Motivic Homotopy Theory

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by

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Certificate

This is to certify that this dissertation entitled Motivic Homotopy Theory towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Arpith Shanbhag at Indian Institute of Science Education and Research under the supervision of Dr.Marc Levine, Professor, Department of Mathematics, University of Duisburg-Essen , during the academic year 2017-2018.

Dr.Marc Levine

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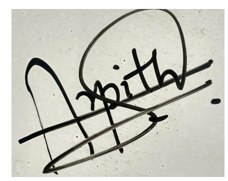
To *Nadapiya*. For being such an excellent role model.

Declaration

I hereby declare that the matter embodied in the report entitled Motivic Homotopy Theory are the results of the work carried out by me at the Department of Mathematics, University of Duisburg-Essen, Indian Institute of Science Education and Research, Pune, under the supervision of Dr.Marc Levine and the same has not been submitted elsewhere for any other degree.



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Abstract

The objective of this thesis is the study of Motivic homotopy theory and Voevodsky's construction of triangulated category of motives. We will construct a model category called the motivic model category which will be the Bousfield localization of level wise model structure on simplicial presheaves on smooth schemes over a base. As an application of this theory, we will look at the representability results for Nisnevich torsors in Motivic homotopy category. In the second part, we will focus on Voevodsky's triangulated category of motives. We will define the motivic cohomology and the category of effective motives. Then we will give a brief overview of the relationship between modules over motivic cohomology spectrum and Voevodsky's category of motives. This provides a relationship between motivic stable homotopy theory and the theory of motives.

Contents

Chapter 1

Model categories

In this section we give a background of model categories, which is a tool for doing homotopy theory. A model category the notion of a 'homotopy category' associated to it. Our motivic homotopy category is the homotopy category of some model category, which we will build. One advantage of model categories is that it gives a way to compute maps in the homotopy category. We refer to [10] and [8] for this section.

1.1 Basic notions and results

Definition 1.1.1. *For a category C:*

1. *A map $f : A \to B$ in C is said to be **retract** of a map $g : C \to D$ in C if there is a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

   where the compositions of horizontal arrows are identity.

2. *A functorial factorization is an ordered pair (α, β) of functors $C^{\to} \to C^{\to}$ such that every map f in C can be factored as $f = \beta(f) \circ \alpha(f)$. Here $C^{\to}$ is the arrow category with objects as maps in C and maps are commutative squares.*

3. Suppose we have the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow{j} & D \end{array}$$

we say that f has left lifting with respect to g (or g has right lifting wrt f) if there is an $h : B \to C$ such that $hf = i$ and $gh = j$

A category is said to be *complete* if it has all small limits. Dually, it is said to be *cocomplete* if it has all small colimits. A category is said to be *bicomplete* if it is both complete and cocomplete.

Definition 1.1.2. *A **model category** is a bicomplete category C with three classes of maps called weak equivalences, fibrations and cofibration which satisfy the following axioms.*

MC1 If f and g are morphisms such that gf is defined and two of f,g and gf are weak equivalences, then so is the third one.

MC2 Weak equivalences, fibrations and cofibrations are closed under compositions and taking retracts(f is a retract of g and if g is a weak equialence(resp. cofibration or fibration) then so is f.

MC3 A trivial cofibration is a map which is both a cofibration and a weak equivalence(we define trivial fibrations similarly). Then the trivial cofibrations have left lifting property with respect to fibrations and trivial fibrations have right lifting property with respect to cofibrations

MC4 There exists two functorial factorization (α, β) and (γ, δ) such that for any morphism f, $\alpha(f)$ is a trivial cofibration, $\beta(f)$ is a fibration, $\gamma(f)$ is a cofibration and $\delta(f)$ is a trivial fibration.

As a model category is bicomplete, we have both the initial object 0 and final object $\star$ in our category. We call an object X, in a model category C, to be fibrant if the map $X \to \star$ is a fibration and is said to be cofibrant if the map $0 \to X$ is a cofibration. By axiom MC4, we can factorize any map $f : Y \to \star$ as $Y \to RY \to \star$ such that $Y \to RY$ is a trivial cofibration and $RY \to \star$ is a fibration and this factorization is functorial. RY is called the fibrant replacement of Y and R is called the fibrant replacement functor. Similary we can define cofibrant replacement QY and cofibrant replacement functor Q using functorial factorization for the map $0 \to Y$.

Remark. If C is a model category then we have an obvious model structure on pointed category $C_\star$ with weak equivalences, fibrations and cofibrations defined by forgetting the point. Refer to [[10], prop 1.1.8]

Lemma 1.1.3. *Suppose C is a model category. Then a map is a fibration(resp.trivial fibration) if and only if it has right lifting property with respect to all trivial cofibrations(resp. cofibrations. Dually, a map is a cofibration(resp. trivial cofibration) if and only if it has left lifting property with respect to all trivial fibrations(resp. fibrations)*

Proof. Refer to [[10], Lemma 1.1.10] □

The above lemma implies that cofibrations(resp. trivial cofibrations) are closed under pushouts. Dually, fibrations(resp.trivial fibrations) are closed under pullbacks.

The advantage of working with model category is that there is a set theoretically well defined notion of localization with respect to weak equivalences. We call this localized category the **homotopy category**. In this section we explicitly define the underlying relation of homotopy. But this homotopy relation is an equivalence relation only for the full subcategory of objects that are both cofibrant and fibrant.

Definition 1.1.4. *Let C be a model category and X be an object of C.*

- *A cylinder object $X \times I$ for X is a factorization of the map $\nabla : X \coprod X \to X$ into a cofibration $X \coprod X \to X \times I$ followed by a weak equivalence $X \times \to X$*
- *A path object X^I for X is a factorization of the map $X \to X \times X$ into a weak equivalence $X \to X^I$ followed by a fibration $X^I \to X \times X$*

We know that cylinder and path objects exist for all X because of the functorial factorizarions.

Now we can define left and right homotopies using the cylinder and path objects. The two definition need not coincide(i,e if two maps are left homotopic then they need not be right homotopic)

Definition 1.1.5. *Let $f, g : X \to Y$ be two maps in a model category C.*

- *We say that they are left homotopic if there is a map $H : cyl(X) \to Y$ from some cylinder object of X such that $Hi_0 = f$ and $Hi_1 = g$ where, i_0 and i_1 are the two possible maps $X \to X \coprod X \to cyl(X)$.*
- *We say that they are right homotopic if there is a map $H : X \to path(Y)$ from some path object of Y such that $p_0K = f$ and $p_1K = g$ where, p_0 and p_1 are the two possible maps $path(Y) \to Y \times Y \to Y$.*

We say that the maps are homotopic if they are both left and right homotopic. A map $f : X \to Y$ is said to be a homotopy equivalence if there is a map $g : Y \times X$ such that fg is homotopic to id_Y and gf is homotopic to id_X.

