Topics in Low Dimensional Topology

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by

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Certificate

This is to certify that this dissertation entitled Topics in Low Dimensional Topology towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Tanushree Shah at Ecole Normale Superieure under the supervision of Etienne Ghys, Professor, Department of Mathematics , during the academic year 2017-2018.

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This thesis is dedicated to IISER Pune mathematics department

Declaration

I hereby declare that the matter embodied in the report entitled Topics in Low Dimensional Topology are the results of the work carried out by me at the Department of Mathematics,Ecole Normale Superieure under the supervision of Etienne Ghys and the same has not been submitted elsewhere for any other degree.

Tanushree Shah

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Abstract

We construct a graph where vertices are 3-manifolds and we join two manifolds if they differ by a Morse surgery. We prove that this graph is connected and unbounded. And then we study how torus bundles are placed in this graph. Before this we look at the classification of surface homeomorphisms and geometrization of surface bundles.

Contents

Introduction

This report is about low-dimensional topology which studies 2,3, and 4 dimensional manifolds. After Perelman solved the long standing conjucture regarding geometrisation of 3-manifolds, one of the next things to study was the construction of these manifolds Ref[6]. One way is using Dehn surgery. We know that one can construct all compact 3-manifolds by doing a single surgery on a link in S^3. Hence all manifolds are just one step away from 3-sphere. So Dehn surgery doesn't tell us how different or similar two manifolds are. But we can consider it's special case called Morse surgery, which is (0,1) Dehn surgery.

In this report we will study Morse surgery and it's connection to cobordism theory. Then we will look at some concrete examples of 3-manifolds which we construct using Morse surgery. In section 4 we introduce a graph of 3-manifolds and rest of report is dedicated to study properties of this graph. It was difficult to study the geometry of this graph so we started by studying it for certain restricted class of 3-manifolds.

But before that we will need a few basics and so we will study about foliations of 3-dimensional manifolds. A foliation of a 3-manifold is a splitting into locally parallel copies of surfaces, just like a deck of cards is foliated by the individual cards. I have followed the book Foliations and Geometry of 3-manifold by Danny Calegari. First we will focus on automorphisms of circles, surfaces and embeddings of surface in 3-manifolds. Then we will study the classification of surface homeomorphism as given by Thurston, for which one needs to know the model geometries in dimension 2 and 3 as well as measured geodesic laminations on surfaces. Laminations are a restricted form of foliations on a closed subsets of the surface. This leads to the geometrization of surface bundles. We will then study minimal surfaces and in particular their existence and compactness theorems [ref 9].

Chapter 1

Surface Bundles

This chapter is recapulation of ideas given in Danny Calegari's book Foliations and the geometry of 3-manifolds. Once we had the classification theorem for surfaces, the next thing to look at was their automorphisms and geometric structure on the mapping tori. In this section we will study Thurston's theory for the same. All surfaces in this section are orientable and closed. Let S be such a surface and let Homeo(S) denote the group of self-homeomorphisms of the surface.

Definition 1.0.1. *Let X, Y be topological spaces and let $Map(X, Y)$ denote the set of continuous maps from X to Y. The compact-open topology on $Map(X, Y)$ is the topology generated by open sets of the form $U_{K,U} = \phi \in Map(X, Y)|\phi(K) \in U$ where $K \in X$ is compact and $U \in Y$ is open.*

Using this as subspace topology we can consider Homeo(S) to be a topological group that is, a topological space for which group multiplication and inverse are continuous maps. The reason why we use this particular topology is because it gives us the following fact. ForX, Y, Z Hausdorff topological spaces and say Map(X, Y) has the compact-open topology. Let Y be locally compact. Then a map $f : X \to Map(Y, Z)$ is continuous if and only if the associated map $F : X \times Y \to Z$ defined by

$$F(x, y) = f(x)(y)$$

is continuous.

Complicated surfaces can be cut open along suitable embedded arcs and loops to simpler ones. These special arcs and loops are called essential ones.

Definition 1.0.1. *An embedded loop is essential if it does not bound a disk or cobound an annulus together with a component of boundary of surface. An embedded arc is essential if there is no other arc such that thses two arc bound a disk.*

If α and β are essential loops in S which intersect transversely, after a small perturbation. Some intersections might be fake a bigon is a properly embedded disk whose interior is disjoint from these loops, and whose boundary consists of two arcs, one in each loop. By isotoping either across disk, one may eliminate at least two points of intersection of α and β. After finitely many such isotopies, we can remove all such bigons and hence we say these loops intersect efficiently. Similarly, if α is an essential loop and β is an essential arc, we say they intersect efficiently if they do not cobound a bigon. If α and β are essential arcs, a semi-bigon is a properly embedded disk whose interior is disjoint from union of α and β and whose boundary consists of three arcs, one in α , one in β , and a third in boundary of S. Again, by proper isotopy, one may eliminate at least one point of intersection.

Let $\text{Homeo}_0(S)$ be the path component of identity in $\text{Homeo}(S)$. Using the fact above we can consider $\text{Homeo}_0(S)$ as the subgroup of $\text{Homeo}(S)$ consisting of all maps isotopic to identity.

1.0.1 Mapping class group

One can check that the path component of identity is normal subgroup in any topological group. Hence one can define the following quotient group

$$MCG(S) = Homeo(S)/Homeo_0(S).$$

This is called the mapping class group of S. Orientation preserving homeomorphism form a subgroup of MCG(S) which we denote by $\text{MCG}^+(S)$. For $\phi \in MCG(S)$ let ϕ_* be the induced outer automorphism of $\pi_1(S)$. One way to study MCG(S) is by Dehn twists.

Definition 1.0.2. *Let $\gamma \subset S$ be an oriented simple closed curve. Parametrize γ by S^1 and let $A = S^1 \times [0,1]$ be parametrized regular neighbourhood of γ. A Dehn twist in γ, τ_γ, is the* *equivalence class in MCG(S), represented by a homeomorphism supported on A, which in (θ, t) coordinates is given by*

$$\tau_{\gamma(\theta,t)} = (\theta - 2\pi t, t).$$

The equivalence class of τ_γ in MCG(S) depends only on the isotopy class of γ and is trivial unless γ is essential. For α, β two essential simle closed curves, there is an identity

$$\tau_\beta \tau_\alpha \tau_\beta^{-1} = \tau_{\tau\beta(\alpha)}.$$

Using this identity Dehn twist along complicated curves can be expressed as product of Dehn twistof simple curves. We can check that if an element of $MCG(S)$ which leaves all the essential simple closed curve invariant upto isotopy class then it is identity. Using these two facts we can show that MCG(S) is generated by Dehn twists in finitely many essential loops.

