{S}_M^{-}(2)+\hat{S}_M^{-}(-2))\right].$$

Therefore,

$$\begin{aligned}
&\frac{1}{r(T)}\sum_{j=1}^{r(T)} N_I(j,x)\\
&=\frac{1}{r(T)}\sum_{j=1}^{r(T)} N_{I'}(j,x)+O\left(\frac{\pi(x)(\log x)^2}{(\log T)^2}\right)\\
&\leq \pi(x)\mu_\infty(I)+S^{+}(M,j)(x)+O\left(\frac{\pi(x)}{M+1}+\frac{\pi(x)(\log x)^2}{(\log T)^2}\right)
\end{aligned}$$

and

$$\frac{1}{r(T)}\sum_{j=1}^{r(T)} N_I(j,x) \geq \pi(x)\mu_\infty(I)+S^{-}(M,j)(x)+O\left(\frac{\pi(x)}{M+1}+\frac{\pi(x)(\log x)^2}{(\log T)^2}\right).$$

Finally, we observe that on averaging over all $1 \leq j \leq r(T)$,

$$\begin{aligned}
&\frac{1}{r(T)}\sum_{1\leq j\leq r(T)}\left(N_I(j,x)-\pi(x)\mu_\infty(I)-S^{\pm}(M,j)(x)\right)\\
&\qquad = \mathrm{O}\left(\frac{\pi(x)}{M+1}+\frac{\pi(x)(\log x)^2}{(\log T)^2}\right).
\end{aligned}$$

Following the same sequence of arguments as in Theorem 1.3.1, the moments

$$\frac{1}{r(T)}\sum_{j=1}^{r(T)}\left(\frac{S^{\pm}(M,j)(x)}{\sqrt{\pi(x)(\mu_\infty(I)-\mu_\infty(I)^2)}}\right)^n$$

are those of a Gaussian, since the main ingredient was the multiplicative relations satisfied by the eigenvalues and these relations are the same for Maass forms and holomorphic cusp forms alike. Thus, we have

$$\begin{aligned}
&\left\langle\left|\frac{N_I(j,x)-\pi(x)\mu_\infty(I)-S^{\pm}(M,j)(x)}{\sqrt{\pi(x)(\mu_\infty(I)-\mu_\infty(I)^2)}}\right|\right\rangle\\
&= \mathrm{O}\left(\frac{\sqrt{\pi(x)}}{(M+1)\sqrt{(\mu_\infty(I)-\mu_\infty(I)^2)}}+\frac{\sqrt{\pi(x)}(\log x)^2}{(\log T)^2\sqrt{(\mu_\infty(I)-\mu_\infty(I)^2)}}\right).
\end{aligned}$$

Choosing $M=\lfloor\sqrt{\pi(x)}\log\log x\rfloor$, we get

Proposition 5.3.5 *Suppose that* $T = T(x)$ *satisfies* $\frac{\log T}{\sqrt{x}\log x} \to \infty$ *as* $x \to \infty$. *Let* $M = \lfloor \sqrt{\pi(x)} \log\log x \rfloor$ *and* $I \subset \mathbb{R}$ *be a fixed interval. Then,*

$$\left\langle \left| \frac{N_I(j,x) - \pi(x)\mu_\infty(I) - S^{\pm}(M,j)(x)}{\sqrt{\pi(x)(\mu_\infty(I) - \mu_\infty(I)^2)}} \right| \right\rangle_{1 \le j \le r(T)} \to 0.$$

Following the line of proof in Section 5.1, since convergence in mean implies convergence in distribution, this proves Theorem 1.3.2.

Chapter 6

Density of solutions to quadratic congruences

6.1 Introduction

The theory of solving a quadratic equation modulo p for p prime has been well studied. Investigating whether a given quadratic equation has solutions, how many there are and calculating what the solutions are, has led to beautiful theorems such as the law of quadratic reciprocity. A related question is the following:
Suppose we fix a quadratic equation $f(x) = x^2 + bx + c$, where $b, c \in \mathbb{Z}$ and would like to know how often the equation $f(x) = 0$ has solutions modulo N if we vary N in a certain range. Let us first look at the case where we vary over primes p not exceeding x. Dirichlet, in 1837, showed that solutions would exist for approximately half the primes. In 1896, this was made precise by de la Vallée-Poussin. Noting that $f(x)$ has exactly two solutions if and only if the discriminant $D = b^2 - 4c$ is a square mod p, what Dirichlet and de la Vallée-Poussin showed was essentially the following:

Proposition 6.1.1 *For a fixed number $D \in \mathbb{Z} - \{0\}$, as $x \to \infty$,*

$$\frac{1}{\pi(x)} \# \left\{ p \le x, p \text{ prime } : \left(\frac{D}{p}\right) = 1 \right\} \sim \frac{1}{2}$$

and
$$\frac{1}{\pi(x)}\#\left\{p \le x, p \text{ prime } : \left(\frac{D}{p}\right) = -1\right\} \sim \frac{1}{2},$$
where $\left(\frac{D}{\cdot}\right)$ *is the Legendre symbol and* $\pi(x)$ *denotes the number of primes not exceeding* x*.*

The main ideas that go into the proof of this result are two classical results: Gauss's law of quadratic reciprocity and Dirichlet's theorem on the infinitude of primes in an arithmetic progression. The latter was proved around 1836. Later, de la Vallée-Poussin proved that for positive integers a, q with $\gcd(a, q) = 1$, the set of primes congruent to a mod q has natural density $\frac{1}{\phi(q)}$. In other words, the number of primes $p \le x$ such that $p \equiv a \mod q$ is asymptotic to $\frac{1}{\phi(q)}\pi(x)$ as $x \to \infty$. Since then, there have been analogues of this theorem in various settings. For example, by applying the Chebotarev density theorem to the case of cyclotomic extensions $\mathbb{Q}(\zeta_n)$ of $\mathbb{Q}$, we obtain Dirichlet's theorem. The analogue in the case of function fields was proved by H. Kornblum and E. Landau in [14]. It is natural to ask if we can extend the result to numbers with k prime factors, $k > 1$. In order to do so, we would first need to talk about the analogue of $\pi(x)$ for numbers with k prime factors, which is defined as follows:

$$\tau_k(x) := \sum_{\substack{n \le x \\ n = p_1 p_2 \cdots p_k}} 1,$$

where $n = p_1 p_2 \ldots p_k$ is the prime factorization of n, with $p_1 \le p_2 \le \ldots \le p_k$. If we add an additional condition that the primes dividing n must be distinct, then we are counting the number of squarefree positive integers not exceeding x, having exactly k prime factors and this quantity is denoted by $\pi_k(x)$.
In 1900, E. Landau [15] proved that

$$\pi_k(x) \sim \tau_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}. \tag{6.1}$$

In 1954, E. M. Wright gave a simpler proof of this in [37], which appears as Theorem 437 in [10]. There have been several attempts since then, at deriving a precise estimate with error terms. An exposition of this can be found in Section 7.4 of [21].