The next proposition([[10], Corollary 1.2.6]) describes that under certain conditions, the idea of left and right homotopies coincide.

Proposition 1.1.6. *Let C be a model category and let X be cofibrant and Y be fibrant. Then the left and right homotopies coincide and are equivalence relations on $Hom_C(X, Y)$. Furthermore, they do not depend on the choice of a cylinder or a path object.*

The next theorem is an analogue of the Whitehead's theorem.

Theorem 1.1.7. *Suppose C is a model category. Then a map between objects that are both cofibrant and fibrant is a weak equivalence if and only if it is a homotopy equivalence.*

A model category is said to be *left proper* if the weak equivalences are preserved under cobase change along cofibrations. Dually, a model category is said to be *right proper* if the weak equivalences are preserved under base change along fibrations. It is called *proper* if it is both left and right proper.

Remark. If C be a model category, then the pointed category C_* has an obvious model structure [[10], prop 1.1.8]. If C is left or right proper then so is C_*

Now we will define the homotopy category of a model category. Refer to [[10], 1.2] for a definition of localization in a general context.

Definition 1.1.8. *Let C be a model category. The homotopy category $Ho(C)$ of C is a category with objects same as C but*

$$Hom_{Ho(C}(X, Y) = Hom_C(RQX, RQY)/\sim$$

where $\sim$ is the homotopy relation. The canonical functor $\gamma : C \to Ho(C)$ is called the localization functor is the identity on objects and takes a map $f : X \to Y$ to $RQ(f) : RQX \to RQY$

Remark. The localization functor is the localization of C with respect to weak equivalences and is universal in the following sense. For any functor $F : C \to D$ which takes weak equivalences in a model category C to isomorphisms in D, there exists a unique functor $G : Ho(C) \to D$ such that $F = G \circ \gamma$. Refer to [[10], lemma 1.2.2].

Remark. Notice that if $f : X \to Y$ is a weak equivalence then $RQ(f) : RQX \to RQY$ is also a weak equivalence and hence a homotopy equivalence by Theorem 1.1.7 and thus is an isomorphism in $Ho(C)$.

1.2 Quillen functors

In this section we will study the maps between model categories which we call *Quillen adjunctions*. We will show that these maps induce maps on the homotopy categories. We will also look a condition under which Quillen adjunctions lead to an equivalence of homotopy categories.

Definition 1.2.1. *Let C and D be two model categories*

1. *We call a functor $F : C \to D$ a* ***left Quillen functor*** *if it is a left adjoint and if it preserves cofibration and trivial cofibration. Dually, a functor $G : D \to C$ is* ***right Quillen functor*** *if it is a right adjoint and preserves fibrations and trivial fibrations.*

2. *Suppose we have an adjunction $F : C \leftrightarrows D : G$ such that F is a left Quillen functor, then it is called a* ***Quillen adjunction****.*

Remark. We get an equivalent definition if G is a right Quillen functor in the above defintion. This statement follows from the lemma [[10], lemma 1.3.4]

Here is a useful lemma. Refer to [[10] 1.1.12] for a proof.

Lemma 1.2.2. *(Ken Brown's lemma) Suppose C is a model category and D is a category with a notion of weak equivalence which satisfies the two out of three axiom. If a functor $F : C \to D$ takes trivial cofibrations between cofibrant objects to weak equiavalences, then F takes weak equivalences between cofibrant objects to weak equivalences. The dual statement for fibrations also holds true.*

Remark. Note that a left Quillen functor preserves cofibrant objects and by Ken Brown's lemma it preserves weak equivalences between cofibrant objects. The dual statement holds true for a right Quillen functor.

The Quillen adjunction describe above induces maps at the level of homotopy categories as follows.

Corollary 1.2.3. *Let $F : C \leftrightarrows D : G$ be a Quillen adjunction. The composite map $F \circ Q$ induces a functor*

$$LF : Ho(C) \to Ho(D)$$

which we call the ***total left derived functor****. Dually, $G \circ R$ induces a functor*

$$RG : Ho(D) \to Ho(C)$$

which we call the ***total right derived functor****. Furthermore, the functors*

$$LF : Ho(C) \leftrightarrows Ho(D) : RG$$

form an adjoint pair.

Proof. The existence of the functor LF follows immediately follows from the previous remark and by the fact that the homotopy category is a category obtained by localizing C with respect to weak equivalences. The adjointness follows from [[10], Lemma 1.3.10] □

Definition 1.2.4. *A Quillen adjunction $F : C \leftrightarrows D : G$ is said to be a Quillen equivalence if , for all cofibrant X in C and fibrant Y in D, a map $f : FX \to Y$ is a weak equivalence in D if and only if $\phi(f) : X \to GY$ is a weak equivalence in C. Here, $\phi : Hom(FX, Y) \tilde{\to} Hom(X, GY)$ is the bijection for the adjoint pair (F, G).*

The following proposition [[10], prop 1.3.13] shows that Quillen equivalences induce adjoint equivalences of homotopy categories

Proposition 1.2.5. *Suppose $F : C \leftrightarrows D : G$ be a Quillen adjunction. Then the following statements are equivalent:*

1. *The composite $X \to GFX \to GRFX$ is a weak equivalence for all cofibrant X, and the composite $FQGX \to FQX \to X$ is a weak equivalence for all fibrant X.*

2. *$LF : Ho(C) \leftrightarrows Ho(D) : RG$ is an equivalence of categories.*

3. *$F : C \leftrightarrows D : G$ is a Quillen equivalence*

Proof. See [[10], prop 1.3.13] □

The following is an important corollary of the above theorem.