1.0.2 Geometric structure on manifolds

A model geometry (G, X) is a manifold X together with a Lie group G of diffeomorphisms of X, so that the following are true

1. X is connected and simply connected
2. G acts transitively on X with compact point stabilizers
3. G is maximal with respect to these properties

A (G, X)-structure on a closed topological manifold M is a homeomorphism $\phi : M \to X/\Gamma$, where Γ is discrete, free, cocompact , properly discontinuous subgroup of G.

In dimension 2 there are three geometries satisfying these conditions modulo scaling, namely spherical, Euclidean and hyperbolic. In third dimension there are more possibilities which are classified in terms of dimension of the point stabilizers in Isom(X). Since these point stabilizers are isomorphic to closed subgroups of $O(3)$ their dimension is either 3 or 1 or 0. The classifiction is done by Thurston and I am listing it below:

1. Spaces whose point stabilizer has dimension 3 are $S^3, \mathbb{E}^3, \mathbb{R}^3$. These are spaces of constant curvature.

2. Spaces with 1 dimensional point stabilizers are product spaces $S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ and twisted product spaces Nil, $\widetilde{SL(2, \mathbb{R})}$.

3. The space with 0 dimensional point stabilizers is solv geometry sol.

1.1 Automorphisms of tori

Let T be the standard 2-dimensional torus. Then $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$. Since this is abelian group $Out(\pi_1(T)) = Aut(\pi_1(T))$. An automorphism of $\mathbb{Z} \times \mathbb{Z}$ is determined by it's action on basis elements. Hence it can be seen as a 2×2 integral matrix. Hence $Out(\pi_1(T)) = Aut(\pi_1(T)) = GL(2, \mathbb{Z})$ and $\text{MCG}^+(T) = SL(2, \mathbb{Z})$.

Theorem 1.1.1. *Let T be a torus and let $\Phi \in Homeo^+(T)$. Then one of the following three holds:*

1. *Φ is **peroidic** i.e. $\Phi^n \simeq Id$.*

2. *Φ is **reducible** i.e. there is some simple closed curve γ in T such that $\Phi(\gamma) = \gamma$, upto isotopy.*

3. *The linear representative of Φ is **Anosov**, i.e. $\phi(F^+) = \lambda F^+$ and $\phi(F^-) = \lambda^{-1} F^-$*

Proof. Let λ and λ^{-1} be eigenvalues of ϕ, where ϕ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence

$$tr(\phi) = a + d = \lambda + \lambda^{-1}.$$

If λ and λ^{-1} are not real then they have to lie on unit circle since trace is real. Hence $|tr(\phi)| < 2$. Since a and d are integers then $|tr(\phi)| = 0$ or $|tr(\phi)| = \pm 1$. In first case order of ϕ is 4 and for second case order is 6. Specifically order of ϕ is finite. If eigenvalues are 1 then $\phi = Id$ or it is conjugate to matrix of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. This fixes the vector $(1, 0)$, which preserves the isotopy class of one of the loops which generates $\pi_1(T)$. Hence such a ϕ is reducible. If eigenvalues are -1, then $\phi = -Id$ or is conjugate to a transformation whihc takes $(1,0)$ to it's inverse, which is also reducible. If the eigenvalues are real such that $\lambda > 1 > \lambda^{-1}$, and let $e^{\pm}$ be the eigenvectors. Let $F^{\pm}$ be the linear foliations of T by lines which are parallel to $e^{\pm}$. Then ϕ takes leaves of $F^{\pm}$ to itself, streching the leaves of F^{+} by factor of λ and streching the other set of foliation by factor of λ^{-1}. These foliations will be perpedndicular for some choice of Euclidean structure on T. □

1.1.1 Geometric structures on mapping tori

For a homeomorphism $\phi : T \to T$, one can form mapping torus M_ϕ, which is

$$M_\phi = T \times I/(s,1) \sim (\phi(s),0).$$

Let us study relationship between geometry of mapping torus and dynamics of ϕ.

Theorem 1.1.2. *Let $\phi : T \to T$ be a homeomorphism of the torus. Then the mapping torus M_ϕ satisfies the following*

1. *If Φ is **peroidic**, M_ϕ admits an $\mathbb{E}^3$ geometry*
2. *If Φ is **reducible**, M_ϕ contains a reducing torus or Klien bottle*
3. *If Φ is **Anosov**, M_ϕ admits a Sol geometry*

Proof. If ϕ is periodic then it has order 2,3,4, or 6. Then ϕ preserves either square or hexagonal Euclidean metric on torus. Hence gluing map can be seen as isometry of $T \times I$, so that mapping torus has Euclidean structure. If ϕ is reducible, then it preserves a simple closed curve γ. Then $\gamma \times I$ gives a closed π_1 injective torus or Klein bottle when it glues under ϕ. If ϕ is Anosov with invariant foliation $F^{\pm}$, then it extends linearly to an automorphism of $\mathbb{R} \times \mathbb{R}$ which is conjugate to diagonal automorphism. Let Sol be 3- dimensional solvable Lie group which is extension of abelian groups

$$0 \to \mathbb{R}^2 \to Sol \to \mathbb{R} \to 0.$$

Here the generator of $\mathbb{R}$, t, acts on $\mathbb{R}^2$ by the following matrix

$$t^{-1}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hence fundamental group is the extension

$$0 \to \mathbb{Z}^2 \to \pi_1(M_\phi) \to \mathbb{Z} \to 0.$$

This short exact sequence includes in the short exact sequence defining Sol in such a way that the generators of $\mathbb{R}^2$ become the eigenvectors of the automorphism ϕ which sits in $\mathbb{R}$ as $\phi \to log(\lambda)$. Hence $\pi_1(M_\phi)$ isa lattice in Sol which induces Sol structure on M_ϕ. □

1.2 Coarse geometry

Coarse geometry is the study of metric spaces from a 'large scale' view point, so that two space which 'look the same from a great distance' are actually equivalent. This equivalence is called quasi-isometry.