With this in mind, it is natural to ask if we can say something analogous to Proposition 6.1.1 when n varies over squarefree numbers. In this chapter, we prove the following:

Theorem 6.1.2 *Let $D \in \mathbb{Z} - \{0\}$ and $k \in \mathbb{N}$. Fix a k-tuple $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$ where each $\varepsilon_i = \pm 1$ for each $i = 1, \ldots, k$. Then*

$$\frac{1}{\pi_k(x)} \# \left\{ n \le x, n = p_1 p_2 \ldots p_k \text{ with } p_1 < p_2 < \ldots < p_k : \left(\frac{D}{p_i} \right) = \varepsilon_i \text{ for each } i \right\} \sim \frac{1}{2^k},$$

where $\pi_k(x)$ denotes the number of squarefree numbers less than x with k prime factors.

The proof involves an analogous version of Dirichlet's theorem, which is the following:

Let us fix $N, k \in \mathbb{N}$ and consider a k-tuple

$$\underline{\mathrm{m}}_{[k]} = (m_1, m_2, \ldots, m_k)$$

where each $m_i \in (\mathbb{Z}/N\mathbb{Z})^\times$, the multiplicative group of units in $\mathbb{Z}/N\mathbb{Z}$. The m_i's are not necessarily distinct.

Consider positive integers $n \le x$ with k prime factors, counted with multiplicity. Represent such n as $n = p_1 p_2 \ldots p_k$ with $p_1 \le p_2 \le \ldots \le p_k$. Let $\tau_{k,\underline{\mathrm{m}}_{[k]}}(x)$ denote the number of positive integers $n \le x$ with k prime factors satisfying $p_i \equiv m_i \bmod N$ for each $i = 1, \ldots, k$. If the primes are distinct, then n is squarefree. Let $\pi_{k,\underline{\mathrm{m}}_{[k]}}(x)$ denote the number of such squarefree $n \le x$. Then we prove

Theorem 6.1.3

$$\pi_{k,\underline{\mathrm{m}}_{[k]}}(x) \sim \tau_{k,\underline{\mathrm{m}}_{[k]}}(x) \sim \frac{1}{\phi(N)^k} \frac{x (\log \log x)^{k-1}}{(k-1)! \log x} \qquad (k \ge 2).$$

Remark 6.1.4 *Note that for $k = 1$, the above theorem is exactly the statement of Dirichlet's density theorem. The prime number theorem, the non-vanishing of $L(1, \chi)$ and the orthogonality relations satisfied by Dirichlet characters are the key results that are used in the proof. Similarly, in the proof of Theorem 6.1.3, Dirichlet's density theorem and Landau's result stated in Equation (6.1) play a significant role. In fact, we essentially use the technique used by Wright in [37] and an orthogonality relation satisfied by the Dirichlet characters to obtain the result.*

6.2 Preliminaries

The following notation will be used in the proof of Theorem 6.1.3:

1. We write $\underline{\mathrm{m}}_{[k]}$ to denote a k-tuple $(m_1, m_2, \ldots, m_k)$.
2. We use $\underline{\mathrm{m}}^i_{[k-1]}$ to denote the tuple $\underline{\mathrm{m}}_{[k]}$ under consideration, with the i^{th} coordinate removed.
3. Henceforth, the sum $\sum\limits_{p_1 p_2 \ldots p_k \leq x}$ is taken over all sets of primes $\{p_1, p_2 \ldots p_k\}$ such that $p_1 p_2 \ldots p_k \leq x$, two sets being considered different even if they differ only in the order of primes.
4. For a fixed $\underline{\mathrm{m}}_{[k]}$, we write

$$\sum_{p_1 p_2 \ldots p_k \leq x} \boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k]}} :=$$
$$\sum_{p_1 p_2 \ldots p_k \leq x} \sum_{\sigma \in S'_k} \sum_{\chi} \overline{\chi(m_{\sigma(1)})} \chi(p_1) \sum_{\chi} \overline{\chi(m_{\sigma(2)})} \chi(p_2) \ldots \sum_{\chi} \overline{\chi(m_{\sigma(k)})} \chi(p_k)$$

   where

   1. The set S'_k is the subset of the symmetric group on k symbols consisting of those permutations that give rise to distinct permutations of $\{m_1, m_2, \ldots, m_k\}$,

   2. The sum $\sum\limits_{\chi}$ runs over the Dirichlet characters modulo N.

Note. We have the following orthogonality relation satisfied by Dirichlet characters mod N:

$$\sum_{\chi} \overline{\chi(m)}\chi(n) = \begin{cases} \phi(N) & \text{if } m \equiv n \bmod N \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that, for a fixed $n = p_1 p_2 \dots p_k$ and $\sigma \in S_k'$, the product

$$\sum_{\chi} \overline{\chi(m_{\sigma(1)})}\chi(p_1) \sum_{\chi} \overline{\chi(m_{\sigma(2)})}\chi(p_2) \dots \sum_{\chi} \overline{\chi(m_{\sigma(k)})}\chi(p_k)$$

is non-zero if and only if $p_i \equiv m_{\sigma(i)}$ for all $i = 1, \dots, k$. The orthogonality relation tells us that this non-zero quantity is $\phi(N)$ for each i. Therefore, for each $n = p_1 p_2 \dots p_k$, the inner double sum is $\phi(N)^k$ if, for *some* $\sigma \in S_k'$, we have $p_i \equiv m_{\sigma(i)}$ for *every* i and zero otherwise. Observe that this can happen for at most one permutation $\sigma \in S_k'$.

The following are auxiliary functions that will appear in the proof:

1.$\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \dfrac{1}{\phi(N)^k} \sum\limits_{p_1 p_2 \dots p_k \le x} \boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k]}}$.

2.$\vartheta_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \dfrac{1}{\phi(N)^k} \sum\limits_{p_1 p_2 \dots p_k \le x} \log(p_1 p_2 \dots p_k) \boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k]}}$.

3.$L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \dfrac{1}{\phi(N)^k} \sum\limits_{p_1 p_2 \dots p_k \le x} \dfrac{1}{(p_1 p_2 \dots p_k)} \boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k]}}$.