Corollary 1.2.6. *Suppose $F : C \leftrightarrows D : G$ be a Quillen adjunction. Then the following statements are equivalent:*

1. *$F : C \leftrightarrows D : G$ is a Quillen equivalence.*

2. *For every fibrant object Y in D, the map $FQGY \to Y$ is a weak equivalence and if $f : X \to Y$ is a map between cofibrant objects such that $F(f) : FX \to FY$ is a weak equivalence, then f is a weak equivalence.*

3. *For every cofibrant object X in C, the map $X \to GRFX$ is a weak equivalence and if $g : X \to Y$ is a map between fibrant objects such that $G(g) : GX \to GY$ is a weak equivalence, then g is a weak equivalence.*

1.3 Cofibrantly Generated Model Categories

Proving that a category has a model structure is usually very difficult. The axioms are a bit redundant in the sense that we only need to specify weak equivalences and fibrations (or cofibrations) and the cofibrations(or fibrations) are determined by left (or right) lifting with respect to trivial fibrations (or trivial cofibrations). This section is devoted to minimizing the number of axioms that need to be checked. One of the important theorems which we won't be stating here is the small object argument [[10], Theorem 2.1.14]

Definition 1.3.1. *Let I be a class of maps in a cocomplete category C.*

1. *$I - inj$ is the class of maps in C which have right lifting with respect to all maps in I.*

2. *$I - cof$ is the class of maps in C which have left lifting with respect to all maps in $I - inj$.*

3. *A map is called a **relative I-cell complex** if it is a transfinite composition of pushout of maps in I. We denoted these maps as I-cell*

Definition 1.3.2. *Let C be a model category. C is called a* ***cofibrantly generated model category*** *if there exist two sets of maps called the generating cofibrations I and generating trivial cofibrations J, with the following axioms.*

1. *I and J satisfy the hypothesis of small object argument. [[10], Theorem 2.1.14]*

2. *A map is a trivial fibration if and only if it has right lifting with respect to every map in I*

3. *A map is a fibration if and only if it has right lifting property with respect to every map in J*

Example: The model category of simplicial sets $sSet$ is a cofibrantly generated model category with

$$I = \{\partial\Delta^n \to \Delta^n | n \in \mathbb{N}\}$$

and

$$J = \{\Lambda^n_k \to \Delta^n | n > 0, 0 \leq k \leq n\}$$

. See [[10], 3.6.5].

Here is a theorem which will help in recognizing cofibrantly generated model categories. [[10], Theorem 2.1.19]

Theorem 1.3.3. *Suppose C is a category with all small limits and colimts. Let W be a subcategory of C and I and J are set of maps in C satisfying the following conditions:*

1. *The subcategory W has two out of three property and is closed under retracts*

2. *Both I and J satisfy the hypothesis of the small object argument.*

3. *$J - cell \subseteq W \cap I - cof$*

4. *$I - inj \subseteq W \cap J - inj$*

5. *Either $W \cap J - inj \subseteq I - inj$ or $W \cap I - cof \subseteq J - cof$*

Then C has a cofibrantly generated model structure, such that W is the class of weak equivalences, I is the set of generating cofibrations and J is the set of generating trivial cofibrations.

Remark. Notice that every cofibrantly generate category satisfies these axioms. Hence the theorem above gives an if and only if condition.

One advantage of cofibrantly generated categories is that it is easier to check whether adjoints are Quillen functors.

Theorem 1.3.4. *Let C and D be two model categories and C is cofibrantly generated with generating cofibrations I and generating trivial cofibrations J. Let*

$$F : C \leftrightarrows D : G$$

be an adjoint pair. Then they are Quillen adjunctions if and only if $F(f)$ is a cofibration for $f \in I$ and is a trivial cofibration for $f \in J$

Proof. See [[10], lemma 2.1.20] □

1.4 Simplicial model categories

The model categories which we will consider will have an extra structure namely, it will be enriched over simplicial sets such that the enrichment is compatible with the model structure. One of the main tool in this section will be that of Bousfield localization which is a way to add more weak equivalences to a sufficiently nice model category.

Definition 1.4.1. *(Simplicial model category) A model category C is called a simplicial model category if there is a* ***mapping space functor***

$$Map : C^{op} \times C \to sSet$$

and an ***action*** *of simplicial sets,*

$$\otimes : sSet \times C \to C$$

satisfying the following conditions:

- *Let $X \in C$ and $K, L \in sSet$ then $(K \times L) \otimes X \cong K \otimes (L \otimes X)$ and $\Delta^0 \otimes X \cong X$*
- *We have adjoint functors*

$$\otimes X : sSets \leftrightarrows C : Map(X, \_)$$

and

$$K \otimes \_ : C \leftrightarrows C : (\_)^K$$

for all $X, Y \in C$ and $K \in sSets$.

- ***MC5*** *For every cofibration $i : X \to Y$ in C and fibration $p : E \to B$ in C,*

$$Map(Y, E) \to Map(X, E) \times_{Map(X,B)} Map(Y, B)$$

is a fibration in sSets which is a weak equivalence if either i or p is a weak equivalence.

Let us first discuss the case of simplicial sets. For details refer to [[11], II. 2-3]. Let X and Y be simplicial sets, then we can define another simplicial set called the **simplicial mapping space** $Map_{sSets}(X, Y)$ such that

$$Map_{sSets}(X, Y)_n := Hom_{sSets}(X \times \Delta^n, Y)$$

. In this case, we have $K \otimes L = K \times L$ and thus we get the adjunction

$$Map(K, L)_n \cong Hom_{sSets}(\Delta^n), Map(K, L) \cong Hom_sSets(\Delta^n \otimes X, Y)$$

Here the first equivalence comes from Yoneda lemma and other by using the adjunctions.

Remark. Let C be a simplicial model category. Then

$$Hom_C(X, Y^{(K)}) \cong Hom_C(K \otimes X, Y) \cong Hom_sSets(K, Map(X, Y))$$

. Hence we have an adjoint pair

$$Map(\_, Y) : C \leftrightarrows sSets^{op} : Y^{(-)}$$

Remark. We could have formulated **MC5** in terms of one of the adjoints of $Map(X, Y)$ (namely, $Y^{(-)}$ or $K \otimes \_$). We call these **SM7a** and **SM7b** respectively. Refer to [[11], II corollary 3.12 and Proposition 3.13] for a proof.

We will state (**SM7b**). For a cofibration $j : X \to Y \in C$ and a cofibration $i : K \to L$ $\in sSets$ then the map

$$K \otimes Y \coprod_{K \times X} L \otimes X \to L \otimes Y$$

is a cofibration in C which is trivial if either i or j is trivial.

We prove that the adjoints in Definition 1.4.1 are infact Quillen adjunctions

Theorem 1.4.2. *Let C be a simplicial model category. Then*

1.
$$K \otimes \_ : C \leftrightarrows C : (\_)^K$$
is a Quillen adjunction

2. *For a cofibrant $X \in C$,*
$$\otimes X : sSets \leftrightarrows C : Map(X, \_)$$
is a Quillen adjunction

3. *For a fibrant $Y \in C$,*
$$Map(\_, Y) : C \leftrightarrows sSets^{op} : Y^{(\_)}$$
is a Quillen adjunction.