Definition 1.2.1. *(Slim triangle) Let $\delta > 0$. A geodesic triangle in a metric space is said to be δ-slim if each of its sides is contained in the δ-neighbourhood of the union of the other two sides. A geodesic space X is said to be δ hypebolic if every triangle in X is δ-slim.*

Definition 1.2.2. *(Quasi-Isometry) Let (X_1, d_1) and (X_2, d_2) be metric spaces. A map $f : X_1 \to X_2$ is called a (λ, ϵ)-quasi-isometric embedding if there exists constants $\lambda \geq 1$ and $\epsilon \geq 0$ such that $\forall x, y \in X_1$*

$$\frac{1}{\lambda} d_1(x, y) - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \epsilon$$

Definition 1.2.3. *(Quasi geodesics) A (λ, ϵ)-quasi-geodesic in a metric space X is a $(\lambda\ ,\epsilon)$-quasi-isometric embedding $c : I \to X$, where I is an interval of the real line (bounded or unbounded).*

Theorem 1.2.1. *For all $\delta > 0, \lambda \geq 1, \epsilon \geq 0$ there exists a constant R with the following property: If X is a δ- hyperbolic geodesic space, c is a(λ, ϵ)-quasi-geodesic in X and [p,q] is a geodesic segment joining the endpoints of c, then the Hausdorff distance between [p,q] and the image of c is less than R.*

Definition 1.2.4. *Let X be metric space and let $p \in X$. The Gromov product of $q, r \in X$ with respect to x is defined to be*

$$(q.r)_p = \frac{1}{2}d(q,p) + d(r,p) - d(q,r)$$

Definition 1.2.5. *Let $\delta \geq 0$. A metric space X is said to be hyperbolic if*

$$(p.q)_r \geq min((p.s)_r, (q.s)_r) - \delta$$

for all p,q,r,s $\in X$

Theorem 1.2.2. *Let X be a geodesic. Definition* 1.2.1 *and definition* 1.2.5 *are equivalent.*

Definition 1.2.6. *Let X be δ-hyperbolic space.Let $p \in X$. A sequence (x_n) in X converges at infinity if $(x_i.x_j)_p \to \infty as$. Two sequences (x_n) and (y_n) are said to be equivalent if $(x_i.y_j)_p \to \infty$ as i,j$\to \infty$. Set of equivalence class of all sequences is called boundary of X.*

Definition 1.2.7. *Let X be hyperbolic space with base point p.*

$$(x.y)_p = supliminf_{i,j\to\infty}(x_1.y_j)_p$$

where supremum is taken over all sequences $(x_i)and(y_j)$ in X such that $x = lim(x_i)$ and $y = lim(y_j)$

Definition 1.2.8. *Let X be hyperbolic space with base point p. A metric d on boundary is called visual metric with parameter a if there exist constants $k_1, k_2 > 0$ such that*

$$k_1 a^{-(\xi.\eta)_p} \leq d(\xi,\eta) \leq k_2 a^{-(\xi.\eta)_p}$$

for all $\xi, \eta \in X$

Let X be hyperbolic space with base point p. Let $\epsilon > 0$

$$\rho_\epsilon(\xi,\eta) = e^{-\epsilon(\xi.\eta)_p}$$

$$d_\epsilon(\xi,\eta) = inf\Sigma\rho_\epsilon(\xi_{i-1},\xi_i)$$

where infimum is taken over all chains $(\xi = \xi_0, ..., \xi_n = \eta)$, no bounds on n.

Theorem 1.2.3. *Let X be (δ)hyperbolic space.Let $\epsilon > 0$ and let $\varepsilon = e^{2\delta\epsilon}$ - 1. If $0 < \varepsilon \leq \sqrt{2} - 1$, then d_ϵ is visual metric on boundary of X and*

$$(1 - 2\varepsilon)\rho_\epsilon(\xi, \eta) \leq d_\epsilon(\xi, \eta) \leq \rho_\epsilon(\xi, \eta)$$

for all $\xi, \eta \in$ boundary of X.

1.3 Geodesic lamination

To study action of an automorphism ϕ of highre genus surface S we need to find an essential 1 dimensional object which is preserved under ϕ upto some equivalence relation. We will study three such objects namely, geodesic lamination, train tracks and singular foliation. Let S be a surface with $\chi(S) < 0$. Then by uniformisation theorem there is a hyperbolic structure on S in every conformal class of metric, which is complete.

Definition 1.3.1. *For S as above, we define $\tau(S)$, Teichmuller space of S is the set of equivalence classes of pairs $(f, \sum)$ where $\sum$ is a hyperbolic surface and $f : S \to \sum$ is orienttion preserving homeomorphism. Also $(f_1, \sum_1) \sim (f_2, \sum_2)$ if and only if there is an isometry $i : \sum_1 \to \sum_2$ for which the composition $i \circ f_1$ is homotopic to f_2.*

We can give a topology on $\tau(S)$ by defining $i : (f_i \sum_i) \to (f, \sum)$ if there is a sequence of $1 + \epsilon_i$ bilipschitz maps $j_i : \sum_i \to \sum$ such that $j_i \circ f_i$ is homotopic to f, where ϵ_i tends to 0. With respect to this topology $\tau(S)$ is homeomorphic to open ball of dimension $6g - 6$.

Geodesic lamination is natural generalisation of the concept of simple closed geodesic.

Definition 1.3.2. *For a hyperbolic surface $\sum$, a geodesic lamination Λ is a union of disjoint embedded geodesics which is closed as a subset of $\sum$. The leaves of λ are the geodesics making up Λ.*

A geodesic lamination is minimal if every leaf is dense in Λ. A simple closed geodesic is a minimal geodesic lamination. A geodesic lamination is full if complementry regions in the surface are all finite sided ideal polygons. If a geodesic lamination is not full then a boundary curve of a tubular neighbourhood of Λ is essential. Also one can check that any two full geodesic laminations have nonempty intersection.

Theorem 1.3.1. *Let $\Phi \in MCG(\sum)$. Then one of the following three possibilities must hold*

1. *Φ has finite order in $MCG(\sum)$*
2. *Φ_* permutes some finite disjoint collection of simple geodesics $\gamma_1, ..., \gamma_n$*
3. *Φ_* preserves a full minimal geodesic lamination.*