By Dirichlet's theorem, we know that for $i \neq j$ the number of primes $p \equiv m_{\sigma(i)} \bmod N$ is asymptotically the same as the number of primes $p \equiv m_{\sigma(j)} \bmod N$. Thus, if we fix a permutation of $\{m_1, m_2, \dots, m_k\}$, then the number of ordered sets $\{p_1, p_2 \dots p_k\}$ so that $p_i \equiv m_i \bmod N$ is equal to $\dfrac{1}{M}\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$, where M is the number of distinct permutations of the multiset $\{m_1, m_2, \dots m_k\}$.

6.3 Towards a generalization of Dirichlet's density theorem

The proof of Theorem 6.1.3, which is a k prime analogue of Dirichlet's original theorem, comes down to proving the following:

Proposition 6.3.1 $\vartheta_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) \sim \dfrac{M}{\phi(N)^k} kx(\log\log x)^{k-1} \quad (k \geq 2).$

The proof of this proposition will follow after a series of lemmas.
First, we prove a recursive relation for $\vartheta_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$:

Lemma 6.3.2 *For $k \geq 1$,*

$$k\vartheta_{k+1,\chi,\underline{\mathrm{m}}_{[k+1]}}(x) = (k+1)\sum_{p\leq x}\frac{1}{\phi(N)}{\sum_i}'\left(\sum_{\chi}\overline{\chi(m_i)}\chi(p)\vartheta_{k,\chi,\underline{\mathrm{m}}^i_{[k]}}\left(\frac{x}{p}\right)\right),$$

where the dash on top of the second summation symbol denotes that only those $i = 1, \ldots, k$ are counted so that the $\underline{\mathrm{m}}^i_{[k+1]}$ are distinct.

Proof.

$$\begin{aligned}
&(k+1)\vartheta_{k+1,\chi,\underline{\mathrm{m}}_{[k+1]}}(x)\\
&= \frac{1}{\phi(N)^{k+1}} \sum_{p_1p_2\ldots p_{k+1}\leq x} (k+1)\log(p_1p_2\ldots p_{k+1})\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}}\\
&= \frac{1}{\phi(N)^{k+1}} \sum_{p_1p_2\ldots p_{k+1}\leq x} \boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}}(\log p_1 + \log(p_2p_3\ldots p_{k+1}) + \log p_2 + \log(p_1p_3\ldots p_{k+1})\\
&\qquad\qquad + \ldots + \log p_{k+1} + \log(p_1p_2\ldots p_k))\\
&= \frac{1}{\phi(N)^{k+1}} \sum_{p_1p_2\ldots p_{k+1}\leq x} \log(p_1p_2\ldots p_{k+1})\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}}\\
&\qquad + \frac{1}{\phi(N)^{k+1}} \sum_{p_1p_2\ldots p_{k+1}\leq x} (\log(p_2p_3\ldots p_{k+1}) + \ldots + \log(p_1p_2\ldots p_k))\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}}\\
&= \frac{1}{\phi(N)^{k+1}} \sum_{p_1p_2\ldots p_{k+1}\leq x} \log(p_1p_2\ldots p_{k+1})\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}}\\
&\qquad + \frac{(k+1)}{\phi(N)^{k+1}} \sum_{p_1p_2\ldots p_{k+1}\leq x} \log(p_2p_3\ldots p_{k+1})\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}}.
\end{aligned}$$

The first sum is just $\vartheta_{k+1,\chi,\underline{\mathrm{m}}}(x)$ and this reduces the left hand side to $k\vartheta_{k+1,\chi,\underline{\mathrm{m}}}(x)$.

In the second sum, observe that the $\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}}$ appearing is a $(k+1)$-tuple. Collecting the terms corresponding to p_1 in $\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}}$, the second term can be written as follows.

$$\sum_{p_1p_2\ldots p_{k+1}\leq x} \log(p_2p_3\ldots p_{k+1})\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k+1]}} =$$

$$\sum\nolimits_i' \sum_{p_1p_2\ldots p_{k+1}\leq x} \log(p_2p_3\ldots p_{k+1})\boldsymbol{\chi}_{\underline{\mathrm{m}}^i_{[k]}} \left(\sum_{\chi} \overline{\chi(m_i)}\chi(p_1)\right).$$

Simplifying, we get

$$k\vartheta_{k+1,\chi,\underline{\mathrm{m}}_{[k+1]}}(x) = (k+1)\sum_{p\leq x} \frac{1}{\phi(N)} \sum\nolimits_i' \left(\sum_{\chi} \overline{\chi(m_i)}\chi(p)\vartheta_{k,\chi,\underline{\mathrm{m}}^i_{[k]}}\left(\frac{x}{p}\right)\right).$$

□

Similarly, we prove a recursion formula for the function $L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$:

Lemma 6.3.3 *Let* $L_{0,\chi,\underline{\mathrm{m}}_{[0]}}(x) = 1$. *Then for* $k \geq 1$,

$$L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \sum_{p\leq x} \frac{1}{p} \sum\nolimits_i' \frac{1}{\phi(N)} \sum_{\chi} \overline{\chi(m_i)}\chi(p) L_{k-1,\chi,\underline{\mathrm{m}}^i_{[k-1]}}\left(\frac{x}{p}\right),$$

where the dash on top of the second summation symbol is as defined in Lemma 6.3.2.

This follows directly from the definitions.

Let

$$f_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \phi(N)^k \vartheta_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) - xk\phi(N)^{k-1} \sum\nolimits_i' L_{k-1,\chi,\underline{\mathrm{m}}^i_{[k-1]}}(x). \qquad (6.2)$$

The idea is to first estimate $f_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$ and $L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$. Plugging in these estimates into Equation (6.2) would then give an asymptotic formula for $\theta_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$ thus proving Proposition 6.3.1. With this in mind, we first prove a recursion formula for $f_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$.

Lemma 6.3.4

$$k f_{k+1,\chi,\underline{\mathrm{m}}_{[k+1]}}(x) = (k+1)\sum_{p\le x}{\sum_i}' \sum_{\chi} \overline{\chi(m_i)}\chi(p) f_{k,\chi,\underline{\mathrm{m}}^i_{[k]}}\left(\frac{x}{p}\right).$$

Proof.

From the definition of $f_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$, we have
$k f_{k+1,\chi,\underline{\mathrm{m}}_{[k+1]}}(x) = k\phi(N)^{k+1}\vartheta_{k+1,\chi,\underline{\mathrm{m}}_{[k+1]}}(x) - xk(k+1)\phi(N)^k {\sum}_i' L_{k,\chi,\underline{\mathrm{m}}^i_{[k]}}(x)$.
We evaluate the two summands using Lemma 6.3.2 and Lemma 6.3.3 proved above.

