Level-rank duality in Chern-Simons theory

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled ”Level-rank duality in Chern-Simons theory” submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents original research carried out by Khilav Majmudar at Tata Institute of Fundamental Research, under the supervision of Prof. Shiraz Minwalla during the academic year 2017-2018.

Student	Supervisor
Khilav	Shiraz
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Declaration

I hereby declare that the matter embodied in the report entitled "Level-rank duality in Chern-Simons theory" are the results of the investigations carried out by me at the Department of Theoretical Physics, Tata Institute of Fundamental Research, under the supervision of Prof. Shiraz Minwalla and the same has not been submitted elsewhere for any other degree.

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Abstract

We begin investigations of the level-rank duality of pure Chern-Simons theory on the torus by counting the number of states on both sides of the duality. We analyse a formula given in [1] for determining the dimensionality of the Hilbert space of $SU(N)$ Chern-Simons theory on the torus. We illustrate a way to find the dimensions of the $U(N)$ theory and show that it matches with the answer expected from duality.

Contents

Chapter 1

Introduction

Chern-Simons theory has proven to be a fruitful field of research in physics and mathematics. It is a topological field theory, meaning that its Lagrangian does not depend on the metric of the spacetime it is considered on. Its non-abelian quantum version in three dimensions was first solved completely by Witten [2] and in the process were obtained deep connections with 2-dimensional rational conformal field theory and the Jones polynomial knot invariants. We discuss some basic properties of the theory below.

1.1 The Lagrangian

The action for Chern-Simons theory is given by the integral of the Chern-Simons 3-form.

$$\mathscr{L} = \frac{k}{4\pi} \int\limits_M Tr(A \wedge dA + A \wedge A \wedge A) \tag{1.1}$$

In index notation it reads as follows [2]:

$$\mathscr{L} = \frac{k}{8\pi} \int\limits_M \epsilon^{ijk} Tr(A_i(\partial_j A_k - \partial_k A_j) + \frac{2}{3} A_i [A_j, A_k]) \tag{1.2}$$

Here, A is a gauge field valued in the Lie algebra of a compact simple gauge group G on an odd-dimensional oriented manifold M. ϵ^{ijk} is the antisymmetric Levi-Civita symbol, with the convention $\epsilon^{012} = 1$. A point to note is that the Lagrangian shown above is independent of the metric, and hence all observables in our theory will be topological invariants. Time evolution is also trivial, as the Hamiltonian is zero.

Under a gauge transformation of winding number m, the Lagrangian changes by a total derivative and a term that is reminiscent of the Wess-Zumino term.

$$\delta\mathscr{L} = \frac{k}{4\pi}\epsilon^{ijk}\partial_i Tr(\partial_j g g^{-1} A_k) + \frac{k}{12\pi}\epsilon^{ijk} Tr(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g) \tag{1.3}$$

The total derivative term vanishes on a manifold without boundary, say S^3. The second term is interesting. It is $2\pi k$ times the winding number density of G, the integral of which can be shown to be an integer [3]. This winding number calculates the number of times S^3 winds around the gauge group. In other words, it measures the π_3 of the map $M \to G$, which is usually isomorphic to $\mathbb{Z}$ for compact simple gauge groups. As quantum field theory requires the consistency of $e^{i\mathscr{L}}$, we deduce that k must be an integer. This argument of the quantisation of k does not hold for the gauge group $U(1)$, which is not a simple group. In that case k turns out to be an integer because of the presence of magnetic monopoles [4].

Thus, as far as gauge invariance goes, Chern-Simons theory is gauge-invariant only under gauge transformations which are connected to the identity, i.e., of winding number 0.

The equations of motion of the Chern-Simons action are given by the equation $F = dA + A \wedge A = 0$. Hence, the phase space of the classical theory is the space of all gauge-inequivalent flat connections. It is also known as the moduli space of flat connections.

1.1.1 4D in disguise

Three dimensional Chern-Simons theory can be viewed as a four-dimensional theory in disguise [5]. The 3-form can be constructed out of a 4-form which lives on a four-manifold. That form is

$$\star F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{1.4}$$

for abelian gauge fields. Where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The non-abelian generalisation is

$$\star F^{a\mu\nu} F^a_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$$

where $F^a_{\mu\nu}$ is the Yang-Mills field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu.$$

We can write $\star FF$ as a total derivative, so that its volume integral over the 4-manifold can be converted by Gauss' law into a surface integral over its boundary.

$$\frac{1}{2} \star F^{\mu\nu} F_{\mu\nu} = \partial_\mu (\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma) \tag{1.5}$$

$$\frac{1}{2} \star F^{\mu\nu a} F^a_{\mu\nu} = \partial_\mu (\epsilon^{\mu\nu\rho\sigma} A^a_\nu \partial_\rho A^a_\sigma + \frac{1}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma) \tag{1.6}$$

The terms in brackets, whose divergences give $\star FF$, are the abelian and non-abelian Chern-Simons terms respectively.

1.2 Observables

We need gauge invariant observables for our theory. Wilson lines serve the purpose, with the additional benefit that they are topological (general covariant) as well, since one does not need a metric to define them. We take an embedding of a circle C in M and compute the holonomy of the gauge field A_i around C. We thus get an element of G, and we then take its trace in an irreducible representation R.

$$W_R(C) = Tr_R Pexp \int_C A_i dx^i \tag{1.7}$$

P denotes that the exponential is path-ordered.

Chapter 2

Quantisation

In this chapter, we will see how the quantisation of Chern-Simons theory works out on different surfaces. To quantise the Chern-Simons thory, one adopts the strategy of first chopping up M into pieces, solving the theory on these pieces, and then gluing them back. M may have Wilson lines running through it. We cut the manifold along a Riemann surface Σ. Near the cut, M looks like $\Sigma \times R^1$.

The case of a manifold of the form $\Sigma \times R^1$ can be solved by the means of canonical quantisation. It produces finite-dimensional Hilbert spaces $\mathscr{H}_\Sigma$. If Wilson lines are present on Σ, we must consider marked points $P_1, ..., P_k$ with a representation R_i attached to every P_i, since every Wilson line has an associated representation.

WE decompose the exterior derivative ($dt\frac{\partial}{\partial t}$ and $\tilde{d}$) and the gauge field (A_0 and $\tilde{A}$) into spatial and temporal components and write the action as [1]

$$\mathscr{L} = -\frac{k}{4\pi}\int dt \int_\Sigma Tr(\tilde{A} \wedge \dot{\tilde{A}}) + \frac{k}{2\pi}\int dt \int_\Sigma Tr(A_0 \tilde{F}). \tag{2.1}$$

Integrating over A_0, we obtain the constraint $\tilde{F} = \tilde{d}\tilde{A} + \tilde{A} \wedge \tilde{A} = 0$.

2.1 $\Sigma = S^2$

When $\Sigma = S^2$, the Hilbert space is one dimensional. There exists only a vacuum state. This is because the space of flat connections is trivial. Flat connections depend on the fundamental group of the manifold. But the 2-

sphere is topologically trivial (simply connected) and so every loop can be shrunk to a point.

2.2 $\Sigma =$ Disc without a source

$\tilde{F} = 0$ implies that $\tilde{A}$ can be set to zero by a gauge transformation in any simply connected region [6]. But the whole disc is simply connected. Thus, the constraint $\tilde{F} = 0$ can be solved to give $\tilde{A} = -\tilde{d}UU^{-1}$, for a single valued function $U : D \times R \to G$. Note that Us are functions of time as well, as the constraint $\tilde{F} = 0$ is valid only on a spatial slice. Implementing this transformation gives

$$\mathscr{L} = \frac{k}{4\pi} \int_{\partial\Sigma} Tr(U^{-1}\partial_\phi U U^{-1}\partial_t U) d\phi dt + \frac{k}{12\pi} \int_{\Sigma} \epsilon^{ijk} Tr(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g) \tag{2.2}$$

Here, ϕ is an angular coordinate on the boundary S^1. This is the standard Wess-Zumino functional plus an off–diagonal kinetic term. No gauge has been fixed in the functional integral. By fixing a gauge, we can change the value of U in the interior, and so the effective action depends only on the boundary values of U.