Proof. Say ϕ does not finite order in $MCG(\sum)$, then there exists some curve γ such that it's iterates $\gamma_i = \phi_*^i(\gamma)$ do not form periodic sequence. For a fixed hyperbolic structure on $\sum$ and number of simple closed geodesics whose length is less than some constant T is finite. Length of γ_i increases without bound and for a fixed n we know that

$$\#\{\gamma_i \cap \gamma_{i+n}\} = K_n < \infty.$$

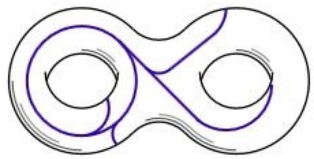
Since the set of geodesic lamination is $\sum$ is compact we will have a subsequence n_i such that $\gamma_{n_i} \to \lambda'$ in Hausdorff topology. Let λ be minimal sublamination of λ'. Following these two results for sufficiently large i the γ_{n_i} contains long segments which spiral around tubular neighbourhood of λ'. If the intersection of $\phi_*^n(\lambda')$ and λ' is transverse for some n, then γ_{n_i} and γ_{n_i+n} will have arbitrarily many points of intersection. This contradicts our earlier estimate and hence the intersection above cannot be transverse. One can check that $\phi_*^i(\lambda) = \lambda$ for some $i \leq 3g - 3$. Hence this case is done. Hence without loss of generality we assume that $\lambda$is not simple closed geodesic. Since it is minimal none of it's leaf is isolated. Hence set of points where λ and $\phi_*(\lambda)$ cross transversely has no isolated points. This set is uncountable if it is nonempty. If so then cardinality of $\gamma_{n_i} \cap \gamma_{n_i} + 1$ is unbounded as $i \to \infty$ which is a contradiction. This contradiction implies that none of the intersection is transverse. Hence there is some $i < 3g - 3$ such that $\phi_*^i(\lambda) = \lambda$. If λ is not full then some boundary curve of tubular neighbourhood of λ is essential in $\sum$. By construction it is periodic and disjoint from it's translates. Otherwise λ is full and hence $\phi_*(\lambda)$ is equal to λ. □

1.4 Train track

Train tracks is very useful tool. It reduces the studyof mapping class group to combinatorics and linear algebra.

Definition 1.4.1. *A train track τ is a finite embedded C^1 graph in a surface with a well-defined tangent space at each vertex.*

Given below is an example of train track on genus 2 surface (Ref[9]).



If we orient the tangent space locally then we can distinguish between rays coming in and rays going out. But this global orientation might not extend to global orientation. Since train track is C^1, it has a well-defined normal bundle and a regular neighbourhood $N(\tau)$ of it in the surface can be foliated by intervals which are transverse to train track.

Definition 1.4.2. *A train track τ is said to carry τ' if τ' can be isotoped in such a way that at the end of the isotopy, τ' is transverse to the interval in this I-bundle on $N(\tau)$. We write $\tau \preceq \tau'$ in this case.*

We can define a map from $N(\tau)$ to τ by collapsing each fibre of the I-bundle structure. If τ carries τ', then after isotopy and projection, we get a map from τ' to τ which is an immersion with respect to the C^1 structure. This is the carrying map.

Illustrated below are two fundamental operations on train track, called splitting and shifting respectively Ref[9].

Definition 1.4.3. *τ is recurrent if for every edge e of τ there is a simple closed curve $c \in S$ which is carried by τ such that, e is contained in the image of c, under the carrying map.*

Let α be union of disjoint simple closed curve in S which is carried by τ. We can define a map from edges of τ to non-negative integers by sending each edge to the number of pre-images in α under the carrying map. This way α determines weight w_α on each edge. Sum of wieghts on incoming edges is equal to sum on outgoing ones with respect to choice of local orientation. This is called switch condition at each vertex. Given a train track and wieghts on each edge which satisfy switch condition we can construct multicurve carried by τ. For each edge e place $w(e)$ number of parallel copies of e transverse to the normal foliation. Glue the ends of these intervals together at each vertex. This results in an embedded 1-manifold because of the switch condition.

1.5 Singular foliation

A foliation of a 3-manifold is a splitting into locally parallel copies of surfaces, just like a deck of cards is foliated by the individual cards. Formally, a p-dimension foliation on n-dimensional Manifold is a covering by charts and maps from each chart to $\mathbb{R}^n$ such that on the intersection these maps overlap. A singular foliation F is foliation on a manifold except on finitly many points called singularities. Open neighbourhood of the singularity is such that the leaves of F look like the level sets in $\mathbb{C}$ of the function Im($z^{n_i/2}$)= constant for some natural number $n_i > 2$ andchoose co-ordinates so that the singular pointis at 0. Exept for singularities the surface is covered by product charts U_i such that leaves of $F \cap U_i$ are taken to factors $point \times I$.

Singular foliation and geodesic lamination are related. Let F be a singular foliation on hyperbolic surface S. Then each non-singular leaf of F is isotopic to a unique embedded geodesic representative. The closure of these geodesics give us geodesic lamination.

1.5.1 Thurston's classification of homeomorphism of orientable surfaces of genus ≥ 2

A map $\phi \in MCG(S)$ is pseudo-Anosov if there are transverse pair of transversely measured singular foliations $F^{\pm}, \mu^{\pm}$ of S such that$\mu^{\pm}$ have no atoms and full support and there exists a real number $\lambda > 1 > lambda^{-1}$ and ϕ takes leaves of F^+ to leaves of F^+ such that μ^+

length of curves are multiplied by λ and μ^- length are multiplied by λ^{-1}. Similarly for F^- as well.

Theorem 1.5.1. *Let $\Phi \in Homeo^+(\sum)$. Then one of the following three possibilities must hold*

1. *Φ is **peroidic**, i.e.$\Phi^n \simeq Id$*
2. *Φ is **reducible**, i.e.there is some finite disjoint collection of disjoint essential simple closed curve in $\sum$ which are permuted up to isotopy by Φ*
3. *Φ is **pseudo-Anosov**, i.e. some $\Psi \simeq \Phi$ acts on $\sum$ by a pseudo-Anosov automorphism.*

Proof. Say ϕ has finite order then it preserves some hyperbolic metric on S upto isotopy. Hence it is isotopic to an isometry hyperbolic surface $\sum$. Such isometry has finite order in $Homeo(\sum)$. Say ϕ has infinite order then it either preserves some multicurve upto isotopy such that the corresponding loop can be homotoped out of any compact region or it preserves a transverse pair of singular foliations $F^{\pm}$ and it preserves the projective class of invariant transverse measures $\mu^{\pm}$. For second case one can exhibit some representative of the isotopy class of ϕ as pseudo-Anosov map. □

One can form mapping torus for higher genus surface as we do for torus

$$M_\phi = \sum \times I/(s,1) \sim (\phi(s),0).$$

We have a correspondance between geometric structure on mapping torus and dynamics of ϕ. Since the proof is similar to previous one we state it here without proof.