By Lemma 6.3.2 we have

$$k\phi(N)^{k+1}\vartheta_{k+1,\chi,\underline{\mathrm{m}}_{[k+1]}}(x) = \phi(N)^{k+1}(k+1)\sum_{p\le x}\frac{1}{\phi(N)}{\sum_i}'\left(\sum_{\chi}\overline{\chi(m_i)}\chi(p)\vartheta_{k,\chi,\underline{\mathrm{m}}^i_{[k]}}\left(\frac{x}{p}\right)\right),$$

which simplifies to

$$(k+1)\sum_{p\le x}{\sum_i}'\sum_{\chi}\overline{\chi(m_i)}\chi(p)\left[\phi(N)^k\vartheta_{k,\chi,\underline{\mathrm{m}}^i_{[k]}}\left(\frac{x}{p}\right)\right].$$

Also using Lemma 6.3.3,

$${\sum_i}' L_{k,\chi,\underline{\mathrm{m}}^i_{[k]}}(x) = \sum_{i=1}^{k+1}\sum_{p\le x}\frac{1}{p}{\sum_j}'\frac{1}{\phi(N)}\sum_{\chi}\overline{\chi(m_j)}\chi(p)L_{k-1,\chi,\underline{\mathrm{m}}^{i,j}_{[k-1]}}\left(\frac{x}{p}\right),$$

where $\underline{\mathrm{m}}^{i,j}_{[k-1]}$ denotes $\underline{\mathrm{m}}^i_{[k]}$ with the j-th coordinate removed and ${\sum}_j'$ denotes that only distinct $\underline{\mathrm{m}}^{i,j}_{[k-1]}$ are counted.
Therefore,

$$xk(k+1)\phi(N)^k{\sum_i}' L_{k,\chi,\underline{\mathrm{m}}^i_{[k]}}(x)$$
$$= (k+1)\sum_{p\le x}{\sum_i}'\sum_{\chi}\overline{\chi(m_i)}\chi(p)\left[k\phi(N)^{k-1}\frac{x}{p}{\sum_j}' L_{k-1,\chi,\underline{\mathrm{m}}^{i,j}_{[k-1]}}\left(\frac{x}{p}\right)\right].$$

Putting the two summands together, we obtain the result. □

Next, we use Lemma 6.3.4 to get an estimate for $f_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$.

Lemma 6.3.5 *Let $k \ge 1$. Then*

$$f_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \mathrm{o}\{x(\log\log x)^{k-1}\}.$$

Proof. We induct on k.

When $k = 1$, writing $\underline{\mathrm{m}}_{[1]} = m$,

$$f_{1,\chi,m}(x) = \phi(N)\vartheta_{1,\chi,m}(x) - x.$$

From Dirichlet's theorem on the density of primes in an arithmetic progression, $\vartheta_{1,\chi,m}(x) \sim \frac{1}{\phi(N)} x$ and so

$$f_{1,\chi,m}(x) = \mathrm{o}(x).$$

Suppose the claim were true for $k = K$, where $K > 1$. This means for any $\varepsilon > 0$, there exists $x_0 = x_0(K, \varepsilon)$ such that

$$|f_{K,\chi,\underline{\mathrm{m}}_{[K]}}(x)| < \varepsilon x (\log\log x)^{K-1} \qquad \forall x \geq x_0.$$

Also, for $1 \leq x < x_0$, from the definition of $f_{K,\chi,\underline{\mathrm{m}}_{[K]}}$, we can find a real number D depending on K, ε so that

$$|f_{K,\chi,\underline{\mathrm{m}}_{[K]}}(x)| < D.$$

Using the above we deduce

1. For $p \leq \frac{x}{x_0}$,

$$\begin{aligned}\sum_{p \leq \frac{x}{x_0}} \left| \sum_{i=1}^{K+1} \sum_{\chi} \overline{\chi(m_i)} \chi(p) f_{K,\chi,\underline{\mathrm{m}}^i_{[K]}}\left(\frac{x}{p}\right) \right| &< (K+1)\phi(N)\varepsilon(\log\log x)^{K-1} \sum_{p \leq \frac{x}{x_0}} \frac{x}{p} \\ &< (K+2)\phi(N)\varepsilon x (\log\log x)^{K},\end{aligned}$$

   for x large enough.

2. For $\frac{x}{x_0} < p \leq x$,

$$\begin{aligned}\sum_{\frac{x}{x_0} < p \leq x} \left| \sum_{i=1}^{K+1} \sum_{\chi} \overline{\chi(m_i)} \chi(p) f_{K,\chi,\underline{\mathrm{m}}^i_{[K]}}\left(\frac{x}{p}\right) \right| &< (K+1)\phi(N) D \pi(x) \\ &< (K+1)\phi(N) D x.\end{aligned}$$

Hence, using Lemma 6.3.4 and the simple inequality $(K+1) < 2K$ for $K > 1$, we have
$K|f_{K+1,\chi,\underline{\mathrm{m}}_{[K+1]}}(x)| < 2K\phi(N)x((K+2)\varepsilon(\log\log x)^k + (K+1)D)$.
Thus, for $x > x_1(D, \varepsilon, K)$ we conclude

$$|f_{K+1,\chi,\underline{\mathrm{m}}_{[K+1]}}(x)| < 2(K+2)\phi(N)\varepsilon x(\log\log x)^K.$$

Since ε was arbitrary, the claim follows for all $k \in \mathbb{N}$ by induction. $\square$

To complete the proof of Proposition 6.3.1, it suffices to prove:

Lemma 6.3.6

$$L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) \sim \frac{M}{\phi(N)^k}(\log\log x)^k.$$

Proof. Recall that

$$L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \frac{1}{\phi(N)^k}\sum_{p_1p_2\ldots p_k\leq x}\frac{1}{(p_1p_2\ldots p_k)}\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k]}}$$

$$= \frac{1}{\phi(N)^k}\sum_{p_1p_2\ldots p_k\leq x}\frac{1}{(p_1p_2\ldots p_k)}\sum_{\sigma\in S_k}\sum_{\chi}\overline{\chi(m_{\sigma(1)})}\chi(p_1)\ldots\sum_{\chi}\overline{\chi(m_{\sigma(k)})}\chi(p_k)$$

and that M is the number of permutations of the (possible) multiset $\{m_1, m_2, \ldots, m_k\}$.