Thus, the states in the spectrum are in representations of the loop group LG. LG is defined as the space of continuous maps $\beta : S^1 \to G$. This is because, as mentioned above, the degrees of freedom of U lie on the boundary of the disc, which is S^1. The phase space of the system is the space of based loops LG/G because U is uniquely defined up to $U \to U \cdot W$, for a constant $W \in G$ [2]. The symplectic structure is

$$\omega = \frac{k}{4\pi} \oint Tr(U^{-1}\delta U) \frac{d}{d\phi} (U^{-1}\delta U) \tag{2.3}$$

Once we have the symplectic structure, we can get the Poisson bracket corresponding to the Lagrangian and this enables us to quantise the theory. Upon quantisation, the boundary A_ϕ become operators that satisfy the Kač-Moody algebra [1]. The Kač-Moody algebra appears as the symmetry algebra when over and above the conformal symmetry, the system enjoys a global symmetry, which is G in this case.

The Lagrangian 2.2 is the chiral WZW Lagrangian. It is called chiral because of the presence of the off-diagonal kinetic term. The full WZW action is

$$\mathscr{L} = \frac{k}{4\pi}\int\limits_{\partial\Sigma} Tr(U^{-1}\partial_\mu U U^{-1}\partial_\mu U) + \frac{k}{12\pi}\int\limits_{\Sigma} \epsilon^{ijk} Tr(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g).$$

It was investigated by Witten in [7] as a bosonization dual of free massless fermions with non-abelian symmetry groups. This full action reduces to the chiral action in light-cone coordinates. It was shown in [7] that quantisation of the chiral WZW theory gives rise to the Kač-Moody current algebra, the currents being $g^{-1}\partial_+ g$ and $g^{-1}\partial_- g$, in light-cone coordinates. These can be obtained by effecting a small variation of the full WZW action and determining the equations of motion [7].

2.3 $\Sigma = T^2$

The most general solution of the constraint $\tilde{F} = 0$ is

$$\tilde{A} = -\tilde{d}UU^{-1} + U\theta(t)U^{-1}. \tag{2.4}$$

θ is a Lie algebra valued one-form which depends only on t, i.e., the R^1 coordinate. Plugging $\tilde{A}$ back into the action, we get

$$\mathscr{L} = -\frac{k}{4\pi}\int Tr(\theta \wedge \dot{\theta}). \tag{2.5}$$

This gives rise to the commutation relations

$$[\theta_1^i, \theta_2^j] = -i\frac{2\pi}{k}\delta^{ij}. \tag{2.6}$$

i and j stand for the Lie algebra indices of the fields. θ_1 and θ_2 run along the a and b cycles of the torus. The independent states are labelled by

$$\lambda \in \frac{\Lambda^W}{W \ltimes k\Lambda^R}, \tag{2.7}$$

Here, Λ^W and Λ^R are the weight and root lattices of G [1]. This formula will be analysed in detail later.

Chapter 3

Canonical Quantisation of $SU(N)$ Chern-Simons theory on $T^2 \times \mathbb{R}$

We will derive an understanding of the quantisation of the theory with $SU(N)$ gauge group on the Torus. This will be important in looking at the distribution of states in the Hilbert space and help in understanding the level-rank duality of the theory.

We look at the situation with a blind eye towards the statistics of the Wilson lines. In other words, the behaviour of states under permutations is ignored.

The gauge fields A and B are traceless diagonal matrices as they belong to the $su(N)$ Lie algebra. We work in the gauge $A_0 = 0$. The Chern-Simons action then reads as

$$S = \frac{k}{2\pi} \int dt \sum_{i=1}^{N} \dot{a}_i b_i \tag{3.1}$$

where a_i and b_i are the diagonal elements of A and B respectively, with the constraints $\sum a_i = \sum b_i = 0$. As in the previous chapter, the lengths of the cycles of the torus have been absorbed into the gauge fields. This gets rid of the integral. Being defined along the cycles of T^2, a_i and b_i enjoy a periodicity

$$\begin{aligned} a_i &\sim a_i + 2\pi\theta_i \\ b_i &\sim b_i + 2\pi\theta_i \\ \theta_i &= \delta_{im} - \delta_{in}. \end{aligned} \tag{3.2}$$

We tackle the problem by first redefining A and B in the Cartan basis $H_1, H_2, \ldots, H_{n-1}$ as

$$\begin{aligned} A &= \sum_{k=1}^{N-1} \alpha_k H_k \\ B &= \sum_{k=1}^{N-1} \beta_k H_k \end{aligned} \tag{3.3}$$

Using the normalisation $Tr(H_k H_{k'}) = \delta_{kk'}$, the action now becomes

$$S = \frac{k}{2\pi} \int dt \sum_{p=1}^{N-1} \dot{\alpha}_p \beta_p \tag{3.4}$$

θ_i now needs to be expressed in the Cartan basis. The required object is a diagonal matrix with 1 in the m^{th} diagonal element and -1 in the n^{th}, i.e., $\delta^{mm} - \delta^{nn}$. We represent it by $\sum\limits_{p=1}^{N-1} A_p^{mn} H_p$, and the α_p and β_p are now periodic with

$$\alpha_p \sim \alpha_p + A_p^{mn}$$

Given the action (4), we can write down the commutation relations

$$[\alpha_p, \beta_q] = i\frac{2\pi}{k}\delta_{pq} \tag{3.5}$$

Now, let us look at the objects of interest in our theory, namely wavefunctions and operators on Hilbert space. Consider the position-space wavefunction $e^{2i\pi\boldsymbol{\zeta}^1\cdot\boldsymbol{\alpha}}$. The periodicity property of the α_p forces the condition $\boldsymbol{\zeta}^1 \cdot A^{mn} \in \mathbb{Z}$. A^{mn}, by our construction, resided in the root lattice of the gauge group $SU(N)$. Therefore, the condition above on $\boldsymbol{\zeta}$ translates into the constraint $\boldsymbol{\zeta}^1 \in \lambda^W$, where λ^W is the weight lattice of G.

On to the operators. Consider the "momentum" operator $e^{2i\pi\boldsymbol{\zeta}^2\cdot\boldsymbol{\beta}}$. This operator should act in a well-defined fashion of the wavefunctions, i.e., it must respect the periodicity property. The momentum $2\pi\boldsymbol{\beta}$ shifts the position $\boldsymbol{\alpha}$ by the amount $\frac{1}{k}\boldsymbol{\zeta^2}$ (in accordance with the commutations relations 3.5). This action is given by

$$e^{2i\pi\boldsymbol{\zeta}^2\cdot\boldsymbol{\beta}} e^{2i\pi\boldsymbol{\zeta}^1\cdot\boldsymbol{\alpha}} e^{-2i\pi\boldsymbol{\zeta}^2\cdot\boldsymbol{\beta}} = exp\left(\frac{2i\pi\boldsymbol{\zeta}^1\cdot\boldsymbol{\zeta}^2}{k}\right) e^{2i\pi\boldsymbol{\zeta}^1\cdot\boldsymbol{\alpha}}$$

So, if $\boldsymbol{\zeta}^1 \in k\lambda^R$, where λ^R is the root lattice of G, then the action is trivial, as $\boldsymbol{\zeta}^2 \in \lambda^W$ by the same argument as that for $\boldsymbol{\zeta}^1$. This places an additional constraint on $\boldsymbol{\zeta}^1$; we must identify all those $\boldsymbol{\zeta}^1$ that differ by k times a root vector and likewise for $\boldsymbol{\zeta}^2$.

The above rules give the Hilbert space of our theory as $\frac{\lambda^W}{k\lambda^R}$.