Proof. Notice that every simplicial set K is cofibrant. Using (**SM7b**) we get that $K \otimes X \to K \otimes Y$ is cofibration(trivial cofibration) for every cofibration(trivial cofibration) $f : X \to Y$. This proves that $K \otimes \_$ is left Quillen. Similarly, (**MC5**) implies that $Map(X, \_)$ is right Quillen for cofibrant $X \in C$ follows from **MC5** and $Map(\_, Y)$ is left Quillen for fibrant Y. □

One advantage of working with a simplicial model category is that the weak equivalences can be detected at the level of mapping spaces.

Lemma 1.4.3. *Let $f : X \to Y$ be a map in a simplicial model category C. Then f is a weak equivalence if and only if the induced map*

$$map(f, Z) : map(Y, Z) \to map(X, Z)$$

is a weak equivalence in simplicial sets for every fibrant object Z in C.

Proof. Refer to [[11], II.Lemma 4.2] □

Now we describe the idea of a Bousfield localization. Throughout this subsection we assume that C is a simplicial model category and $map(X, Y) := Map(QX, RY)$, where Q is a cofibrant replacement functor and R is a fibrant replacement functor. Note that because of axiom **MC5**, for a fibrant object $Y \in C$, $map(X, Y)$ is weak equivalent to $Map(QX, Y)$.

Definition 1.4.4. *Let S be a class of morphisms in C. A fibrant object Z in C is said to be **S-local** if for every morphism $f : X \to Y$ in C the morphism*

$$map(f, Z) : map(Y, Z) \to map(X, Z)$$

is a weak equivalence of simplicial sets.

*A morphism $f : X \to Y$ is said to be a **S-local weak equivalence** if for every S-local object Z, the morphism*

$$map(f, Z) : map(Y, Z) \to map(X, Z)$$

is a weak equivalence of simplicial sets.

Remark. Every $f \in S$ is an S-local weak equivalence which is clear by the definition of an S-local object and every weak equivalence in C is an S-local weak equivalence which follows from Lemma 1.4.3

Definition 1.4.5. *(Left Bousfield localization) Let S be a class of morphisms. A model structure $\boldsymbol{L}_S C$ on C is called a left Bousfield localization of C w.r.t S if:*

- *weak equivalences in $\boldsymbol{L}_S C$ are S-local weak equivalences*
- *cofibrations in $\boldsymbol{L}_S C$ are cofibrations in C.*

The following theorem gives the existence of left Bousfield localizations under certain conditions. Note that the theorem requires the model category to be cellular model category but one can find the definition in [Hir, Chapter 12] or [[10], chapter 2]

Theorem 1.4.6. *Let C be a left proper cellular model category and let S be a set of morphisms. Then the left Bousfield localization with respect to S exists and has the following properties.*

1. *$L_S C$ is a left proper cellular simplicial model category*
2. *The fibrant objects of $L_S C$ is exactly the S-local objects in C*

Proof. Refer to [Hir, Theorem 4.1.1] □

Chapter 2

Unstable motivic homotopy category

We will formulate unstable motivic homotopy category by 'refining' the levelwise projective model structure on simplicial presheaves on Sm_S to include the Nisnevich topology and we will contract $\mathbb{A}^1$ by Bousfield localization. The homotopy category of our model category will be equivalent to Morel-Voevodsky's construction under the assumption that the base scheme is Noetherian with finite Krull dimension.

2.1 Simplicial presheaves with descent

Let C be an essentially small category. Let $sPre(C)$ be the functor category $Func(C^{op}, sSets)$. These can also be considered as simplicial objects in the category of presheaves over C. This has a **levelwise projective model structure** given by the following data:

- Levelwise weak equivalences: A map $f : \mathcal{F} \to \mathcal{G}$ is a levelwise weak equivalence if for every $X \in Sm_S$, $\mathcal{F}(X) \to \mathcal{G}(X)$ is weak equivalence of simplical sets.
- Levelwise fibrations: A map $f : \mathcal{F} \to \mathcal{G}$ is a levelwise fibration if for every $X \in Sm_S$, $\mathcal{F}(X) \to \mathcal{G}(X)$ is a fibration of simplical sets.
- Projective cofibrations: A map $f : \mathcal{F} \to \mathcal{G}$ is a projective cofibration if it has left lifting property w.r.t levelwise fibrations that are also levelwise weak equivalences(trivial levelwise fibrations)

Proposition 2.1.1. *The category of simplicial presheaves with weak equivalences, fibrations and cofibrations given as above is a combinatorial left-proper simplicial model category.*

Proof. Refer to [[9], Proposition A.2.8.2]. See [[9], Remark A.2.8.4] for left properness. □

We will 'refine' the above model structure to reflect the topology (Nisnevich topology in our case). To achieve this we start with Voevodsky's definition of cd-structures, which is a convenient way to topologize a category. We then construct a '*t-local*' model category by inverting the t-covering sieves via the process of Bousfield localization.

Definition 2.1.2. *(cd-structures) Let C be a small category with an initial object ϕ. A cd-structure on C is a collection P of commutative diagrams such that if $Q \in P$ and $Q^{'}$ is isomorphic to Q, then $Q^{'} \in P$.*

Let P be a cd-structure on C. The Grothendieck topology t_P generated by P is the coarsest topology such that:

- *The empty sieve covers ϕ*
- *Let $Q \in P$ be a square of the form*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

  then the sieve generated by $X \to Z$ and $Y \to Z$ is a t_P-covering sieve of Z

Here are the few examples of the cd-structures which we will be using throughout.

- The Zariski cd-structure on Sm_S is given by squares of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

  where, $U \to X$ and $V \to X$ are open embeddings such that $U \bigcup V = X$.

- The Nisnevich cd-structure on Sm_S is given by squares of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

where, $U \to X$ is an open embedding and $V \to X$ is an etale map such that $V \times_X Z \to Z$ is an isomorphism, with Z being the complement of U with reduced induced subscheme structure.

- The affine Zariski cd-structure on the category of smooth affine schemes over a base S Sm_S^{aff} is given by squares of the form

$$\begin{tikzcd} Spec(R_{fg}) \arrow[r] \arrow[d] & Spec(R_g) \arrow[d] \\ Spec(R_f) \arrow[r] & Spec(R) \end{tikzcd}$$

  where $f, g \in R$ and generate the unit ideal.