We observe that the following hold:
Given a squarefree number n with k factors, if each prime p dividing n satisfies $p \leq x^{1/k}$ then $n \leq x$. This leads us to write

$$L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) \geq M\prod_{i=1}^{k}\sum_{p\leq x^{1/k}}\frac{1}{p}\left(\frac{1}{\phi(N)}\sum_{\chi}\overline{\chi(m_i)}\chi(p)\right),$$

i.e.,

$$L_{k,\chi,\underline{\mathrm{m}}_{[k]}} \geq M\prod_{i=1}^{k}\sum_{\substack{p\leq x^{1/k}\\ p\equiv m_i \bmod N}}\frac{1}{p}.$$

Similarly, if $n = p_1 p_2 \dots p_k$ is less than x then each $p_i \leq x$, which gives us an upper bound:

$$L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) \leq M \prod_{i=1}^{k} \sum_{p \leq x} \frac{1}{p} \left(\frac{1}{\phi(N)} \sum_{\chi} \overline{\chi(m_i)} \chi(p) \right) = M \prod_{i=1}^{k} \sum_{\substack{p \leq x \\ p \equiv m_i \mod N}} \frac{1}{p}.$$

It is known (see for example [30]) that for any a coprime to N,

$$\sum_{\substack{p \leq x \\ p \equiv a \mod N}} \frac{1}{p} \sim \frac{1}{\phi(N)} \log \log x.$$

Thus, $L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)$ is bounded below and above by functions that are each asymptotic to $\frac{M}{\phi(N)^k} (\log \log x)^k$, implying that

$$L_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) \sim \frac{M}{\phi(N)^k} (\log \log x)^k.$$

□

Finally, Proposition 6.3.1 follows by using Lemma 6.3.5 and Lemma 6.3.6 in Equation (6.2).

Remark: Some care needs to be taken while applying Lemma 6.3.6. The term $\sum_{i=1}^{k} L_{k-1,\chi,\underline{\mathrm{m}}^i_{[k-1]}}(x)$ appearing in Equation (6.2) involves number of distinct permutations of $\underline{\mathrm{m}}^i_{[k-1]}$, whereas M appearing in Proposition 6.3.1 is the number of distinct permutations of $\underline{\mathrm{m}}_{[k]}$. This is resolved by using the following simple fact:
Let $k_1 + k_2 + \ldots + k_m = n$. Then

$$\frac{n!}{k_1! k_2! \ldots k_m!} = \frac{(n-1)!}{(k_1 - 1)! k_2! \ldots k_m!} + \frac{(n-1)!}{k_1! (k_2 - 1)! \ldots k_m!} + \ldots + \frac{(n-1)!}{k_1! k_2! \ldots (k_m - 1)!}.$$

We are now ready to prove the theorem.

6.4 Proof of Theorem 6.1.3

By partial summation we have

$$\vartheta_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)\log x - \int_2^x \frac{\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(t)}{t}dt.$$

Clearly, $\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(t) = \mathrm{O}(t)$ and therefore,

$$\int_2^x \frac{\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(t)}{t}dt = \mathrm{O}(x).$$

Hence, for $k \geq 2$, by Proposition 6.3.1,
$\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = \dfrac{\vartheta_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)}{\log x} + \mathrm{O}\left(\dfrac{x}{\log x}\right) \sim \dfrac{M}{\phi(N)^k}\dfrac{kx(\log\log x)^{k-1}}{\log x}$. Thus,

$$\frac{1}{M}\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) \sim \frac{1}{\phi(N)^k}\frac{kx(\log\log x)^{k-1}}{\log x}. \tag{6.3}$$

We now relate this to the functions $\pi_{k,\underline{\mathrm{m}}_{[k]}}(x)$ and $\tau_{k,\underline{\mathrm{m}}_{[k]}}(x)$. It is easy to see that

$$k!\pi_{k,\underline{\mathrm{m}}_{[k]}}(x) \leq \frac{1}{M}\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) \leq k!\tau_{k,\underline{\mathrm{m}}_{[k]}}(x).$$

We have two cases to consider.

Case 1: The units $m_1, m_2, \ldots, m_k$ are distinct.

Then $\boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k]}} = 0$ unless $p_1, p_2, \ldots, p_k$ are all distinct. This forces the following equality:

$$k!\pi_{k,\underline{\mathrm{m}}_{[k]}}(x) = \frac{1}{M}\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x) = k!\tau_{k,\underline{\mathrm{m}}_{[k]}}(x),$$

so using Equation (6.3) we are done.

Case 2: At least two of the m_i are equal.

Certainly, in this case we include those $n = p_1 \ldots p_k$ so that at least two of the primes are equal. The number of such $n \leq x$ is $\tau_{k,\underline{\mathrm{m}}_{[k]}}(x) - \pi_{k,\underline{\mathrm{m}}_{[k]}}(x)$. These n can be expressed in the form $n = p_1 \ldots p_k$ with $p_{k-1} = p_k$ and $\underline{\mathrm{m}}_{[k]}$

with $m_{k-1} = m_k$. Therefore, we have

$$\begin{aligned}\tau_{k,\underline{\mathrm{m}}_{[k]}}(x) - \pi_{k,\underline{\mathrm{m}}_{[k]}}(x) &\leq \frac{1}{M} \sum_{p_1 p_2 \ldots p_{k-1}^2 \leq x} \frac{1}{\phi(N)^k} \boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k]}} \\ &\leq \frac{1}{M} \sum_{p_1 p_2 \ldots p_{k-1} \leq x} \frac{1}{\phi(N)^{k-1}} \boldsymbol{\chi}_{\underline{\mathrm{m}}_{[k]}} \\ &= \frac{1}{M} \Pi_{k-1,\chi,\underline{\mathrm{m}}_{[k-1]}}(x).\end{aligned}$$

Since $\frac{1}{M}\Pi_{k-1,\chi,\underline{\mathrm{m}}_{[k-1]}}(x)$ is $\mathrm{o}\left(\frac{1}{M}\Pi_{k,\chi,\underline{\mathrm{m}}_{[k]}}(x)\right)$, from our observation above, we have

$$\pi_{k,\underline{\mathrm{m}}_{[k]}}(x) \sim \tau_{k,\underline{\mathrm{m}}_{[k]}}(x) \sim \frac{1}{\phi(N)^k} \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} \qquad (k \geq 2)$$

thus proving the theorem in this case as well.

6.5 Proofs of Proposition 6.1.1 and Theorem 1.3.4

In order to prove Proposition 6.1.1, we note that it suffices to prove the result for p odd, since 2 is the only even prime and the density of finite sets is zero. Thus we will assume that p is odd in the proof.

Proof of Proposition 6.1.1.