We may now write the action of the position and momentum operators on an abstract vector in the Hilbert space as follows:

$$\begin{aligned} |\zeta\rangle &= e^{2i\pi\boldsymbol{\zeta}\cdot\boldsymbol{\alpha}} \\ e^{2i\pi\boldsymbol{\zeta}^1\cdot\boldsymbol{\alpha}}\,|\zeta\rangle &= |\zeta+\zeta^1\rangle \\ e^{2i\pi\boldsymbol{\zeta}^2\cdot\boldsymbol{\beta}}\,|\zeta\rangle &= e^{2i\pi\frac{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}^2}{k}}\,|\zeta\rangle \\ |\zeta\rangle &\sim |\zeta+kR\rangle \end{aligned} \tag{3.6}$$

We recover formula 2.7 by including permutations and identifying states which differ by an action of the Weyl group, as they are gauge equivalent. The Weyl group for $SU(N)$ is S_N, the permutation group of N elements. It merely changes the order of the a_is (or b_is) in the A (or B) gauge fields via a unitary transformation. We illustrate this by the $SU(2)$ case.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}$$

Chapter 4

Level-rank duality

There is a remarkable duality between two different Chern-Simons theories. It applies when the level k and the rank of the gauge group are interchanged in a certain manner. We are looking at the duality on the torus as there already are non-trivial states of pure Chern-Simons theory. We would eventually like to see how these states interact with particle dynamics when matter is added to the theory. The duality consists of the following two relations

$$\begin{aligned} (\mathbf{I}) \quad U(N)_k &\longleftrightarrow (\mathbf{I}) \quad U(k)_{-sgn(k)N} \\ SU(N)_k &\longleftrightarrow (\mathbf{II}) \quad U(k)_{-sgn(k)N} \end{aligned} \tag{4.1}$$

($\mathbf{I}$) and ($\mathbf{II}$) stand for Type I and Type II theory respectively. Type I $U(N)$ theory means that the gauge group is $SU(N)_k \times U(1)_{N\kappa}$. Type II $U(N)$ theory means that the gauge group is $SU(N)_k \times U(1)_{Nk}$. κ is the renormalised level $k + sgn(k)N$. Essentially, κ is the level on both sides of the Type I duality and k is the level for the other statement of the duality.

Dualities help in easing out difficult calculations. Quantities computed on both sides of the duality must be same for it to hold. The quantities commonly used are the partition function of the theory, the free energy, dimensionality of the Hilbert spaces, etc. Level-rank duality of Chern-Simons theory has also been found to lift to a Bose-Fermi duality, wherein a theory with fundamental bosons coupled to Chern-Simons theory is dual to fermions coupled to the level-rank dual Chern-Simons theory. In this thesis, we check that the level-rank duality of pure Chern-Simons theory indeed holds.

4.1 Check for Type I duality

A check for the duality would be to show that the number of states obtained for both theories are the same, i.e., the Hilbert spaces are of the same dimension on both sides.

Our action is

$$\mathscr{L} = \frac{\kappa}{4\pi} \int dt \int_{T^2} d^2x \epsilon^{ij} A_i \dot{A}_j \tag{4.2}$$

.

A_1 and A_2 are periodic gauge fields now. Let us take the lengths of the a and b cycles of T^2 as l_1 and l_2 respectively. The gauge fields being $u(N)$ matrices, the constraint $\tilde{F} = 0$ implies that they must be diagonal and independent of space. Diagonality will ensure $[A_1, A_2] = 0$ and space-independence will impose $\epsilon^{ij}\partial_i A_j = 0$. These conditions together mean $\tilde{F} = 0$.

As a result of A_1 and A_2 being independent of space, the area integral can be carried out trivially in the action to give $l_1 l_2$. These can be absorbed into the gauge fields so that they are now periodic with period 2π. Set $\tilde{A}_1 = l_1 A_1$ and $\tilde{A}_2 = l_2 A_2$. This reduces the action (4.2) to

$$\mathscr{L} = \frac{\kappa}{2\pi} \int dt \tilde{A}_1 \dot{\tilde{A}}_2$$

The antisymmetric ϵ symbol has been implemented and the action integrated by parts to get it in the canonical form. This is why the 4π becomes 2π. The commutation relations therefore are

$$[\tilde{A}_2{}^a, \tilde{A}_1{}^b] = \frac{2\pi i}{\kappa} \delta^{ab} \tag{4.3}$$

Now, we are dealing with the quantisation of a compact Hilbert space, in which the coordinate and momenta both are compact and periodic. To this end, we construct the $\kappa \times \kappa$ matrix operators $U = e^{i\tilde{A}_1}$ and $V = e^{i\tilde{A}_2}$. Their forms are chosen to be

$$\begin{aligned} U_{\mu\nu} &= e^{\frac{2\pi i}{\kappa}(\mu-1)} \delta_{\mu\nu} \\ V_{\mu\nu} &= \delta_{\mu,\nu-1} \end{aligned} \tag{4.4}$$

The form given for V is that of the shift matrix. It looks like this

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{4.5}$$

U and V satisfy the relation $VU = e^{\frac{2\pi i}{\kappa}}UV$. After κ iterations of UV on a state, we come back to the same state. Thus, the Hilbert space is κ dimensional.

But, there is more work to be done. Certain phenomenological considerations by Aharony et al. have shown that Chern-Simons theory describes particles of fermionic statistics [8]. They identified states in the Hilbert space that differed by permutations times their sign, i.e., by $W \times \text{sgn}(W)$. We understand this in the following manner.

The information about about a particular fermion is contained in the Lie algebra index a of the gauge fields A_i^a. There are N such $a's$ for $U(N)$. The particles being fermions, each κ can be occupied by at most one N. The identification shown above means that the eigenvalues of the U matrix can be interchanged among each other without any change in the physics of the theory. So, the actual dimensionality of the Hilbert space is $\binom{\kappa}{N}$, where $\kappa = k + N$. This formula is invariant under interchange of N and k, and hence holds for the other side of the duality too.

4.2 Type II dualilty

The fact that the gauge fields now lie in the $su(N)$ algebra complicates matters, as this introduces the constraint of tracelessness. The U eigenvalues are no longer on an equal footing, and so the straightforward analysis of the previous section cannot be carried over. We dissect the formula 2.7 and see what structure it gives.

Chapter 5

Counting of $SU(N)$ states

Using abstract techniques, Witten in [2] showed that the Hilbert space of Chern-Simons theory on the torus is the space of vacuum conformal blocks of the torus. The vacuum conformal blocks are characters of the Kač-Moody algebra and so are labelled by allowed primary operators of the Kač-Moody algebra. These characters are in turn labelled by representations of $SU(N)$. However, not every representation labels a character. Characters are labelled by integrable representations, which are well-known to be labelled by Young tableaux with no greater than k columns [9]. Note that for $SU(N)$, the number of rows of these tableaux cannot exceed $N-1$.

Putting this together, it follows that number of states of $SU(N)$ Chern-Simons theory on the torus is the number of Young tableaux of the above description.

We use a recursion relation to find this number. Imagine our Young tableaux with k columns, and cut out the last column. This leaves us with a $k-1$ column tableaux. The number of all possible tableaux, $A(N-1,k)$, at each k can be found out by the equation

$$A(N-1,k) = \sum_{j=1}^{N-1} A(j,k-1) \tag{5.1}$$

Using the fact that $A(N-1,1) = N-1$, this gives the final answer as

$$A(N-1,k) = \frac{(N+k-1)!}{k!(N-1)!} = \binom{N+k-1}{k}. \tag{5.2}$$

Our aim is to get this number from the counting formula given in [1].

$$\frac{\lambda^W}{W \ltimes k\lambda^R}, \tag{5.3}$$

where λ^W and λ^R are the weight and root lattices of the gauge group respectively, and W is the Weyl group. Let us construct these lattices for $SU(N)$. W for $SU(N)$ is S_N.

The weights of a Lie algebra are the eigenvalues of its fundamental representation. This, for $su(N)$, consists of the matrices themselves. We pick a basis whose elements are diagonal matrices with $e_k = \delta_{kk}$. There are N of these. But, as will be explained below, this system enjoys an invariance under shifts by the identity matrix. Thus, let the eigenvalues, which can be normalised to be positive integers, be labelled by $m \in \{0, 1, \dots, N-1\}$.

A basis of the root lattice is determined by the Cartan subalgebra. For $su(N)$, this consist of traceless diagonal matrices, the basis for which is given by the following $N-1$ diagonal matrices

$$\begin{aligned} T_i &= \delta_{ii} - \delta_{i+1,i+1} \\ i &\in \{1, 2, \dots, N\} \end{aligned} \tag{5.4}$$

A typical such matrix would be $\mathrm{diag}(0, 0, \dots, 1, -1, 0, \dots, 0)$. Hence, a root vector is given by the N-tuple $(x_1, x_2, \dots, x_N)$ such that the coordinates add up to $m = 0$. This gives us a picture of the root lattice as being a hyperplane passing through origin. A weight vector is likewise obtained when the coordinates add up to any other value of m.