Theorem 1.5.2. *Let $\phi : \sum \to \sum$ be a homeomorphism of the surface. Then the mapping torus M_ϕ satisfies the following*

1. *If Φ is **peroidic**, M_ϕ admits an $\mathbb{H}^2 \times \mathbb{R}$ geometry*
2. *If Φ is **reducible**, M_ϕ contains a essential tori or Klien bottles*
3. *If Φ is **pseudo-Anosov**, M_ϕ admits a $\mathbb{H}^3$ geometry.*

Chapter 2

Minimal surfaces

We will study the basics of minimal surfaces which will lead to existence and compactness theorems. Just like we have monotonicity in the theory of groups of homeomorphisms of 1-manifold, we have monotonicity properties of codimension one minimal surfaces like barrier surfaces, maximum principle etc. We will first review the theory of Riemannian geometry.

Definition 2.0.1. *Let M be a smooth manifold. A connection on M is a linear map*

$$\nabla : \Gamma(TM) \otimes_{(R)} \Gamma(TM) \to \Gamma(TM)$$

which is given by

$$\nabla_X(Y) = \nabla(X, Y) \in \Gamma(TM)$$

where X, Y are vector fields on M satisfying the following properties:

1.∇ is a C^∞-linear(i.e.tensorial) in the first factor

$$\nabla_{fX}Y = f\nabla_X Y$$

for all smooth functions f on M.

2.∇ satisfies a Leibniz rule with respect to the second factor that is

$$\nabla_X fY = X(f)Y + f\nabla_X Y.$$

The value of $\nabla_X Y$ at a point depends on the value of X at that point and on the germ of Y along some smooth path such that the tangent space at the starting point is X. Using theory of ODEs for this path we can see that any vector $v \in T_{c(0)}M$ there is a unique vector field Y along the path c such that $Y(0) = v$ and $\nabla_{c'} Y \equiv 0$. This Y is parallel transport of v along c. This parallel transport gives a linear map

$$P_c : T_{c(0)}M \to T_{c(1)}M$$

given by

$$P_c(Y(0)) = Y(1)$$

where Y is a vector field.

2.0.2 Minimal surfaces in $\mathbb{R}^3$

For a smooth surface $S \subset \mathbb{R}^3$ the Gauss map takes each point p to its unit normal in the unit sphere S^2. One can think of S and S^2 as Riemann surfaces corresponding to their conformal structure which they inherit from $\mathbb{R}^3$. Hence for S to be minimal is equivalent to saying that Gauss map is holomorphic or antiholomorphic.

Existence theorems

There is a lot of literature about existence of minimal surfaces in 3-manifolds. I will be stating a few theorems here without proof.

Theorem 2.0.3. *(Douglas, Rado) For a Jordan curve* Γ *in* $\mathbb{R}^3$ *there exists a simply-connected immersed minimal surface bounded by* Γ.

For a more general immersed surface, we have

Theorem 2.0.4. *(Schoen-Yau) For a compact Riemannian manifold M, and a surface S of genus≥1, let f:S→M be a continuous map. If map induced by f on* π_1 *is injective tehn there is minimal immersion* $h : S \to M$ *such that map induced by h on* π_1 *is same as f. We can choose h to be homotopic to f if* $\pi_2(M) = 0$.

We have to be a little more careful for sphere. We have the following theorem in homotopy category.

Theorem 2.0.5. *(Sacks-Uhlenbeck,Meeks-Yau) For a closed 3-manifold M, where $\pi_2(M)$ is nonzero, there exists a map of f which has least area amongst the set of all maps from S^2 to M representing nontrivialelements of $\pi_2(M)$. Further f is smooth embedding or a double cover of a smoothly embedded projective plane.*

We can construct minimal surfaces via geometric measuretheory as well.

Theorem 2.0.6. *(Meeks-Simon-Yau) Let M be a closed orientable irreducible 3-manifold. Then every incompressible surface S is isotopic to a globally least area minimal surface.*

We have similar statement for embedded spheres.

Theorem 2.0.7. *(Meeks-Simon-Yau) For a closed orientable reducible 3-manifold M there is a globally least area essential embedded sphere.*

compactness theorem

For this section M will denote compact Riemannian 3-manifold. Let S_i be a sequence of embedded minimal surfaces in M. Say we have global bound $area(S_i) \leq C_1$ and pointwise bound $|A_{S_i}|^2 \leq C_2$ for constants C_1 and C_2. There is a uniform upper bound on the sectional curvature of M because M is compact. Hence S_i have two sided curvature bounds. Let $p \in M$ be accumulation point of S_i and $v \in T_p(M)$ be limit point of normal vectors to the S_i. Then for a ball B whose radius depends on C_2 and curvature of M and the injectivity radius of M at p, there is a subsequence S_i' and the local coordinates on B in a way that suitable sheets of $S_i' \cap B$ can be realised as a family of graphs of functions over a fixed planar domain which has uniform bounds on the first and second derivative. By the Arzela-Ascoli theorem we can go to further subsequence and check that the surfaces converge locally to a C^1 limiting surface S. Using Elliptic regularities we know that higher derivatives of S are determined by first derivative. Hence S is C^∞ and so is the convergence. Therefore S is a minimal surface. Let total curvature of S to be $\int_S |A_S|^2$.

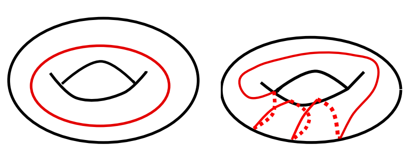
Theorem 2.0.8. *(Choi-Schoen) For a 3-manifold M let $S_i \subset M$ be a sequence of complete embedded minimal surfaces of genus g such that $area(S_i) \leq C_1$ and $\int_S |A_S|^2 \leq C_2$. Then* *there exists a finite set of points $P \subset M$ and a uniformly converging subsequence S_1' in the C^l topology for $l < \infty$ on compact subsets of M-P to a minimal surface S. Furthermore S is smooth in M and has genus atmost g and satisfys same bounds as above for area and total curvature.*

Chapter 3

Morse surgery on 3-manifolds

3.1 Morse Surgery

In this section result about connctedness and unboundedness and theorem 3.5.4 are not published anywhere before. Theorem 3.5.1 and the three proposition following it are proved by Poincare. Morse surgery is one way to create new manifolds from old ones. Let V be a smooth 3-manifold without boundary. Let S^1_{emb} be an embedded circle in V. Then look at some tubular neighbourhood T in V which decomposes in direct product $T = S^1 \times \mathbb{D}$, since V is orientable. We define meridian as an essential curve in the boundary torus which bounds a disk in this solid torus. Longitude is a curve on the torus that does not bound a disk in the solid torus and algebraic intersection number with the meridian is exactly one. This is not well defined as seen in figure below.