Let $D = \pm q_1^{a_1} q_2^{a_2} \ldots q_m^{a_m}$ be the decomposition of D. Then, by the multiplicative property of the Legendre symbol, we have

$$\begin{aligned}\left(\frac{D}{p}\right) &= \left(\frac{\pm 1}{p}\right)\left(\frac{q_1}{p}\right)^{a_1}\left(\frac{q_2}{p}\right)^{a_2}\cdots\left(\frac{q_m}{p}\right)^{a_m} \\ &= \pm\left(\frac{q_1}{p}\right)\left(\frac{q_2}{p}\right)\cdots\left(\frac{q_m}{p}\right).\end{aligned}$$

We have two possibilities:

Case (i): $2 \nmid D$

Then, either $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$. If $p \equiv 1 \bmod 4$ then by quadratic reciprocity, $\left(\frac{q_i}{p}\right) = \left(\frac{p}{q_i}\right)$. Also, $\left(\frac{\pm 1}{p}\right) = 1$. If $p \equiv 3 \bmod 4$, $\left(\frac{q_i}{p}\right) = \pm\left(\frac{p}{q_i}\right)$, depending on whether $q_i \equiv 1$ or $3 \bmod 4$ and so we can write

$$\left(\frac{D}{p}\right) = \pm\left(\frac{p}{q_1}\right)\left(\frac{p}{q_2}\right)\cdots\left(\frac{p}{q_m}\right).$$

Since $p \nmid q$, we know that p is a unit mod q, so it is congruent to one of the $q-1$ units in $\mathbb{Z}/q\mathbb{Z}$. We also know that if q is an odd prime, then there are $\frac{q-1}{2}$ squares in $(\mathbb{Z}/q\mathbb{Z})^{\times}$, therefore we conclude that for each q_i, the equations

$$\left(\frac{p}{q_i}\right) = 1$$

and

$$\left(\frac{p}{q_i}\right) = -1$$

each have $\frac{q_i-1}{2}$ solutions for $p \bmod q_i$.

Let S_i^+ denote the set of $\frac{q_i-1}{2}$ congruences mod q_i that solve $\left(\frac{p}{q_i}\right) = 1$ and S_i^- denote the set of $\frac{q_i-1}{2}$ congruences mod q_i that solve $\left(\frac{p}{q_i}\right) = -1$.

Clearly,

$$\left(\frac{D}{p}\right) = 1 \Leftrightarrow \left(\frac{p}{q_1}\right)\left(\frac{p}{q_2}\right)\cdots\left(\frac{p}{q_m}\right) = 1, \tag{6.4}$$

Now, the equations

$$x_1x_2\ldots x_m = 1 \text{ and } x_1x_2\ldots x_m = -1$$

each have $M = 2^{m-1}$ solutions in $\{-1,1\}^m$.

Let us enumerate them as

$$X_1 = (x_{11}, x_{12}, \ldots, x_{1m}) \qquad Y_1 = (y_{11}, y_{12}, \ldots, y_{1m})$$

$$X_2 = (x_{21}, x_{22}, \ldots, x_{2m}) \qquad Y_2 = (y_{21}, y_{22}, \ldots, y_{2m})$$

$$\vdots \qquad \text{and} \qquad \vdots$$

$$X_M = (x_{M1}, x_{M2}, \ldots, x_{Mm}) \qquad Y_M = (y_{M1}, y_{M2}, \ldots, y_{Mm})$$

respectively, where each of the x_{ij}, y_{ij} are 1 or -1. Depending on whether we need the product in Equation (6.4) to be 1 or -1, we solve using X_i's or Y_j's.

Without loss of generality let us assume that we need the product to be 1. Then, for each solution X_j, $j = 1, \ldots, M$ we need to solve the following system :

$$p \equiv 1 \mod 4$$

$$\left(\frac{p}{q_i}\right) = x_{ji}, \qquad i = 1, \ldots m.$$

For each i, the equation $\left(\frac{p}{q_i}\right) = x_{ji}$ will involve choosing a congruence relation among $S_i^{\pm}$ depending on the parity of x_{ji}. This gives us a total of $\prod_{i=1}^{m} \frac{q_i - 1}{2}$ systems of congruences for each X_j. By the Chinese Remainder Theorem, each system will give rise to a unique solution. Thus, the total number of solutions we obtain is

$$M \prod_{i=1}^{m} \frac{q_i - 1}{2} = 2^{m-1} \prod_{i=1}^{m} \frac{q_i - 1}{2} = \frac{1}{2} \prod_{i=1}^{m} (q_i - 1).$$

Similarly we get $\frac{1}{2} \prod_{i=1}^{m} (q_i - 1)$ solutions coming from the parallel case of $p \equiv 3 \mod 4$.

So, in total we have $\prod_{i=1}^{m} (q_i - 1)$ number of solutions $(\mathrm{mod}\, 4q_1 q_2 \ldots q_m)$.

If we denote $Q = 4q_1 q_2 \ldots q_m$, then $\left(\frac{D}{p}\right) = 1$ has $\frac{1}{2}\phi(Q)$ number of solutions mod Q.

Case (ii): $2|D$.

Without loss of generality, we may assume that $q_1 = 2$ and q_i is odd for $i = 2, \ldots, k$.

Therefore, we need to find solutions to the equation

$$\begin{aligned}\left(\frac{D}{p}\right) &= \pm\left(\frac{2}{p}\right)\left(\frac{q_2}{p}\right)\cdots\left(\frac{q_m}{p}\right)\\ &= \pm\left(\frac{2}{p}\right)\left(\frac{p}{q_2}\right)\cdots\left(\frac{p}{q_m}\right).\end{aligned}$$

The only difference in this case is that instead of considering the congruence $p \equiv 1$ or $3 \mod 4$, we further consider congruences mod 8:

If $p \equiv 1 \mod 4$, we have

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \mod 8 \\ -1 & \text{if } p \equiv 5 \mod 8. \end{cases}$$

Thus in this case, for each $i = 2, \ldots m$, we have $\frac{q_i - 1}{2}$ number of congruences $\mod q_i$ and one congruence $\mod 8$ corresponding to $i = 1$. Therefore, for every X_j (or Y_j, depending on whether we need the product to be 1 or -1) we get $\prod_{i=2}^{m} \frac{q_i - 1}{2}$ number of solutions. Hence the total number of solutions is

$$\prod_{i=2}^{m}(q_i - 1).$$

Similarly, if $p \equiv 3 \mod 4$, then we use

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 7 \mod 8 \\ -1 & \text{if } p \equiv 3 \mod 8 \end{cases}$$

and obtain another set of $\prod_{i=2}^{m}(q_i - 1)$ solutions.