Let us for a moment go back to chapter 3 Notice that if $\boldsymbol{\zeta}^1$ is shifted by the identity matrix, then we get the new wavefunction as $e^{2i\pi(\boldsymbol{\zeta}^1 \cdot \boldsymbol{\alpha} + Tr(\boldsymbol{\alpha}))}$. But, as $\boldsymbol{\alpha} \in su(N)$, $\boldsymbol{\alpha}$ is traceless. Thus, we must impose a quotienting of λ^W by shifts by the identity element. This shift by the identity matrix in each coordinate direction for λ^W has an interesting geometric interpretation. Take the root lattice hyperplane (which is just the $m = 0$ component of the weight lattice), and move by one unit in each coordinate direction. Take the origin as the reference point. This action, when performed on all points of the root lattice, gives us a hyperplane that is at a distance of $N\sqrt[2]{N}$ units away from the origin. This hyperplane is supposed to be identified with the root lattice. Doing this for every value of m, we find that our setup consists of N hyperplanes having equations $\sum x_i = m$, with $m \in \{0, 1, \dots, N-1\}$.

Thus, for given values of m, N and k the following rules must be implemented to find all the possible states that exist in the theory. The important constraints to keep in mind are that states differing by k times a root lattice vector must be identified and also that the sum of the coordinates of each point must be non-negative.

1. The positive number k should not appear with any negative integer, as under a $k\lambda^R$ shift, the pair will reduce to a pair with the largest number of lower modulus. Eg.: $(k, -1\ldots) \sim (0, k-1, \ldots)$.

2. $k-1$ should appear with an integer no lower than -1, as $(k-1, -2, \ldots) \sim (-1, k-2, \ldots)$.

3. $k-2$ should appear with an integer no lower than -2, and so on.

Now that we have the geometry clear, we would like to connect it with the Young tableaux counting. We outline a simple approach which assigns to each state given by 5.3 a Young tableaux, and vice versa.

5.1 The correspondence

We know that a tableaux with a single box transforms in the fundamental representation of $SU(N)$. Consider the centre of the $SU(N)$ group. This is given by $\mathbb{Z}_N$. Its elements are αI, where α is an N^{th} root of unity and I is the $N \times N$ identity matrix. Under the action of the centre, a tableaux of n boxes transforms with an eigenvalue

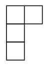
$$e^{\frac{2\pi i}{N}n}$$

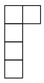
But, $\mathbb{Z}_N$ is cyclic. So, we split the tableaux into equivalence classes with n in the following classes

$$\begin{aligned}
&n \in 0, N, 2N, 3N, \ldots \\
&n \in 1, N+1, 2N+1, \ldots \\
&\vdots \\
&n \in N-1, 2N-1, 3N-1, \ldots
\end{aligned}$$

These equivalence classes are our planes that were constructed above. $m = 0$ corresponds to the first equivalence class, $m = 1$ to the next and so on. Now we construct our tableaux.We illustrate the construction via a couple of examples.

We assign the fundamental representation to the state $(1,0,0,\ldots)$. The adjoint is assigned the state $(1,-1,0,0,\ldots)$. The adjoint of $SU(N)$ looks as follows.

$(1,0,0,\ldots) \longrightarrow$  for any N. $(1,-1,0) \longrightarrow$  for $N=3$.

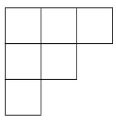
$(1,-1,0,0) \longrightarrow$  for $N=4$.

Taking cues from this, we formulate the rules of constructing a Young tableaux given an allowed state. Let an arbitrary state be an N-tuple of the form

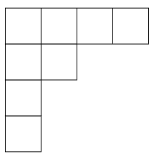
$$(-b_1,-b_2,\ldots,-b_r,0,\ldots,0,a_1,a_2,\ldots,a_p)$$

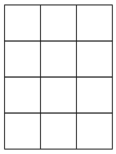
Here, a_i and b_i are positive and the numbers have been arranged in an ascending order. Consider these two sets of numbers as giving two different tableaux. The a_is give a tableaux with a_p boxes in the first row, a_{p-1} boxes in the second row until we have a_1 boxes in the last row. The a_is can have degeneracies, in which case rows of the same length will be repeated.

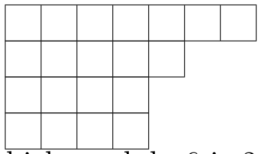
A "dual" tableaux for the b_is is created in a similar way, except for the distinction that now we start building from the bottom. Thus, the last row has length b_1, the row above has length b_2 and so on. But here is the important part. We *subtract* this tableaux from an $N \times b_1$ size tableaux and obtain the final form required for our construction. For example, this dual tableaux for the part $(-3,-3-2,-1)$ of a state for $N=5$ would look like this



Once we have obtained the tableaux constructed out of the positive and negative parts, we glue them together by placing the positive tableaux to the right of the negative tableaux. This is the full tableaux for a given state. For example, consider the state $(-3,-3,4,2,1,1)$ for $N=6$ and $k=7$. The sum of coordinates equals 2, and hence this lies on the $m=2$ plane.

The positive tableaux is 

The negative tableaux is 

Gluing these together gives . Note that the number of boxes in the tableaux is 20, which modulo 6 is 2.

Let us do a consistency check. Across the junction between the negative and positive parts, the number of column elements should decrease. But the left of the junctions contains information of the number negative numbers in the state, and likewise for the right side and positive numbers. Let n_s be the number of negative numbers in the state. Then the number of boxes in the column to the left of the junction is $N - n_s$. Let p_s be the number of positive numbers. Then

$$N - n_s \geq p_s \implies n_s + p_s \leq N$$

This is always true in a state. The equality is saturated by zeroes if required.

We need to check a few more things to be certain that the Young tableaux we are constructing are indeed allowed by the theory.

1. Each tableaux has at most $N - 1$ rows.

   - If there is at least one negative number is the state, then there is at least one box less than N in the leftmost column of the dual tableaux. This means that there will be at most $N - 1$ rows in the whole tableaux.
   - If there are no negative numbers in the state, then the state with maximum number of rows is the one with $N - 1$ ones. There can't be N ones, as this will get the sum of coordinates to be N and so this would no longer be an allowed state. It would be degenerate to a state in the $m = 0$ plane.

2. Each tableaux has at most k columns.

   - The length of first and largest row of the tableaux is equal to $|a_p| + |b_1|$ or the value of the largest positive number plus the modulus of the maximally negative number. Following the rules outlined in list above this section, the possible pairs are $(k, 0)$,

$(k-1, |-1|)$, $(k-2, |-2|)$ and so on. Every time, the pairs sum up to k. So, every tableaux can have at most k columns.

Below we show that there is a one-to one correspondence between states and Young tableaux following our construction above. This amounts to showing that every state gives rise to a Young tableaux and that every tableaux gives a unique allowed state.

5.1.1 State to tableaux

This proof follows almost tautologically from the construction. Given an allowed state

$$(-b_1, -b_2, \ldots, -b_r, 0, \ldots, 0, a_1, a_2, \ldots, a_p)$$

we can construct a Young tableaux of the specified prescription for $k \geq |b_1| + |a_p|$.

5.1.2 Tableaux to state

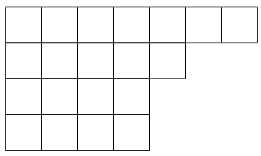
This situation can be tricky depending on the number of boxes in the tableaux. We must fix N first to remove ambiguity of tableaux, otherwise there will be a many-to-one relationship between tableaux and states. For example, the states $(1, -1, 0, 0)$ and $(2, 1, 1, 0)$ can come from the same tableaux. But if N is set to be 4, then $(2, 1, 1, 0)$ isn't allowed as the sum of its coordinates adds up to N.

To determine which state a tableaux corresponds to, it is imperative to know where to make the cut which separates the positive and negative parts.Every successive cut reduces the sum of coordinates of the corresponding state by N. This can be seen by the following simple argument. Let there be p boxes which have gone from the positive side to negative side to a shift in the cut. They all lie in the same column. Thus the positive value removed from the state is p, while the negative part added to the state value is $N - p$. Summing these numbers gives the value by which the total value of the state has decreased, N.