Now we remove the tubular neighbourhood of the embedded circle. Then identify the boundary of $V \setminus int(T)$ and boundary of $S^1 \times \mathbb{D}^2$ by a diffeomorphism which takes meridian of one torus to longitude of the other torus.

There is another way to look at Morse surgery which comes from Morse theory. Take a 4-Manifold and a Morse function f and say c is a critical value with exactly one critical point in its pre-image. If the index of this critical point is 2 then the level set $f^{-1}(c-\epsilon)$ is our 3-manifold before surgery and $f^{-1}(c+\epsilon)$ is our 3-manifold after surgery. This means that there is a cobordism between these two 3-manifolds. As a fact any two cobordant 3-manifolds can be obtained from one another using a series of Morse surgery.

To understand this concept better let us look at a set of spaces which one can construct using Morse surgery from 3-sphere.

3.2 Lens Spaces

Lens space $L(p,q)$, for p and q coprime can be constructed by an action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 and taking it's quotient. This identification is given by (Z_1, Z_2) identified to $(e^{2\pi i/p}.Z_1, e^{2\pi iq/p}.Z_2)$. The Lens space $L(p,q)$ has genus 1 Heegaard decomposition, $L(p,q) = S^1 \times \mathbb{D}^2 \cup_A S^1 \times \mathbb{D}^2$ fro $A \in SL(2,\mathbb{Z})$. This A takes meridian of one boundary torus to p,q curve on the other boundary torus. Here by p,q curve we mean a curve which wraps around the meridian p times and q times around longitude. To construct $L(p,q)$ spaces using Morse surgery we need to know a bit more about these p,q curves or in other words torus knots.

3.2.1 Torus knot

The torus knot $T_{p,q}$ of type p,q is the knot which wraps around the solid torus T in the longitudinal direction p times and q times around the meridian. Since S^3 can be broken into two solid tori identified to each other through their boundary torus, we can break the complement of tori knot in S^3 in similar way. Then using Van Kampen theorem one can see that fundamental group of the complement of the knot $T_{p,q}$ is $< x, y \mid x^p = y^q >$

Now observe that the $T_{p,\pm 1}$ and $T_{\pm 1,q}$ are trivial knots. Fundamental group of trivial knot is $\mathbb{Z}$. Hence any other torus knot is not trivial knot. Now we know the form of all the trivial torus knots.

Now observe that the type of $T_{p,q}$ is unchanged by changing the sign of p or q or by interchanging p and q upto orientation. And by theorem of Schreier which says if $1 < p < q$ then the fundamental group determines the pair p, q. To see the proof see ref[1]. This gives complete description of $T_{p,q}$ in terms of it's fundamental group.

Now coming back to constructing Lens spaces. Since $T_{p,\pm 1}$ and $T_{\pm 1,q}$ are trivial knots, their compliment in S^3 is a solid torus. So to construct $L(p, 1)$ consider the knot $T_{p,1}$ in S^3. Consider it's tubular neighbourhood and remove it. Now consider this $p, 1$ curve as the longitude on the removed solid torus and map it to the meridian of the boundary torus of S^3. This gives us the Morse surgery construction of $L(p, 1)$. Similarly one can construct $L(1, q)$.

3.3 Graph of 3-manifolds

Sometimes it is interesting to look at 3-manifolds as a collection rather than looking at them individually. Let us consider the graph in which each vertex is a closed orientable 3-manifold up to oriented homeomorphism and we join two manifolds if they differ by one Morse surgery. We can make this a metric graph by declaring that each edge has length one.

This graph has countably infinite vertices which is an obvious fact. This graph is connected. As we noted above, this is just another proof of the fact that all 3-manifolds are cobordant. To see this we need to know another type of surgery, called Dehn surgery. Using this we can go from one 3-manifold to another. Consider a 3-manifold and a simple closed curve in it. Consider a tubular neighbourhood around that curve. Remove this tubular neighbourhood and stich it back using some diffeomorphism of the boundary torus. To define this diffeomorphism it is enough to define where the meridian goes say, $meridian \mapsto a \times meridian' + b \times longitude$. Then the surgery coefficient is defined as b/a. The fundamental theorem of surgery on 3-manifolds says that every closed, orientable 3-manifold can be obtained from S^3 via a surgery on a link in S^3. Morever we can find a surgery presentation where each component of link is unknotted and has surgery coefficient ± 1.

The distance between two manifolds in this graph signifyies how similar the two manifolds are in terms of their topology. Hence if two manifolds are near each other on this graph then some of their topological invariants are similar. Let us look at their first homology groups as an example. One way to look at Morse surgery is that we take a manifold M and remove a solid torus to get M' and stich back the solid torus to get M''. Hence we can write $M = M' \cup (\mathbb{D}^2 \times S^1)$ and $M'' = M' \cup (\mathbb{D}^2 \times S^1)$. Let's write the reduced $\mathbb{Z}$ homology Mayer-Vietoris sequence for M and M'. Here $M' \cap (\mathbb{D}^2 \times S^1)$ is a torus.

$$... \xrightarrow{\delta} H_1(M' \cap (\mathbb{D}^2 \times S^1)) \xrightarrow{\phi} H_1(M') \oplus H_1(\mathbb{D}^2 \times S^1) \xrightarrow{\psi} H_1(M) \xrightarrow{\delta} 0$$

Here δ is a boundary map and ϕ and ψ are group homomorphisms defined as follows: $\phi(x) = (x, -x)$ and $\psi(x, y) = i(x) + j(y)$. where i and$j$ are the respective inclusion maps.

For us the above long exact sequence is

$$... \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\phi} H_1(M') \oplus \mathbb{Z} \xrightarrow{\psi} H_1(M) \xrightarrow{\delta} 0$$

Since this a long exact sequence ψ is an onto map. Hence

$$H_1(M) = \frac{H_1(M') \oplus \mathbb{Z}}{Im(\phi)}.$$

Since ϕ is a group homomorphism, $Im(\phi)$ can be 0, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$. Hence difference in Betti number between M and M' is atmost 1. Similarly difference in Betti number between M' and M'' is also atmost 1. Hence $|b_1(M) - b_1(M'')| \leq 2$. We know that given any finitely generated abelian group G, there exists a 3-manifold M such that $H_1(M) = G$. Hence our graph of 3-manifolds is unbounded.

Now let us look at another class of graphs.

3.4 Cayley graph of conjugacy class of SL$(2, \mathbb{Z})$

For any finitely generated group Γ, generated by S, one can construct its Caley graph G_Γ. The set of vertices is Γ We draw an edge between $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma$ if $\gamma_1 = \gamma_2$.s where $s \in S$.