- The affine Nisnevich cd-structure on the category Sm_S^{aff} of smooth affine schemes over a base S is given by squares of the form

$$\begin{tikzcd} Spec(A_f) \arrow[r] \arrow[d] & Spec(A) \arrow[d] \\ Spec(B_f) \arrow[r] & Spec(B) \end{tikzcd}$$

  where, $f \in B$ and $Spec(A) \to Spec(B)$ is etale such that it induces an isomorphism $A/f \cong B/f$

  Remark. The theorems [[3], 2.3.2] and [[3], 2.1.3] show that the cd-structures generates the Nisnevich and Zariski topologies.

Let C be a small category and t be a Grothendieck topology on C. Let S_t be the set of covering sieves of the form $R \to X$ considered as simplicial presheaves with representable X.

Theorem 2.1.3. *Left Bousfield localization of $sPre(C)$ with respect to S_t exists. We will denote this category by $L_t sPre(C)$.*

Proof. This follows because $sPre(C)$ is a left proper, combinatorial simplicial model category. See Theorem 1.4.6. □

Definition 2.1.4. *We say that a simplicial presheaf $\mathcal{F}$ satisfies **t-descent**(or is t-local) if it is S_t-local.(See Definition 1.4.4) and we call a morphism of simplicial presheaves $\mathcal{F} \to \mathcal{G}$ a **t-local weak equivalence** if it is an S_t-local weak equivalence.*

There is a convenient way to characterize t-local objects in terms of pre-topologies.

Let $\mathfrak{U} = \{\ U_i \to X\}_{\ i\in I}$ be a family of maps in C. The 'Čech nerve' $\check{C}(\mathfrak{U})$ is a simplicial presheaf over C such that $\check{C}(\mathfrak{U})_n := \coprod U_{i_0 \dots i_n}$ where $U_{i_0 \dots i_n} := U_{i_0} \times \dots \times U_{i_n}$

Theorem 2.1.5. *Let (C, τ) be a small Grothendieck site and Cov_τ be the family of coverings (i,e $\mathfrak{U} \in Cov_\tau(X)$ if $\mathfrak{U}$ is a cover of X in the sense of Grothendieck topology). Let*

$$S = \{\ \check{C}(\mathfrak{U}) \to X | X \in C, \mathfrak{U} \in Cov_\tau(X)\}$$

A simplicial presheaf $\mathcal{F} \in sPre(C)$ is τ-local if and only if it is S-local.

Proof. See [[3], Lemma 3.1.3] □

We give a description of fibrant objects in the $L_\tau sPre(C)$ in terms of cd-structures via a property called P-excision. Our main aim will be to state a variant of theorem by Voevodsky, which states that under certain conditions the notions of P-excision and τ-descent coincide.

Definition 2.1.6. *Let C be a small category with an initial object ϕ. Let P be a cd-structure on C. We say that $\mathcal{F} \in sPre(C)$ satisfies $P-excision$ if:*

- *$\mathcal{F}(\phi)$ is a contractible simplicial set*
- *$\mathcal{F}(Q)$ is homotopy Cartesian for every $Q \in P$.*

Remark. The notion of excision appears in [1] and [12] as Brown-Gersten property and Voevodsky calls it P-flasque in [13].

Now we state the variant of a theorem by Voevodsky [13]. The proof is given in [[3], 3.2.5]. Notice that the examples of cd-structures mentioned earlier satisfy the hypothesis of this theorem.

Theorem 2.1.7. *Let C be a small category with strictly initial object. Let P be a cd-structure on C such that:*

- *Every square in P is cartesian*
- *Pullback of squares in P exists and belong to P*

- *For every square in P of the form*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

$Y \to Z$ is a monomorphism.

- *For every P-square of the form above*

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\Delta"'] & X \arrow[d, "\Delta"] \\ W \times_Y W \arrow[r] & X \times_Z X \end{tikzcd}$$

is also in P

Let $\mathcal{F}$ be a simplicial presheaf on C. $\mathcal{F}$ satisfies P-excision if and only if it satisfies τ-descent.

2.2 Site with an interval and $Sing^I$-construction

Definition 2.2.1. *Let C be a small category. A representable interval object in C is a quadruple (I, m, i_0, i_1) consisting of a presheaf I on C, a map $m : I \times I \to I$ and maps $i_0, i_1 : * \to I$, satisfying the following conditions:*

1. *for every $X \in C$, $X \times I$ is representable*
2. *$m(id \times i_0) = m(i_0 \times id) = i_0 p$ and $m(id \times i_1) = m(i_1 \times id) = id$ where, $p : I \to *$ is the canonical map.*
3. *the map $i_0 \bigsqcup i_1 : * \bigsqcup * \to I$ is a monomorphism.*

Definition 2.2.2. *A simplicial presheaf $\mathcal{F}$ over C is called I-invariant if for all $X \in C$, the projection map $X \times I \to X$ induces a weak equivalence $\mathcal{F}(X) \to \mathcal{F}(X \times I)$ of simplicial sets. In other words, $\mathcal{F}$ is S_I-local where, S_I is the set of all projections $X \times I \to X$.*

For a representable interval object I we can define a cosimplicial presheaf $I^\bullet$ on C with $I^n := I^{\times n}$. We define a functor $Sing^I : sPre(Sm_S) \to sPre(Sm_S)$ as follows

$$Sing^I(\mathcal{F}(X)_n := (\mathcal{F}(X \times I^n))_n$$

We have a natural map $\mathcal{F} \to Sing^I(\mathcal{F})$ which is an S_I-weak equivalence and $Sing^I(\mathcal{F})$ is S_I-local. [Refer to, [1], 2.3.5 and 2.3.8]

Theorem 2.2.3. *Consider the commutative diagram of bisimplicial sets*

$$\begin{array}{ccc} \mathcal{F}_{12} & \longrightarrow & \mathcal{F}_1 \\ \downarrow & & \downarrow \\ \mathcal{F}_2 & \longrightarrow & \mathcal{F}_0 \end{array}$$

such that it is levelwise homotopy cartesian. If $\pi_0(\mathcal{F}_0)$ and $\pi_0(\mathcal{F}_1)$ are constant simplicial presheaves then,

$$\begin{array}{ccc} diag(\mathcal{F}_{12}) & \longrightarrow & diag(\mathcal{F}_1) \\ \downarrow & & \downarrow \\ diag(\mathcal{F}_2) & \longrightarrow & diag(\mathcal{F}_0) \end{array}$$

is homotopy cartesian.