Now, assume that the tableaux has less than N boxes. Then any cut will make the sum of coordinates of its corresponding state go negative, which is not allowed. Thus, this kind of tableaux represents a state with only positive entries, them being the number of boxes in each row of the tableaux. Note that the number of boxes in the first row fixes a lower bound on the ks for which the tableaux may represent a meaningful state.

Now let the tableaux have more than N boxes. This means that it must represent a state with negative entries present too. So we make cuts in the tableaux starting from the left, and end at the one which gives the sum of coordinates to be between 0 and $N-1$. An example will be useful in clarifying matters.

Consider the same tableaux as before at $N = 6$ and $k \geq 7$.



1. If no cuts are made, then the state obtained is $(7, 5, 4, 4, 0, 0)$, which adds up to 20, making it lie in the $m = 2$ equivalence class. But it is not an allowed state in the theory.

2. Now make a cut between the first and second columns from the left. That would give $(6, 3, 3, 3)$ for the positive part and $(-1, -1)$ for the negative part, giving the full state as $(6, 4, 3, 3, -1, -1)$ which sums up to 14, lying in the same equivalence class as before but still not an allowed state.

3. A further cut gives the state $(5, 3, 2, 2, -2, -2)$, summing up to 8.

4. The next cut brings us to the actual state this tableaux corresponds to, i.e., $(4, 2, 1, 1, -3, -3)$. These coordinates sum up to 2 and by that virtue, this allowed state represents the tableaux.

The other superfluous states generated by the tableaux lie in the same equivalence class as the allowed one. In other words, by subtracting 1 from each entry in the state as many times as required, it can be brought inside one of the planes of the weight lattice.

Thus, we have shown that every state gives a unique Young tableaux, and every tableaux gives a unique allowed state.

In the following sections and the appendix, we construct these states by hand and the check that they indeed give the number of states expected from the tableaux counting formula.

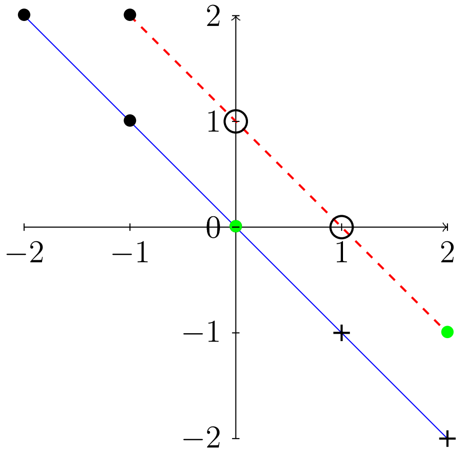


Figure 5.1: $N = 2$, $k = 3$

5.2 $N = 2$, arbitrary k

This case can be presented through some figures shown on the following pages. The blue line represents the root lattice ($m = 0$). The red dashed line is the $m = 1$ component of the weight lattice. All dots, crosses and circles represent states, with certain identifications. The black dots can be obtained from the other points by permutations and hence are not included in the final counting. On each line, points with crosses are identified, and similarly for circles. The identification will be illustrated via an example.

Take (2,-1) in figure 5.2. Permute its coordinates to obtain (-1,2), shown by a black dot. Now shift it by (4,-4) to obtain (3,-2). This completes the modding out procedure of (5.3). Thus, (2,-1) and (3,-2) are to be counted as one state. The green dots show states that go to themselves under the identification.

There is always one such zero-length cycle on the $m = 0$ line. The other lies on the same line if k is even, as then the remaining $k - 1$ states give rise to $\frac{k}{2} - 1$ pairs and the point $(\frac{k}{2}, \frac{-k}{2})$. For odd k, the $k - 1$ states apart from the origin at $m = 0$ pair up, and the zero length cycle appears on the $m = 1$ line. Thus, there are $\frac{k+1}{2}$ states on each line if k is odd. When k is even, there are $\frac{k}{2} + 1$ states at $m = 0$ and $\frac{k}{2}$ states at $m = 1$. In both cases, we get $k + 1$ states in total, as required by (5.2).

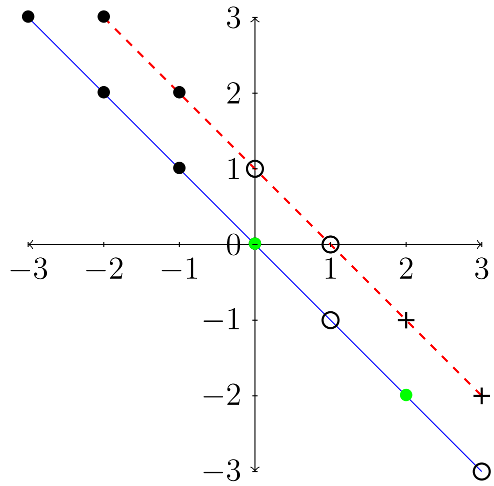


Figure 5.2: $N = 2$, $k = 4$

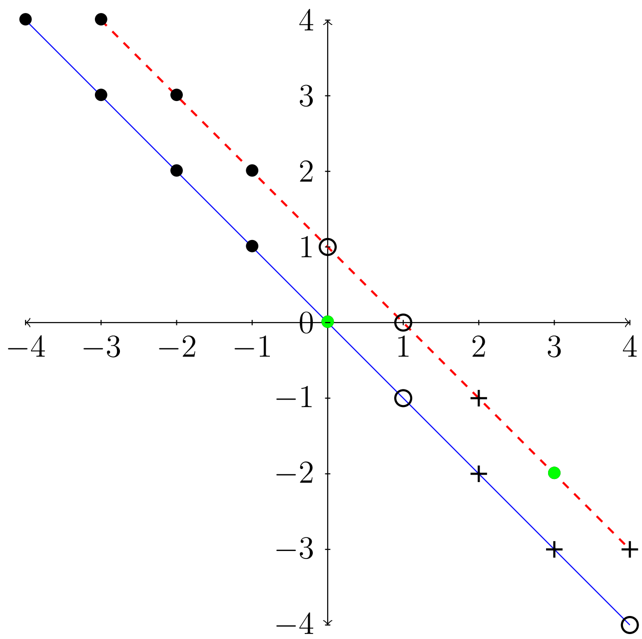


Figure 5.3: $N = 2$, $k = 5$

For $N = 3$ this analysis becomes quite cumbersome. It would be nice to see if the cycles of states under permutations follow a specific curve. Higher Ns require powers of visualisation beyond those possessed by ordinary mortals. We thus rewrite the problem as one of algebra and solve it manually for small values of N and k and provide conjectures for bigger values. We see interesting structure depending on the value of m.

5.3 $k = 2$, arbitrary N

The formulation of the problem can be done as follows. We want the sum of the coordinates of a point (which signifies a state) to be equal to a positive integer m, where m signifies the order, from the origin, of the hyperplane on which the point lies, starting from $m = 0$ for the plane passing through origin, which is also the root lattice. Let $x_i \in \mathbb{Z}$ be these coordinates.

$$\begin{aligned} \sum_{i=1}^{N} x_i &= m \\ m &= 0, 1, \ldots, N-1 \\ x_i &\sim x_i + k \end{aligned} \tag{5.5}$$

Another constraint that needs to be applied is that of invariance under permutations, given by W in equation 5.3. The Weyl group for $SU(N)$ is S_N, which is the permutation group of N elements. While applying the third of the constraints above, which imposes the quotient by $k\lambda^R$, one has to be a little careful. Every time we shift a coordinate by k, we must shift another by $-k$. This is imposed by the structure of the root lattice, the coordinates of a point on which add up to zero. States are represented by an N-tuple of coordinates x_i, subject to the conditions above.

Let us examine the situation $N = 4$.

m	States	Number of states
0	(0,0,0,0), (1,-1,0,0), (1,-1,1,-1)	3
1	(1,0,0,0), (1,1,-1,0)	2
2	(1,1,0,0), (1,1,1,-1), (2,0,0,0)	3
3	(1,1,1,0), (2,1,0,0)	2

This gives the total number of states as 10, which is what was expected from the Young tableaux formula 5.2, i.e., $\frac{N(N+1)}{2}$.