But if you change the generators then their corresponding Cayley graphs are quasi isolongest word in this translation. Hence these two metrics are Lipschitz equivalent since $\lambda^{-1} d_S(\gamma_1, \gamma_2) \leqslant d_{S'}(\gamma_1, \gamma_2) \leqslant \lambda d_S(\gamma_1, \gamma_2)$.Hence the identity map from (X, d_S) to $(X, d_{S'})$ is quasi isometric.

We can also construct a graph of conjugacy classes $[\gamma_i]$, where vertices are conjugacy classes and there is an edge between two classes $[\gamma_1]$ and $[\gamma_2]$ if there exist $g_1, g_2 \in \Gamma$ and $s \in S$ s.t. $g_2 \gamma_2 g_2^{-1} = g_1 \gamma_1 g_1^{-1} s$. Let us consider an example of free group generated by a and b. Here all the words in the same conjugacy class are cyclic words i.e. one word is a different cyclic form of the other word. Any word which is not of this form defines another conjugacy class. We join two conjugacy classes by an edge if they differ by an generator.

Similarly as above if you change the generators then their corresponding graph of conjugacy classes are quasi isometric. This follows from the above proof for Caley graph and the fact that conjugacy classes of a group also form a group.

We can consider a metric on this graph by assigning length one to each edge. Hence distance between two vertices $[\gamma_1]$ and $[\gamma_2]$ is the length of shortest path between them, denoted by $d_S([\gamma_1], [\gamma_2])$. We write s there to show that distance depends on the generating set you choose.

Now let us look at specific 3-manifolds which will connect the above two graphs.

3.5 Torus Bundle

Consider the following construction of 3-manifolds.Let T^2 be $S^1 \times S^1$.

$$T_A^3 = T^2 \times I/((x, y, 0) \sim (A(x, y), 1))$$

for $A \in Sl(2, \mathbb{Z})$. Following is a theorem by Poincare and we will study the proof given by Ghys. Ref[8]

Theorem 3.5.1. *(Poincare) T_A^3 is diffeomorphic to T_B^3 if and only if A is conjugate to $B^{\pm 1}$ in $Gl(2, \mathbb{Z})$.*

Proof. First the if side. Applying a diffeomorphism $f : T^2 \to T^2$ in each slice $T^2 \times y$ of

$T^2 \times I$ has the effect of conjugating A by f when we form the quotient T^3_A. By switching the two ends of $T^2 \times I$ we see that T^3_A is diffeomorphic to $T^3_{A^{-1}}$.
For converse, notice that $\pi_1(T^3_A)$ is the following group of transformations of $\mathbb{R}^3$

$$(x, y, t) \mapsto (x + 1, y, t)$$

$$(x, y, t) \mapsto (x, y + 1, t)$$

$$(x, y, t) \mapsto (A(x, y), t + 1).$$

We can define this group as set of triples $(a, b, c) \in \mathbb{Z}$ with multiplication operation given by $(a, b, c).(a', b', c') = (a + a', A^a(b', c') + (b, c))$. Since T^3_A is torus fibration over S^1 we have the short exact sequence of their corresponding fundamental groups

$$0 \to \mathbb{Z} \times \mathbb{Z} \to \pi_1(T^3_A) \to \mathbb{Z} \to 0$$

Here the first map is $(a, b) \mapsto (0, a, b)$ and the second map is $(a, b, c) \mapsto (a)$. Let H be image of $\mathbb{Z} \times \mathbb{Z}$ in $\pi_1(T^3_A)$.

Proposition 3.5.1. *$Dim_{\mathbb{Q}} H_1(T^3_A, \mathbb{Q}) = 3$ if and only if A is identity matrix.*
$Dim_{\mathbb{Q}} H_1(T^3_A, \mathbb{Q}) = 2$ if and only if A is conjugate to $\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}\}$, where $n \neq 0$.
$Dim_{\mathbb{Q}} H_1(T^3_A, \mathbb{Q}) = 1$, otherwise.

Proof. Let $[\pi_1(T^3_A),\pi_1(T^3_A)]$ be commutator subgroup of $\pi_1(T^3_A)$. Due to the short exact sequence above $rank([\pi_1(T^3_A), \pi_1(T^3_A)])$ can be $0, 1$ or 2. Say 1 is eigenvalue of A. Then A can be written as $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ in certain base. If $a = 0$ then $rank([\pi_1(T^3_A), \pi_1(T^3_A)]) = 0$. If $a \neq 0$ then $rank([\pi_1(T^3_A), \pi_1(T^3_A)]) = 1$. One can check that for $x \in H$ then $centralizer(x) = H$ and for $x \notin H$ the $centralizer(x) = \mathbb{Z}$. If A does not have eigenvalue 1 then $[\pi_1(T^3_A), \pi_1(T^3_A)]$ is of finite index in H. Hence the group $\pi_1(T^3_A)/[\pi_1(T^3_A), \pi_1(T^3_A)]$ is an extension of $\mathbb{Z}$ by a finite group. □

Proposition 3.5.2. *If $H_1(T^3_A, \mathbb{Q})$ has dimension 1, any automorphism of $\pi_1(T^3_A)$ preserves the image of subgroup H in $\pi_1(T^2)$.*

Proof. An automorphism of $\pi_1(T^3_A)$ preserves the commutator subgroup. Let A be hyperbolic. The commutator group has a finite index in H. Then the automorphism must preserve

H. ☐

This implies that any homeomorphism of T^3_A can be extended in $T^2 \times \mathbb{R}$. Similarly any homeomorphism of T^3_B can be extended in $T^2 \times \mathbb{R}$.

Proposition 3.5.3. *The only automorphism of $\pi_1(T^3_A)$ (for $|tr(A)| > 2$) are:*

$$(m, n, p) \mapsto (m, (I + A + ... + A^{m-1})(b) + B(n, p)) \ for \ m \geq 0$$

$$(m, n, p) \mapsto (m, (I + A^{-1} + ... + A^{-m+1})(b) + B(n, p)) \ for \ m \leq 0$$

where $b \in \mathbb{Z} \oplus \mathbb{Z}$ and $B \in GL(2, \mathbb{Z})$ is such that $AB = BA$.

$$(m, n, p) \mapsto (m, (I + A + ... + A^{m-1})(b) + B(n, p)) \ for \ m \geq 0$$

$$(m, n, p) \mapsto (m, (I + A^{-1} + ... + A^{-m+1})(b) + B(n, p)) \ for \ m \leq 0$$

where $b \in \mathbb{Z} \oplus \mathbb{Z}$ and $B \in GL(2, \mathbb{Z})$ is such that $ABA = B$.