Proof. See [[3],4.2.1] □

Using the above theorem, we immediately get the following result. [[4], 2.2.1]

Theorem 2.2.4. *Let C be a small category. Consider a homotopy fiber sequence of simplicial presheaves over C*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

such that $\pi_0(\mathcal{H})$ is I-invariant. Then

$$Sing^I(\mathcal{F}) \to Sing^I(\mathcal{G}) \to Sing^I(\mathcal{H})$$

is a homotopy fiber sequence.

Proof.

$$\begin{tikzcd} \mathcal{F}(X \times I^\bullet) \arrow[r] \arrow[d] & \mathcal{G}(X \times I^\bullet) \arrow[d] \\ * \arrow[r] & \mathcal{H}(X \times I^\bullet) \end{tikzcd}$$

is levelwise homotopy cartesian. As $\pi_0(\mathcal{H})$ is a I-invariant, $\pi_0(\mathcal{H})(X \times I^\bullet)$ is levelwise constant. The statement follows from Theorem 2.2.3. $\square$

2.3 $\mathbb{A}^1$-homotopy category

Let Sm_S be the category of smooth schemes over a quasi compact and quasi separated base S. We consider the simplicial presheaves on Sm_S under Nisnevich topology. As noted before, the cd- structure in given by the squares of the form

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\ U \arrow[r] & X \end{tikzcd}$$

with $U \to X$ being an open immersion and π is etale such that, $V \times_X Z \cong Z$ where Z is the complement of U with reduced induced subscheme structure.

Definition 2.3.1. *The category $L_{Nis}sPre(Sm_S)$ is called **Nisnevich local model category** and we will denote it by Spc_S. We have a map $sPre(Sm_S) \to Spc_S$ given by the identity map. We denote the left derived functor of this map as L_{Nis}.*

By the theorem [[8], Proposition 3.4.1], we know that the fibrant objects in Nisnevich local model category are precisely the levelwise fibrant objects that satisfy τ-descent.

We perform a further left Bousfield localization of Spc_S to get the $\mathbb{A}^1$-homotopy category.

Let I be the class of maps $X \times_S \mathbb{A}^1 \to X$ in Spc_S where X is a representable presheaf. As Sm_S is essentially small, we have a subset $J \subseteq I$ containing maps $X \times_S \mathbb{A}^1 \to X$ for every representative element of isomorphism class in Sm_S.

Theorem 2.3.2. *The left Bousfield localization of Spc_S with respect to the set of maps J exists. We denote this as $Spc_S^{\mathbb{A}^1}$.*

Proof. The Bousfield localization exists as Spc_S inherits a simplicial, combinatorial, left proper model structure from $sPre(Sm_S)$ $\square$

This is usually called the $\mathbb{A}^1$ model structure on the presheaf category and the homotopy category the unstable $\mathbb{A}^1$ homotopy category.

Remark. Fibrant objects in $Spc_S^{\mathbb{A}^1}$ will be precisely the fibrant objects in Spc_S which are $\mathbb{A}^1$-invariant (i,e $\mathcal{F}(X) \to \mathcal{F}(X \times_S \mathbb{A}^1)$ is a weak equivalence of simplicial sets for all $X \in Sm_S$)

Remark. We denote $[X, Y]_{\mathbb{A}^1}$ as the set of maps in homotopy category of $Spc_S^{\mathbb{A}^1}$ from X to Y. Similarly $[X, Y]_{Nis}$ be the set of maps in homotopy category of Spc_S from X to Y and $[X, Y]_s$ be the set of maps in homotopy category of $sPre(Sm_S)$

2.4 Stable motivic homotopy theory

In this section we will recall Jardine's category of motivic spectra and motivic symmetric spectra. There is a 'stable' model structure on these categories which makes them a proper closed simplicial model categories. We will show a Quillen equivalence between these two model categories and the homotopy category will be the motivic stable homotopy category. The motivation behind motivic symmetric spectra is that there is a well defined smash product at the level of spectra, making it into a symmetric monoidal category.

The 'sphere' we consider the spectra will be a combination of topological and Algebro-geometric spheres. Let S^1 be the simplical presheaf which takes every $U \in Sm_S$ to the simplicial set $\Delta^n/\partial\Delta^n$. Let $\mathbb{G}_m := \mathbb{A}^1 - \{0\}$ considered as a simplicial presheaf. We define the Tate-object $T := S^1 \wedge (\mathbb{G}_m, 1)$. Notice that homotopy pushout of

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

is $S^1 \wedge \mathbb{G}_m$ and now in $Spc_S^{\mathbb{A}^1}$, we can replace one of the $*$ with $\mathbb{A}^1$ and hence T is weakly equivalence to $\mathbb{A}^1/(\mathbb{A}^1 - 0)$ in $Spc_S^{\mathbb{A}^1}$.

Definition 2.4.1. *The category of motivic spectra is defined as follows. A motivic spectrum consists of a sequence of pointed simplicial presheaves $E_0, E_1, ..., E_n, ...$ with 'bonding maps'*

$\sigma_n : T \wedge E_n \to E_{n+1}$. *A map between spectra* $f : E \to F$ *is a collection of maps* $f_n : E_n \to F_n$ *such that it commutes with the bonding maps. We denoted this category as* $Spt_T(Sm_S)$*.*

We have a levelwise model structure on spectra which makes it a proper simplicial model category. We will refine this model category by adding more weak equivalences. Our definition differs from that of Morel-Voevodsky. They define stable weak equivalence using stable homotopy groups. But both the categories turn out to be Quillen equivalent.

We say a map between spectra $f : E \to F$ is:

- A level cofibration if $f_n : E_n \to F_n$ is a cofibration in $Spc_S^{\mathbb{A}^1}$ for all $n \geqslant 0$
- A level fibration if $f_n : E_n \to F_n$ is a fibration in $Spc_S^{\mathbb{A}^1}$ for all $n \geqslant 0$
- A level weak equivalence if $f_n : E_n \to F_n$ is a weak equivalence in $Spc_S^{\mathbb{A}^1}$ for all $n \geqslant 0$

Let *cofibrations* be maps which have left lifting with respect to level fibrations that are also level weak equivalences.