The states for $N = 5$ look as follows.

m	States	No. of states
0	(0,0,0,0,0), (1,-1,0,0,0), (1,-1,1,-1,0)	3
1	(1,0,0,0,0), (1,1,-1,0,0), (1,1,-1,1,-1)	3
2	(1,1,0,0,0), (1,1,1,-1,0), (2,0,0,0,0)	3
3	(1,1,1,0,0), (1,1,1,1,-1), (2,1,0,0,0)	3
4	(1,1,1,1,0), (2,1,1,0,0), (2,2,0,0,0)	3

Therefore, the total number of states is 15, as expected from (5.2).

Tables for $N = 6$ and $N = 7$ follow.

m	States	No. of states
0	(0,0,0,0,0,0), (1,-1,0,0,0,0), (1,-1,1,-1,0,0), (1,-1,1,-1,1,-1)	4
1	(1,0,0,0,0,0), (1,1,-1,0,0,0), (1,1,-1,1,-1,0)	3
2	(1,1,0,0,0,0), (1,1,1,-1,0,0), (1,1,1,-1,1,-1), (2,0,0,0,0,0)	4
3	(1,1,1,0,0,0), (1,1,1,1,-1,0), (2,1,0,0,0,0)	3
4	(1,1,1,1,0,0), (1,1,1,1,1,-1), (2,1,1,0,0,0), (2,2,0,0,0,0)	4
5	(1,1,1,1,1,0), (2,1,1,1,0,0), (2,2,1,0,0,0)	3

Table 5.1: $k = 2$, $N = 6$

m	States	No. of states
0	(0,0,0,0,0,0,0), (1,-1,0,0,0,0,0), (1,-1,1,-1,0,0,0), (1,-1,1,-1,1,-1,0)	4
1	(1,0,0,0,0,0,0), (1,1,-1,0,0,0,0), (1,1,-1,1,-1,0,0), (1,1,-1,1,-1,1,-1)	4
2	(1,1,0,0,0,0,0), (1,1,1,-1,0,0,0), (1,1,1,-1,1,-1,0), (2,0,0,0,0,0,0)	4
3	(1,1,1,0,0,0,0), (1,1,1,1,-1,0,0), (1,1,1,1,-1,0,0), (2,1,0,0,0,0,0)	4
4	(1,1,1,1,0,0), (1,1,1,1,1,-1), (2,1,1,0,0,0), (2,2,0,0,0,0)	4
5	(1,1,1,1,1,0,0), (1,1,1,1,1,1,-1), (2,1,1,1,0,0), (2,2,1,0,0,0)	4
6	(1,1,1,1,1,1,0), (2,1,1,1,1,0), (2,2,1,1,0,0), (2,2,2,0,0,0,0)	4

Table 5.2: $k = 2$, $N = 7$

We would like to draw the reader's attention to a pattern here. If N is even, then at each even m, we get $\frac{N}{2} + 1$ states. Any odd m has $\frac{N}{2}$ states. If N is odd, then each m has $\frac{N+1}{2}$ states. This pattern is observed in explicit computations with higher values of N, and can in fact be rigorously proven. Calculations are shown in the appendix. The key idea is to count the $1, -1$ pairs and the number of 2s, with 2 not appearing with a -1, as a (2,-1) combination is equivalent to (0,1), due a shift by 2 times a root lattice vector.

States for $k = 3$ and higher have been shown in the appendix. An interesting pattern is seen emerging in the number of states at each level m which depends on the relationship between the numerical values of N and k.

5.4 Counting without statistics

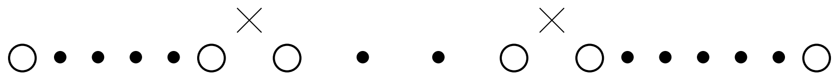
The action of modding out by the Weyl group is an imposition of statistics of the underlying states. In the case of $SU(N)$, it turns out to be bosonic, as one does not pick up a minus sign upon identifying states by permutations. If one chooses to not impose statistics on the states and hence not mod out by the Weyl group, then something remarkable happens. We get, for given values of N and k, the same number of states at each m. That number is k^{N-1}, giving the total number of states as Nk^{N-1}.

This value can be calculated from the states shown previously by taking each state and finding out its possible permutations. While doing so, it must be kept in mind that some permutations are still related to others by a $k\lambda^R$ vector. Let us illustrate this fact by using the state (1,-1,1,-1) for $k = 2$. It might be thought of as having $\frac{4!}{2!2!} = 6$ possibilities under permutations. But, any state with a different positioning of 1s and -1s can be obtained by a $k\lambda^R$ shift. For example, $(1, -1, 1, -1) \longrightarrow (-1, 1, 1, -1)$, while this new state was among the 6 possibilities counted above. Thus, in essence, $(1, -1, 1, -1)$ represents just one state even without taking permutations into account. This mixing between the operations of W and $k\lambda^R$ would explain the appearance of the semidirect product in 5.3.

Nk^{N-1} is a strange number to have for the dimension of the Hilbert space of a theory. One would expect to be able to write it as a tensor product of single particle Hilbert spaces. k^{N-1} does possess this feature. We do not have a resolution for this conundrum yet.

5.5 $U(N)$ counting

The $U(N)$ state counting proceeds in the following manner. We have k number of states for each N. And the statistics are bosonic as shown above, which means we can have any number of particles in a state. Imagine a series on N circles, and insert $k - 1$ crosses in the spaces between the circles.



Interpret the circles lying to the left of the first cross as the number of states that can be occupied by the first cross. The number of circles between the first and second crosses is the number of states that can be occupied by the second cross. Going on in this fashion, the number of circles between the $(k-1)^{th}$ cross and the end is the number of states that can be occupied by the k^{th} cross. Thus we are led to finding out the number of ways in which $k-1$ crosses can be inserted in the spaces between N circles. The answer for which is

$$\binom{N+k-1}{k-1} = \binom{N+k-1}{N} \tag{5.6}$$

This answer matches the one expected by the duality with $SU(k)_N$

We would like to see this as some sort of combination of the $SU(N)$ and $U(1)$ parts of $U(N)$. The decomposition of $U(N)$ actually occurs in the following manner

$$U(N) = \frac{SU(N) \times U(1)}{\mathbb{Z}_{\mathbb{N}}} \tag{5.7}$$

Work is underway to unravel the substructure of the $U(N)$ theory.

Chapter 6

Discussion

We have uncovered some structure of the Hilbert space of $SU(N)$ Chern-Simons theory on the torus. Apart from the usual N and k we can describe states in terms of another parameter m, which arises because of the structure of the weight lattice of $SU(N)$. We have shown that the duality holds at least at the level of the Hilbert spaces.

Our aim is to add matter to the theory and see how it interacts with these already existing states. We would like to see if a Bose-Fermi duality holds for the torus like it does for the sphere. We can check this by writing down the Schrŏdinger equation of the corresponding massive excitations on both sides of the duality.

On a mathematical note, what was attempted in chapter 5 can be seen as a genaralisation of the problem of finding out the partitions of a natural number. There exist approximation formulae for finding out the number of partitions of a natural number, most notable the Hardy-Ramanujan-Rademacher formula. We encountered the situation in which negative numbers were allowed in the partition, along with a cap on the maximum absolute value of any number in the partition. A solution of this problem would involve techniques of modular arithmetic as well.

Chapter 7

Miscellaneous

This chapter contains aspects of my reading which do not have any direct connection to the project, but are interesting nonetheless.

7.1 The Jones polynomial

Witten [2] used the partition function or path integral (Z) of Chern-Simons theory to derive the Jones polynomial invariant for knots on 3-manifolds. By following procedures of surgery and Dehn twisting, one can obtain the partition function on complicated manifolds, with and without Wilson lines, which characterise knots now. One cuts the manifold in question along a tubular neignbourhood of a Wilson line, performs a homeomorphism on this surface, and glues it back in. After a number of iterations of this kind, one can obtain S^3 or $S^2 \times S^1$, on which it is simple to do calculations.