So we have a total of

$$2\prod_{i=2}^{m}(q_i - 1) = \frac{1}{2}\phi(4q_1q_2 \ldots q_m) = \frac{1}{2}\phi(Q)$$

solutions, which is the same number as in Case 1.
To summarize, for a fixed number D, the number of odd primes $p \mod Q$ so that $\left(\frac{D}{p}\right) = 1$ is $\frac{1}{2}\phi(Q)$. Coming back to our problem, we wish to calculate

$$\#\left\{\text{primes } p \leq x : \left(\frac{D}{p}\right) = 1\right\}.$$

By Dirichlet's density theorem, we know that for any positive integer a which is coprime to n,

$$\#\{p \leq x, p \text{ prime } | p \equiv a \mod n\} \sim \frac{1}{\phi(n)}\pi(x).$$

Let $B(1) := \{b_i, i = 1, \ldots, b_{\frac{\phi(Q)}{2}}\}$ denote the set solutions $\bmod Q$ obtained from the discussion above and $B(-1) := \{b'_i, i = 1, \ldots, b'_{\frac{\phi(Q)}{2}}\}$ denote the remaining residue classes that correspond to the primes $p \bmod Q$ so that $\left(\frac{D}{p}\right) = -1$. Then, $\left(\frac{D}{p}\right) = 1$ if and only if p is congruent to any one of the elements in the set $B(1)$. So we have

$$\#\left\{\text{primes } p \leq x : \left(\frac{D}{p}\right) = 1\right\}$$

$$= \sum_{i=1}^{\frac{\phi(Q)}{2}} \#\{p \leq x, p \text{ prime } | p \equiv b_i \mod Q\} \sim \sum_{i=1}^{\frac{\phi(Q)}{2}} \frac{1}{\phi(Q)}\pi(x) = \frac{1}{2}\pi(x).$$

Hence, the asymptotic density of primes p for which $\left(\frac{D}{p}\right) = 1$ is $\frac{1}{2}$.

Using the set $B(-1)$, the same proof can be used to show that

$$\#\left\{\text{primes } p \leq x : \left(\frac{D}{p}\right) = -1\right\} \sim \frac{1}{2}\pi(x),$$

implying that the density of primes p for which $f(x)$ has no solution $\bmod p$ is $\frac{1}{2}$.
We now use this proposition to prove

Theorem 6.5.1 *Let $D \in \mathbb{Z} - \{0\}$ and $k \in \mathbb{N}$. Fix a k-tuple $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$ where each $\varepsilon_i = \pm 1$ for each $i = 1, \ldots, k$. Then*

$$\frac{1}{\pi_k(x)} \# \left\{ n \leq x, n = p_1 p_2 \ldots p_k \text{ with } p_1 < p_2 < \ldots < p_k : \left(\frac{D}{p_i}\right) = \varepsilon_i \text{ for each } i \right\} \sim \frac{1}{2^k},$$

where $\pi_k(x)$ denotes the number of squarefree numbers less than x with k prime factors.

Remark: From the statement of Proposition 6.1.1 and Theorem 6.5.1, it is clear that we are counting only those squarefree numbers with k-prime factors which are coprime to the discriminant D of $f(x)$.

Proof of Theorem 6.5.1. We first prove the statement for n odd.
In this case, using Proposition 6.1.1 we conclude that the condition

$$\left(\frac{D}{p_i}\right) = \varepsilon_i \text{ for each } i$$

will hold if and only if every prime p_i dividing n belongs to the set $B(\varepsilon_i)$.
Let us represent the (odd) squarefree number as a tuple $(p_1, p_2, \ldots, p_k)$ with $p_1 < p_2 < \ldots < p_k$ and choose any k-tuple $(m_1, m_2, \ldots, m_k)$ where each $m_i \in B(\varepsilon_i)$. Since $|B(\pm 1)| = \dfrac{\phi(Q)}{2}$, the number of k-tuples such that

$$(p_1, p_2, \ldots, p_k) \equiv (m_1, m_2, \ldots, m_k) \mod Q \tag{6.5}$$

component-wise is $\left(\dfrac{\phi(Q)}{2}\right)^k$. Therefore, appplying Theorem 1.3.3, we have

$$\# \left\{ \text{Odd } n \leq x, n = p_1 p_2 \ldots p_k \text{ with } p_1 < p_2 < \ldots < p_k : \left(\frac{D}{p_i}\right) = \varepsilon_i \text{ for each } i \right\}$$

$$\sim \frac{1}{\phi(Q)^k} \frac{x (\log \log x)^{k-1}}{(k-1)! \log x} \left(\frac{\phi(Q)}{2}\right)^k,$$

settling the odd case.
Note: Even n are counted only if D is odd.
The even case follows by counting the number of odd squarefree $n \leq x/2$

with $k-1$ prime factors. From the argument for the odd case, we have

$$\#\left\{n \leq x, n = 2p_2 \dots p_k, \text{ with } 2 = p_1 < p_2 < \dots < p_k : \left(\frac{D}{p_i}\right) = \varepsilon_i \text{ for each } i\right\}$$

$$\sim \frac{1}{\phi(Q)^{k-1}} \pi_{k-1}(x/2) \left(\frac{\phi(Q)}{2}\right)^{k-1}.$$

Noting that $\dfrac{\pi_{k-1}(x/2)}{2^{k-1}} = \mathrm{o}\left(\dfrac{\pi_k(x)}{2^k}\right)$, the result follows.

Corollary 6.5.2 *The density of squarefree numbers n with k prime factors so that a quadratic equation has exactly 2^k solutions mod n is $\dfrac{1}{2^k}$.*

Proof. This easily follows from Theorem 1.3.4 by taking D as the discriminant of the quadratic equation and $\underline{\varepsilon}$ with $\varepsilon_i = 1$ for each i. □

Note: We may ask what happens when n has k prime factors counted with multiplicity, i.e., when $n = p_1 p_2 \dots p_k$ is not necessarily squarefree. We observe that in this case, the k-tuple $\underline{\mathrm{m}}$ will neccesarily have $m_i = m_j$ whenever $p_i = p_j$. Therefore, for such n, the number of k-tuples satisfying Equation 6.5 will be bounded by $\left(\dfrac{\phi(Q)}{2}\right)^k$ and equal to it if and only if n is squarefree. Hence, we deduce the following:

Corollary 6.5.3 *Let $D \in \mathbb{Z} - \{0\}$ and $k \in \mathbb{N}$. For any k-tuple $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k)$ where each $\varepsilon_i = \pm 1$ for each $i = 1, \dots, k$, we have*

$$\#\left\{n \leq x : n = p_1 p_2 \dots p_k \text{ with } p_1 \leq p_2 \leq \dots \leq p_k : \left(\frac{D}{p_i}\right) = \varepsilon_i \text{ for each prime } p_i | n\right\}$$

$$= \mathrm{O}\left(\frac{1}{2^k} \tau_k(x)\right),$$

where $\tau_k(x)$ is the function defined in the introduction.