Theorem 2.4.2. *The category* $Spt_T(Sm_S)$ *with class of cofibrations, levelwise fibrations and level wise weak equivalences forms a proper closed simplicial model category.*

Proof. Refer to [[14], 2.1] □

Let J be the fibrant replacement functor in this model category. For a simplicial presheaf we have the T-loop space functor Ω_T which is right adjoint for T-smash $\_ \wedge T$.

$$\Omega_T(X) = \mathbf{Hom}_*(T, X)$$

. In general, the functor $\mathbf{Hom}_*(K, \_)$ is a right adjoint of the functor $\_ \wedge K$ for any $K \in sPre(Sm_S)$. This induces a *loop space* functor $\Omega_T : Spt_T(Sm_S) \to Spt_T(Sm_S)$ taking a motivic spectrum E to $\Omega_T(E)$ defined as

$$\Omega_T(E)_n = \Omega_T(E_n) = \mathbf{Hom}_*(T, E_n)$$

with bonding maps $T \wedge \Omega_T(E)_n \to \Omega_T(E_{n+1})$ given by taking a adjoint of the composition map

$$T \wedge \Omega_T(E_n) \wedge T \xrightarrow{id \wedge ev} T \wedge \Omega_T(E_{n+1}) \to E_{n+1}$$

Here ev is the evaluation map $\mathbf{Hom}_*(T, E_n) \wedge T \to E_n$, which in turn is the map corresponding to $id_{\mathbf{Hom}_*(T,E_n)}$ via the adjunction

$$Hom(\mathbf{Hom}_*(T, E_n), \mathbf{Hom}_*(T, E_n)) \xrightarrow{\sim} Hom(\mathbf{Hom}_*(T, E_n) \wedge T, E_n)$$

There is another functor Ω^l_T which is called the *fake loop space* functor[[14], 2.3], which is defined as follows: $\Omega^l_T(E)_n = \Omega_T(E_n)$ and the bondings maps are adjoints of the map

$$\Omega_T(\sigma^*_n) : \Omega_T(E_n) \to \ \Omega_T(\mathbf{Hom}_*(T, E_{n+1}))$$

where σ^*_n is the adjoint of the map $E_n \wedge T \cong T \wedge E_n \xrightarrow{\sigma_n} E_{n+1}$

Because of the bonding maps above, we have maps

$$E \to \Omega^l_T(E[1]) \to (\Omega^l_T)^2(E[2]) \to ...$$

where, $E[m]_n := E_{m+n}$. Let $Q_T(E)$ denote the colimit of the above diagram. The functor Q_T is called the stabilization functor. We have a natural map $\eta : E \to Q_T(E)$.

Definition 2.4.3. *A map $f : E \to F$ between motivic spectra is*

- *A stable weak equivalence if the natural map $Q_T(JE) \to Q_T(JF)$ is a levelwise weak equivalence. The functor J is the fibrant replacement in the level wise model structure (i,e JE is levelwise fibrant)*
- *A stable fibration if it has right lifting property with respect to maps that are simultaneousy cofibrations(in the level wise model) and stable weak equivalences.*

With the above maps, the category of T-spectra has a proper simplicial model structure. This is proved in [[14], Theorem 2.9]

Theorem 2.4.4. *The category $Spt_T(Sm_S)$ of motivic spectra along with stable weak equivalences, stable fibrations and cofibrations forms a proper simplicial model category.*

We now consider the category of motivic symmetric spectra[[14], chap.4]. The motivation behind symmetric spectra is that there is a well defined smash product on it, with respect to which it becomes a symmetric monoidal category. We have a stable model structure, with respect to which the category is Quillen equivalent to the stable model category of motivic spectra.

Definition 2.4.5. *A motivic symmetric spectra is a motivic spectra E with an action of symmetric group $\Sigma_n \times E_n \to E_n$ such the maps $T^{\wedge p} \times E_m \to E_{m+p}$ is $\Sigma_p \times \Sigma_m$-equivariant(action of $\Sigma_p \times \Sigma_m$ on E_{m+p} comes from viewing $\Sigma_p \times \Sigma_m$ as a subgroup of Σ_{m+p}). A map of motivic symmetric spectra $f : E \to F$ is a map of motivic spectra which is levelwise equivariant. We denote this category as $Spt_T^{\Sigma}(Sm_S)$.*

We have a levelwise model structure similar to motivic spectra, with levelwise weak equivalence, levelwise cofibration and injective fibrations, which are the maps with right lifting with respect to levelwise cofibrations that are levelwise weak equivalences, resulting in the following theorem.[[14], theorem 4.2]

Theorem 2.4.6. *The category of $Spt_T^{\Sigma}(Sm_S)$ with levelwise weak equivalence, levelwise cofibration and injective fibrations forms a proper closed simplicial model category.*

We have a forgetful functor $U : Spt_T^{\Sigma}(Sm_S) \to Spt_T(Sm_S)$ with left adjoint V, which is called the symmetrization functor. This functor takes cofibrations in motivic spectra to level cofibration in symmetric spectra[[14], Lemma 4.3]. This allows us to define the stable model structure on $Spt_T^{\Sigma}(Sm_S)$

Chapter 3

Affine representability results in $\mathbb{A}^1$-homotopy theory

3.1 Representability result for Nisnevich G-torsors

Aim of this section is to prove the following theorem.

Theorem 3.1.1. *Suppose $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is a homotopy fiber sequence of pointed simplicial presheaves over Sm_S satisfying the following conditions:*

1. *$\mathcal{G}$ and $\mathcal{H}$ satisfy Nisnevich excision.*
2. *$\pi_0 G$ and $\pi_0 H$ are $\mathbb{A}^1$-invariant on affine schemes.*

Then, $\mathcal{F}$ is $\mathbb{A}^1$-naive(See section 3.1.2).

We will further specialize this theorem to the case of Nisnevich G-torsors. Using this, we prove affine representability of Nisnevich G-torsors under the assumption that $H^1_{Nis}(\_\_ , G)$ is $\mathbb{A}^1$-invariant.

3.1.1 Affine Zariski(Nisnevich-)descent

We introduced affine Zariski and affine Nisnevich cd-structures in definition 2.1.2. Also, we considered the Grothendieck topology generated by the cd-structures. The following theorem shows that the topology generated by affine Nisnevich(Zariski-)squares is in fact the Nisnevich(Zariski-)topology restricted to the category Sm_S^{aff} of affine schemes over the base S.

Theorem 3.1.2. *The topology τ_{affNis} generated by affine Nisnevich cd-squares coincides with the Nisnevich topology on Sm_S^{aff}.*

Proof. Refer to [[3], Proposition 2.3.2] $\square$

Let $i : Sm_S^{aff} \to Sm_S$ be the inclusion functor. This induces adjunction of the form

$$i^* : sPre(Sm_S) \rightleftharpoons sPre(Sm_S^{aff}) : i_*$$

which is a Quillen adjunction as i^* preserves cofibrations and weak equivalences. Let $(i^*, \mathbf{R}i_*)$ the total derived functors on the homotopy category.