We can calculate Z and get a wavefunction on our manifold. If a 3–manifold M is the connected sum of two 3–manifolds M_1 and M_2, joined along a 2–sphere S^2, then the following equation holds:

$$\frac{Z(M)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)} \cdot \frac{Z(M_2)}{Z(S^3)} \tag{7.1}$$

$Z(S^3)$ denotes the partition function of S^3 without any knots. The ratios appearing in 7.1 above are the knot invariants given by Jones, and 7.1 says that these invariants multiply when one takes disjoint sums of knots [2].

7.2 Connection to $1+1$ dimensional RCFT

The Hilbert space of $2+1$ dimensional Chern-Simons theory is in a one–to–one correspondence with the space of conformal blocks of 2D RCFT. This can be seen directly when the 2D manifold is $D \times R$, as discussed before. The Hilbert space of this theory furnishes representations of the Loop group LG, which arise in 2d CFT. The conserved currents of the boundary ($S^1 \times R$) Wess–Zumino–Witten theory obey the chiral Kač-Moody algebra [7]. The topology of the spatial slice Σ (which was D for the situation just described) has a role to play in how the 2D CFT will be obtained. If Σ is compact, i.e., a manifold without boundary, then $\mathscr{H}_\Sigma$ is identifiable with the space of conformal blocks of the CFT on Σ. Conformal blocks are that part of CFT correlators which is not fixed by the Ward identities. If Σ has a boundary, then $\mathscr{H}_\Sigma$ is a representation of the chiral algebra of the conformal field theory.

Thus, by considering different manifolds and boundaries, we can generate other 2D CFTs.

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Appendix A

$SU(N)$ state counting

A.1 Young tableaux formula

We know that $A(N-1,1) = N-1$. This is a tableaux with just one column of $N-1$ boxes. Let $N-1 = p$ for convenience. Thus,

$$A(p,k) = \sum_{j=1}^{p} A(j,k-1)$$

Let us find out the first few terms for the sum.

$$\begin{aligned} A(p,2) &= \sum_{j=1}^{p} A(j,1) \\ &= \sum_{j=1}^{p} j \\ &= \frac{p(p+1)}{2} \end{aligned}$$

$$\begin{aligned} A(p,3) &= \sum_{j=1}^{p} A(j,2) \\ &= \sum_{j=1}^{p} \frac{j(j+1)}{2} \\ &= \frac{p(p+1)(p+2)}{3!} \end{aligned}$$

Therefore, we get

$$A(p,k) = p + \frac{p(p+1)}{2!} + \frac{p(p+1)(p+2)}{3!} + \ldots + \frac{p(p+1)\ldots(p+k-1)}{k!}$$

This sum can be shown by induction to be

$$A(N-1,k) = \frac{(N+k-1)!}{k!(N-1)!} \tag{A.1}$$

A.2 Checks of the state counting (continued)

A.2.1 $k = 3$

Explicit computations lend further support to (5.3) by reproducing (5.2).

$N = 3$

m	States	No. of states
0	(0,0,0), (1,-1,0), (2,-1,-1), (1,1,-2)	4
1	(1,0,0), (1,1,-1), (2,-1,0)	3
2	(1,1,0), (2,0,0), (2,1,-1)	3

Formula 5.2 reduces to $\frac{N(N+1)(N+2)}{3!}$ here. The counting above matches with the answer expected.

$N = 4$

m	States	Number of states
0	(0,0,0,0), (1,-1,0,0), (1,-1,1,-1), (2,-1,-1,0), (1,1,-2,0)	5
1	(1,0,0,0), (1,1,-1,0), (2,-1,0,0), (2,-1,1,-1), (-2,1,1,1)	5
2	(1,1,0,0), (1,1,1,-1), (2,0,0,0), (2,1,-1,0), (2,2,-1,-1)	5
3	(1,1,1,0), (2,1,0,0), (2,1,1,-1), (2,2,-1,0), (3,0,0,0)	5

The counting matches as expected. Notice that we get the same number of states at each m.

$N = 5$

m	States	No. of states
0	(0,0,0,0,0), (1,-1,0,0,0), (1,-1,1,-1,0), (2,-1,-1,0,0), (2,-1,-1,1,-1), (-2,1,1,0,0), (-2,1,1,1,-1)	7
1	(1,0,0,0,0), (1,1,-1,0,0), (1,1,-1,1,-1), (2,-1,0,0,0), (2,-1,1,-1,0), (-2,1,1,1,0), (2,-1,2,-1,-1)	7
2	(1,1,0,0,0), (1,1,1,-1,0), (2,0,0,0,0), (2,1,-1,0,0), (2,1,-1,1,-1), (-2,1,1,1,1), (2,-1,2,-1,0)	7
3	(1,1,1,0,0), (1,1,1,1,-1), (2,1,0,0,0), (2,1,1,-1,0), (2,2,-1,0,0), (2,-1,2,-1,1), (3,0,0,0,0)	7
4	(1,1,1,1,0), (2,1,1,0,0), (2,1,1,1,-1), (2,2,0,0,0), (2,2,1,-1,0), (2,2,2,-1,-1), (3,1,0,0,0)	7

Again, we get the same number of states at each m. We will show one more instance of this counting, that for $N = 6$, to drive home a point.

$N = 6$

m	States	No. of states
0	(0,0,0,0,0,0), (1,-1,0,0,0,0), (1,-1,1,-1,0,0), (1,-1,1,-1,1,-1), (2,-1,-1,0,0,0), (2,-1,-1,1,-1,0), (2,-1,2,-1,-1,-1), (-2,1,1,0,0,0), (-2,1,1,1,-1,0), (-2,-2,1,1,1,1)	10
1	(1,0,0,0,0,0), (1,1,-1,0,0,0), (1,1,-1,1,-1,0), (2,-1,0,0,0,0), (2,-1,1,-1,0,0), (2,-1,1,-1,1,-1), (2,-1,2,-1,-1,0), (-2,1,1,1,0,0), (-2,1,1,1,1,-1)	9
2	(1,1,0,0,0,0), (1,1,1,-1,0,0), (1,1,1,-1,1,-1), (2,0,0,0,0,0), (2,1,-1,0,0,0), (2,1,-1,1,-1,0), (2,-1,2,-1,0,0), (2,-1,2,-1,1,-1), (-2,1,1,1,1,0)	9
3	(1,1,1,0,0,0), (1,1,1,1,-1,0), (2,1,0,0,0,0), (2,1,1,-1,0,0), (2,1,1,-1,1,-1), (2,-1,2,-1,2,-1), (2,2,-1,0,0,0), (2,2,-1,1,-1,0), (-2,1,1,1,1,1), (3,0,0,0,0,0)	10
4	(1,1,1,1,0,0), (1,1,1,1,1,-1), (2,1,1,0,0,0), (2,1,1,-1,1,0), (2,2,0,0,0,0), (2,2,1,-1,0,0), (2,2,1,-1,1,-1), (2,2,2,-1,-1,0), (3,1,0,0,0,0)	9
5	(1,1,1,1,1,0), (2,1,1,1,0,0), (2,1,1,1,1,-1), (2,2,1,0,0,0), (2,2,1,1,-1,0), (2,2,2,-1,0,0), (2,2,2,-1,1,-1), (3,1,1,0,0,0), (3,2,0,0,0,0)	9

Notice the similarity of distribution of states between $m = 3$ and $m = 6$, and also between $m = 4$ and $m = 5$. When N is a multiple of 3, we get $\frac{N(N+3)}{6} + 1$ states at all m's divisible by 3, and $\frac{N(N+3)}{6}$ states at all m's not divisible by 3. When N is not a multiple of 3, we get $\frac{(N+1)(N+2)}{3!}$ states at each m.

This pattern is observed for higher values of N as well. A calculation that tries to show the dependence of the number of states on m for a given N is shown in section A.3.2. Getting a final analytic expression has not been possible due to the presence of numerous sub-cases and due to series which are functionals of the integer-value function.

A.2.2 $k = 4$ and higher

One would naively expect the same kind of pattern to go through, with a neat dependence of the number of states at each m respecting the splitting of Ns as multiples or not of 4. We find a deeper structure. The same kind of counting as above leads to the conjecture that there is an equal number of states at each m only when N and k are *coprime*.