Proof. We know that automorphism preserves the image of $\pi_1(T^2)$. Hence there is a $B \in \mathrm{GL}(2, \mathbb{Z})$ such that $(0, n, p)$ maps on $(0, B(n, p))$. Say (m, n) is image of $(1, 0, 0)$ for $a \in \mathbb{Z}$ $b \in \mathbb{Z} \oplus \mathbb{Z}$. Then the image of (m, n, p) is $(ma, (I + A^a + ... + A^{(m-1)a})(b) + B(n, p))$. Since we are interested in automorphism we have $a = \pm 1$. To be a group homomorphism if $a = 1$ then $AB = BA$ and if $a = -1$ then $ABA = B$. ☐

We have T^3_A is diffeomorphic to T^3_B via the map h and hence $\pi_1(T^3_A)$ is isomorphic to $\pi_1(T^3_B)$ via the induced map h_*. Now, let A and B act on respective $\mathbb{Z} \times \mathbb{Z}$

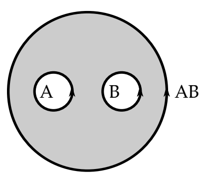
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \pi_1(T^3_A) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \, . \\ & & & & \downarrow{\scriptstyle h_*} & & & & \\ 0 & \longrightarrow & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \pi_1(T^3_B) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

And by using the propositions above, we can conclude that there exists M such that $A = MBM^{-1}$. When $tr(A) = 0$ there is just one conjugacy class. For $|tr(A) = 1|$ there are two conjugacy classes one for $tr(A) = -1$ and one for $tr(A) = 1$. For $tr(A) = 2$ we have infinite conjagy classes of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n \in \mathbb{Z}$. A simple calculation shows that any two matrices of this form, say $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & n' \\ 0 & 1 \end{pmatrix}$ are conjugate if and only if $n = \pm n'$. Say we have two matrices $A = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ which are not conjugate then we have to prove that T^3_A is not homeomorphic to T^3_B. Let us calculate their first homology group. Let $x = (1, 0, 0)$ and $x^{-1} = (-1, 0, 0)$. By the above group operation $x(0, a, b)x^{-1} = (0, A(a, b))$. For the above case we have $A(a, b) = (a, b)$ which means $nb = 0$, that is $b \in \mathbb{Z}/n\mathbb{Z}$. Hence first homology group $H_1(T^3_A) = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}$. Similarly we have for $tr(A) = -2$.

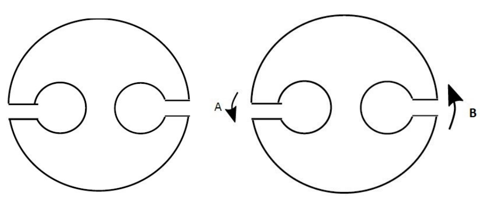
□

Consider the group $\mathrm{SL}(2, \mathbb{Z})$ with generating set $S = \{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$. We can map a matrix A to manifold T^3_A. This gives a map from graph of conjugacy classes of $\mathrm{SL}(2, \mathbb{Z})$ upto A and A^{-1} to the graph of 3-manifolds. Also this map of graphs is Lipschitz. To prove this we need to study construction of cobordism between T^3_A and T^3_B. From $A, B \in SL(2, \mathbb{Z})$ we can contruct a 4-manifold $M_{A,B}$ which is a fibration over a pair of pants with fibers as torus and such that the monodromies of the boundary components are A, B, AB.

Hence the boundary of $M_{A,B}$ is $T^3_{AB} - T^3_A - T^3_B$. It is represented as shown in figure below



Another way to see this construction is to start with $T^2 \times \mathbb{D}^2$ as shown in figure a below. Then we identify the two set of intervals using matrix A and B. This is exactly how we constructed T^3_A and T^3_B. Hence we get a four manifold with three boundary components T^3_A, T^3_B and T^3_{AB}.

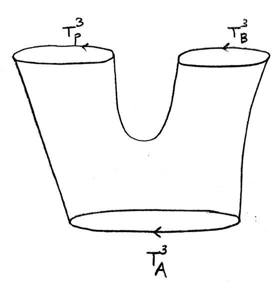


Theorem 3.5.4. *The map defined above from conjugacy graph of $Sl(2,\mathbb{Z})$ to the graph of 3-manifolds is Lipschitz.*

Proof. To prove this we need to prove that $dist(T^3_A, T^3_B) \leq Kdist(A, B)$ for some constant K.

It is enough to prove for A and B such that $B = A.P$ for P in finitely generating set of $Sl(2,\mathbb{Z})$. To see this assume that the above statement is true for all A, B such that $B = AP$. Let $B' = APP'$ in the reduced form, then $dist(T^3_A, T^3_{AP}) \leq Kdist(A, AP)$ and $dist(T^3_{AP}, T^3_{APP'}) \leq Kdist(AP, APP')$. Adding these two equations and using triangle inequality we get $dist(T^3_A, T^3_{APP'}) \leq K(dist(A, AP)+dist(AP, APP'))$. But since $B' = APP'$ in the reduced form then $dist(A, AP) + dist(AP, APP') = dist(A, APP')$. Hence we can safely asssume that $dist(A, B) = 1$.

Let $B = A.P$. Using the above construction we can construct a 4-manifold with boundary as T^3_A, T^3_P and T^3_B. Now consider the projection of this 4-manifold on pair of pants. We have a Morse function from pair of pants with one critical point. Composing these two we get Morse function from our 4-manifold with one critical point. By morse lemma one critical point of Morse function implies we need one Morse surgery to go from T^3_A to T^3_P union T^3_B (see fig. below)



Now we cap off T^3_P with a 4-manifold whose boundary is T^3_P. This gives us a cobordism between T^3_A, T^3_B. Let K be the number of surgeries needed to go from S^3 to T^3_P. Hence $dist(T^3_A, T^3_B) \leq K + 1$. Since $Sl(2,\mathbb{Z})$ has two generators P_1 and P_2, we would have corresponding K_1 and K_2. Choose $K = max\{K_1, K_2\}$. This gives us the desired bound. □

Chapter 4

Conclusion

There are a lot of questions about this graph which are still unexplored. I will end this report by stating a few intresting ones. Since Homology spheres are topologically very similar to the 3-sphere, is it true that homology spheres are at a bounded distance from the 3-sphere? Like the construction of $L(1,p)$ one can study the Morse surgey construction of $L(p,q)$ in general. And an ambition one would be to find the global geometry of this graph.

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