Bibliography

[1] Barnet-Lamb, T., D. Geraghty, M. Harris and R. Taylor, "A family of Calabi-Yau varieties and potential automorphy II." *Publications of the Research Institute for Mathematical Sciences*, No. 1, 47 (2011), 29-98.

[2] Bromwich, T. J., *Introduction to the theory of infinite series*. London: Macmillan Co., 1926.

[3] Bucur A., C. David, B. Feigon and M. Lalín, "Statistics for ordinary Artin-Schreier covers and other p-rank strata." *Transactions of the American Mathematical Society*, 368 (2016), 2371-2413.

[4] Bucur A., C. David, B. Feigon, M. Lalín and K. Sinha, "Distribution of zeta zeroes of Artin-Schreier curves." *Mathematical Research Letters*, No. 6, 19 (2012), 1329-1356.

[5] Bucur A. and K. Kedlaya, "An application of the effective Sato-Tate conjecture." *Frobenius Distributions: Lang-Trotter and Sato-Tate conjectures*, Contemporary Mathematics 663, American Mathematical Society, Providence, RI, (2016), 45-56.

[6] Conrey, J. B., W. Duke and D. W. Farmer, "The distribution of the eigenvalues of Hecke operators." *Acta Arithmetica* 78 (1997), 405-409.

[7] Deligne, P., "La conjecture de Weil. I." *Institut des Hautes Études Scientifiques. Publications Mathématiques*, 43 (1974), 273-307.

[8] Faifman, D. and Z. Rudnick, "Statistics of the zeros of zeta functions in families of hyperelliptic curves over a finite field." *Compositio Mathematica*, 146 (2010), 81-101.

[9] Goldfeld, D. and J. Hundley, *Automorphic Representations and L-functions for the General Linear Group Volume I*, Cambridge studies in advanced mathematics 129, Cambridge University Press, 2011.

[10] Hardy, G. H. and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1985.

[11] Jolley, L. B. W., *Summation of Series*, Dover, New York, 1961.

[12] Knightly, A. and C. Li, *Traces of Hecke Operators*, Mathematical Surveys and Monographs 133, American Mathematical Society, 2006.

[13] Koblitz, N., *Introduction to Elliptic Curves and Modular Forms*, Graduate Texts in Mathematics 97, Springer-Verlag, 1984.

[14] Kornblum, H. and E. Landau, " Über die Primfunktionen in einer Arithmetischen Progression." *Mathematische Zeitschrift* 5 (1919), 100–111.

[15] Landau, E., "Sur quelques problèmes relatifs à la distribution des nombres premiers." *Bulletin de la SMF*, 28 (1900) 25–38.

[16] Langlands, R. P., "Problems in the theory of automorphic forms." *Lectures in modern analysis and applications*, Lecture Notes in Mathematics 170 (1970), Springer, Berlin, 18-61.

[17] Lau, Y.K. and Y. Wang, "Quantitative version of the joint distribution of eigenvalues of the Hecke operators." *Journal of Number Theory*, 131 (2011), 2262-2281.

[18] Martin, G., "Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$." *Journal of Number Theory*, 112 (2005), 298-331.

[19] Kim, H. and P. Sarnak, Appendix 2 in "Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2." *Journal of the American Mathematical Society* , 16 (2003), 139-183.

[20] Montgomery, H. L., *Ten lectures on the interface between analytic number theory and harmonic analysis*, CBMS Regional Conference Series in Mathematics 84, Washington, DC, Conference Board of the Mathematical Sciences, 1994.

[21] Montgomery, H. L. and R. C. Vaughan, *Mutliplicative Number Theory I. Classical Theory*, Cambridge Studies in Advanced Mathematics 97, Cambridge University Press, 2007.

[22] Murty, M. R., *Problems in Analytic Number Theory*, Second Edition, Graduate Texts in Mathematics 206, Springer-Verlag, 2008.

[23] Murty, V. K., "Explicit formulae and the Lang-Trotter conjecture." *Rocky Mountain Journal of Mathematics*,No. 2, 15 (1985), 535-551.

[24] Murty, M. R. and V. K. Murty, "Some remarks on automorphy and the Sato-Tate conjecture." *Advances in the theory of numbers*,Fields Institute Communications 77 (2015) 159-168.

[25] Murty, M. R. and K. Sinha., "Effective equidistribution of eigenvalues of Hecke operators." *Journal of Number Theory*, No. 3, 129 (2009), 681-714.

[26] Murty, M.R. and K. Sinha, "Fatoring Newparts of Jacobians of certain modular curves." *Proceedings of the American Mathematical Society*, 138 (2010), 3481-3494.

[27] Nagoshi, H., "Distribution of Hecke Eigenvalues." *Proceedings of the American Mathematical Society* 134 (2006), 3097-3106.

[28] Rouse, J. and J. Thorner, "The explicit Sato-Tate conjecture and densities pertaining to Lehmer-type questions." to appear in *Transactions of the American Mathematical Society.*

[29] Sarnak, P., "Statistical properties of eigenvalues of the Hecke operators." *Analytic number theory and Diophantine problems*, (Stillwater, OK, 1984), Progr. Math., 70, Birkhäuser Boston, Boston, MA, 1987, 321-331.

[30] Pomerance, C., "On the distribution of amicable numbers", *Journal für die reine und angewandte Mathematik*, 293/294 (1977), 217-222.

[31] Selberg, A., "Harmonic Analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *Journal of the Indian Mathematical Society*, 20 (1956), 47-87.

[32] Serre, J.-P., *Abelian l-adic representations and elliptic curves.* McGill University Lecture Notes Written with the collaboration of Willem Kuyk and John Labute. W. A. Benjamin Inc, New York (1968).

[33] Serre, J.-P., "Répartition Asymptotique des Valeurs Propres de l'Operateur de Hecke T_p." *Journal of the American Mathematical Society*, No. 1, 10 (1997): 75-102.

[34] Sinha, K., "Effective Equidistribution of eigenvalues of Hecke Operators", *PhD thesis, Queen's University at Kingston*, 2006.

[35] Vaaler, J. D., "Some extremal functions in Fourier analysis." *Bulletin of the American Mathematical Society*, 12 (1985) 183-216.

[36] Wang, Y., "The quantitative distribution of Hecke eigenvalues." *Bulletin of the Australian Mathematical Society* 90 (2014), 28-36.

[37] Wright, E. M., “ A simple proof of a theorem of Landau.” *Proceedings of the Edinburgh Mathematical Society*, No. 2, 9 (1954), 87-90.

[38] Xiong, M., “Statistics of the zeros of zeta functions in a family of curves over a finite field.” *International Mathematics Research Notices*, 18 (2010), 3489-3518.