We are led to the coprime conjecture because of the distribution of states for $N = 4$, $N = 5$, $N = 6$, and $N = 8$ for $k = 4$. They are as follows:

N	Distribution of states
4	$10, 8, 9, 8$
5	$14, 14, 14, 14, 14$
6	$22, 20, 22, 20, 22, 20$
8	$43, 40, 42, 40, 43, 40, 42, 40$

The distribution for $N = 4$ is $10, 8, 9, 8$, while that for $N = 6$ is $22, 20, 22, 20, 22, 20$ and $N = 8$ goes as $43, 40, 42, 40, 43, 40, 42, 40$. One sees a periodicity of 4, while also seeing traces of a periodicity in 2, which is a factor common to $4, 6, 8$ and 4.

A.3 Calculations using the lattice formula

A.3.1 $k = 2$ and arbitrary N

The essence of this calculation is to count the number of (1,-1) pairs and 2s appearing in each state. The rest of the spaces will be saturated by 1s or 0s. We will reduce the calculation to cases where N is either odd or even.

Odd N

No 2s appear until $m = 2$. The number of 2s can be determined by the remainder when m is divided by 2. Let S denote the total number of states obtained. The order of appearance of a term reflects its contribution at the respective m.

$$\begin{aligned}
S = &\left(1 + \frac{N-1}{2}\right) + \left(1 + \frac{N-1}{2}\right) + \left(1 + \left(\frac{N-3}{2} + 1\right)\right) + \\
&\left(1 + \left(\frac{N-3}{2} + 1\right)\right) + \left(1 + \left(\frac{N-5}{2} + 2\right)\right) + \\
&\left(1 + \left(\frac{N-5}{2} + 2\right)\right) + \ldots + \left(\frac{N-1}{2} + 1\right) \\
=&2\left(\frac{N+1}{2}\right) + 2\left(\frac{N-1}{2} + 1\right) + 2\left(\frac{N-3}{2} + 2\right) \\
&+ 2\left(\frac{N-5}{2} + 3\right) + 2\left(\frac{N-7}{2} + 4\right) + \ldots + \left(\frac{N+1}{2}\right) \\
=&2\left(\frac{N+1}{2}\right) + 2\left(\frac{N+1}{2}\right) + \ldots \left(\frac{N-1}{2} times\right) + \left(\frac{N+1}{2}\right) \\
=&\frac{N(N+1)}{2}
\end{aligned}$$

Even N

$$\begin{aligned}
S = &\left(\frac{N}{2} + 1\right) + \left(\frac{N-2}{2} + 1\right) + \left(\left(\frac{N-2}{2} + 1\right) + 1\right) + \\
&\left(\left(\frac{N-4}{2} + 1\right) + 1\right) + \left(\left(\frac{N-4}{2} + 1\right) + 2\right) + \left(\left(\frac{N-6}{2} + 2\right) + 1\right) + \ldots \\
= &\left(\left(\frac{N}{2} + 1\right) + \left(\frac{N}{2} + 1\right) + \ldots\right)\left(\frac{N}{2} times\right) + \left(\frac{N}{2} + \frac{N}{2} + \ldots\right)\left(\frac{N}{2} times\right) \\
= &\frac{N(N+1)}{2}
\end{aligned}$$

The second step was obtained by clubbing together the odd and even numbered terms, as they respectively have the same form.

A.3.2 $k = 3$ and arbitrary N

We will present a formula for the number of states and then explain each term.

$$S = \left(1 + \left[\frac{N-m}{2}\right]\right) + \left(\left[\frac{N-m}{3}\right] + \sum_{q=1}^{\left[\frac{N-m}{3}\right]} \left[\frac{(N-q)-(m+2q)}{2}\right]\right) +$$
$$\sum_{q=1}^{\left[\frac{N+m}{3}\right]} \left(\left[\frac{(N-q)-|m-2q|}{2}\right] + 1\right) + \sum_{q=1}^{\left[\frac{m}{3}\right]} \left(\left[\frac{m-3q}{2}\right] + 1\right) \tag{A.2}$$

Here, $[h]$ represents the integer part of a real number h. The first term counts the number of states formed by just 1s and -1s. The solitary 1 stands for the case with only 1s. The other term counts the number of $(1, -1)$ pairs that appear to make up the coordinates, whilst effectively adding zero to their sum. The next big bracket signifies the presence of the set of numbers $(-2, -1, 0, 1)$ in a state. q represents the number of -2s. $N - q$ counts the effective number of coordinates left to be filled after q number of -2s have been filled, and $m - 2q$ is the remaining numerical value to be filled to make to sum of coordinates to be m. The term not under the sum counts the states with only -2s and 1s. The summed over term includes -1s as well. Limits are decided by requiring the numerator to be positive. The third big term counts states with coordinates from the set $(2, 1, 0, -1)$. The modulus function comes in as it is possible to have a 2 in states at all values of m, and not just for those the ones which satisfy $m \geqslant 2q$. The last bracket denotes involvement of 3 in a state. Once the number of 3s has been fixed, it is a matter of counting the number of 2s, which is done by the term in the integer valued function. The addition of 1 is for the purposes of counting states with only 3s and 1s.

The goal is to add up these terms and get the formula required by the Young tableaux procedure. We need to get rid of the variables q and m to achieve this purpose. However, due to the existence of mod 2 and mod 3 in the equation above, and not least due to the involvement of the integer valued function, it is difficult to get rid of m, and we are reduced to giving the answer as a bunch of cases and unsummed series.

Bracket 2

$$B_2 = \left[\frac{N-m}{3}\right] + \sum_{q=1}^{\left[\frac{N-m}{3}\right]} \left[\frac{(N-m3q)}{2}\right] \tag{A.3}$$

$N-m$ **being even**

$$B_2 = \left(\frac{N-m+2}{2}\right)\left[\frac{N-m}{3}\right] - \left(2+3+5+6+8+9+\ldots\left[\frac{N-m}{3}\right] terms\right)$$

$N-m$ **being odd**

$$B_2 = \left(\frac{N-m-1}{2}\right)\left[\frac{N-m}{3}\right] - \left(1+3+4+6+7+9+\ldots\left[\frac{N-m}{3}\right] terms\right)$$

Bracket 3

$$B_3 = \sum_{q=1}^{\left[\frac{N+m}{3}\right]}\left(\left[\frac{(N-q)-|m-2q|}{2}\right]+1\right) \tag{A.4}$$

<u>$m \geqslant 2q$</u>

$$B_3 = \sum_{q=1}^{\left[\frac{N+m}{3}\right]}\left(\left[\frac{(N-m+q}{2}\right]+1\right)$$

$N-m$ **being even**

$$B_3 = \left(\frac{N-m+2}{2}\right)\left[\frac{N+m}{3}\right] + \left(0+1+1+2+2+3+3+\ldots\left[\frac{N+m}{3}\right] terms\right)$$

$N-m$ **being odd**

$$B_3 = \left(\frac{N-m+1}{2}\right)\left[\frac{N+m}{3}\right] + \left(0+0+1+1+2+2+\ldots\left[\frac{N+m}{3}\right] terms\right)$$

<u>$m < 2q$</u>

$$B_3 = \sum_{q=1}^{\left[\frac{N+m}{3}\right]}\left(\left[\frac{(N+m-3q}{2}\right]+1\right)$$

$N+m$ **being even**

$$B_3 = \left(\frac{N+m+2}{2}\right)\left[\frac{N+m}{3}\right] - \left(2+3+5+6+8+9+\ldots\left[\frac{N+m}{3}\right] terms\right)$$

$N+m$ **being odd**

$$B_3 = \left(\frac{N+m+1}{2}\right)\left[\frac{N+m}{3}\right] - \left(1+3+4+6+7+9+10+\ldots\left[\frac{N+m}{3}\right] terms\right)$$

Bracket 4

$$B_4 = \sum_{q=1}^{\left[\frac{m}{3}\right]}\left(\left[\frac{m-3q}{2}\right]+1\right) \tag{A.5}$$

m **being even**

$$B_4 = \left(\frac{m}{2}+1\right)\left[\frac{m}{3}\right] - \left(2+3+5+6+8+9+\ldots\left[\frac{m}{3}\right] terms\right)$$

m **being odd**

$$B_4 = \left(\frac{m+1}{2}\right)\left[\frac{m}{3}\right] - \left(1+3+4+6+7+9+10+\ldots\left[\frac{m}{3}\right] terms\right)$$