Lemma 3.1.3. *The functor $\boldsymbol{R}i_*$ is fully faithful*

Proof. Proving $\mathbf{R}i_*$ is fully faithful is equivalent to proving that the counit map $i^*\mathbf{R}i_* \to id$ is an isomorphism. Recall that $\mathbf{R}i_*$ is a right Kan extension given by $\mathbf{R}i_*\mathcal{F}(X) = holim_{Y \in Sm_S^{aff}/X}\mathcal{F}(Y)$ where $X \in Sm_S$ and $\mathcal{F} \in sPre(Sm_S^{aff})$. So if X is affine, then $\mathbf{R}i_*\mathcal{F}(X) \cong \mathcal{F}(X)$,because X is the final object in the category Sm_S^{aff}/X. $\square$

Lemma 3.1.4. *Let τ be Zariski or Nisnevich topology. The functors i^* and $\boldsymbol{R}i_*$ preserve τ-local objects and is an equivalence when restricted to full subcategories of τ-local objects in $Ho(sPre(Sm_S^{aff})$ and $Ho(sPre(Sm_S))$.*

Proof. The claim that i^* preserves τ-local objects immediately follows from the characterization of local objects in terms of Čech descent(Theorem 2.1.5). Since $\mathbf{Map}(\mathcal{G}, \mathbf{R}i_*(\mathcal{F})) \simeq \mathbf{Map}(i^*(\mathcal{G}), \mathcal{F})$, $\mathbf{R}i_*$ preserves τ-local objects iff for every covering sieve $R \to X$ on Sm_S, $i^*(R) \to i^*(X)$ is a τ-local weak equivalence. It is easy to check that for a presheaf considered as a constant simplicial presheaf, being τ-local is same as being a τ-sheaf. Hence, i^* sends covering sieves to isomorphism of τ-sheaves. This is in turn equivalent to i_* preserving τ-sheaves of sets which follows by [[16], Exp 3, 2.2]. Thus the derived adjunction $(i^*, \mathbf{R}i_*)$ makes sense when restricted to full subcategories of τ-local objects, with the right adjoint $\mathbf{R}i_*$ being fully faithful. Now it's enough to prove that i^* is conservative.

Let f be a morphism of τ-local simplicial presheaves over Sm_S such that $i^*(f)$ is a weak equivalence. Since every separated scheme admits a τ-cover whose Čech nerve consists of affine schemes, it follows that f is a weak equivalence on Sm_S^{sep} and since every scheme admits a τ-cover whose Čech nerve consists of separated schemes, f becomes a weak equivalence on Sm_S.

$\square$

Theorem 3.1.5. *Let $\mathcal{F}$ be a simplicial presheaf on Sm_S.*

- *If $\mathcal{F}$ satisfies affine Zariski excision, then, $\mathcal{F}(X) \to R_{Zar}\mathcal{F}(X)$ is a weak equivalence for every $X \in Sm_S^{aff}$. Here R_{Zar} is a the fibrant replacement in the Zariski-local model category.*
- *If $\mathcal{F}$ satisfies Nisnevich excision, then $R_{Zar}\mathcal{F}$ is Nisnevich-local.*

Proof. As $\mathcal{F}$ satisfies affine Zariski excision, it is affine Zariski-local by theorem 2.1.7, $i^*(\mathcal{F})$ is Zariski local as i^* preserves Zariski-local objects. Hence, the map $i^*(\mathcal{F}) \to R_{Zar}i^*(\mathcal{F})$ is a weak equivalence. By previous lemma, $i^*(R_{Zar}\mathcal{F})$ is Zariski-local. Hence, $i^*(R_{Zar}\mathcal{F}) \to R_{Zar}i^*(R_{Zar}\mathcal{F})$ is a weak equivalence. Consider the following diagram,

$$\begin{tikzcd} i^*(\mathcal{F}) \arrow[r] \arrow[d] & i^*(R_{Zar}\mathcal{F}) \arrow[d] \\ R_{Zar}i^*(\mathcal{F}) \arrow[r] & R_{Zar}i^*(R_{Zar}\mathcal{F}) \end{tikzcd}$$

As $\mathbf{R}i_*$ preserves Zariski-local objects by previous lemma, i^* preserves Zariski-local weak equivalences, the bottom horizontal map in the diagram above is a weak equivalence and hence the top horizontal map is a weak equivalence by 2-out of-3 property. This proves the first part.

For the second part, notice that as $\mathcal{F}$ is Nisnevich local, $i^*(\mathcal{F})$ is also Nisnevich local. By the first part, $i^*(R_{Zar}\mathcal{F})$ is Nisnevich local. Now, $\mathbf{R}i_*i^*(R_{Zar}\mathcal{F}) \cong R_{Zar}\mathcal{F}$ because $\mathbf{R}i_*$ is fully faithful. As $\mathbf{R}i_*$ preserves Nisnevich local objects, $R_{Zar}\mathcal{F}$ is Nisnevich local.

□

The following lemma develops a relationship between affine Zariski(resp. affine Nisnevich) excision and $Sing^{\mathbb{A}^1}$ functor. The statement is true in a more general context. See [[3] 4.2.3].

Lemma 3.1.6. *Let $\mathcal{F}$ be a simplicial presheaf on Sm_S^{aff}. If $\pi_0(\mathcal{F})$ is $\mathbb{A}^1$-invariant and if $\mathcal{F}$ satisfies affine Zariski(resp. affine Nisnevich) excision then, $Sing^{\mathbb{A}^1}(\mathcal{F})$ satisfies affine Zariski(resp. affine Nisnevich) excision.*

Proof. See [[3], 4.2.4] □

Remark. If $\mathbf{I}$ is a small diagram and F: $\mathbf{I} \to sPre(Sm_S)$ such that F(i) satisfies affine Zariski(*resp.* affine Nisnevich) excision for every i $\in \mathbf{I}$. Then $holim(F(i))$ satisfies affine

Zariski(*resp.* affine Nisnevich) excision. This is a consequence of commutativity of homotopy limits.

3.1.2 Naive $\mathbb{A}^1$-homotopy classes

Let $\mathcal{F}$ be a simplicial presheaf on Sm_S. We have a canonical map

$$\pi_0(Sing^{\mathbb{A}^1})\mathcal{F}(X) \to [X, \mathcal{F}]_{\mathbb{A}^1}$$

The left hand side is called the set of naive $\mathbb{A}^1$-homotopy classes of maps from X to $\mathcal{F}$. It is the equivalence class of set of maps $X \to \mathcal{F}$ by the equivalence relation generated by $\mathbb{A}^1$-homotopies. In general this need not be a bijection for all $X \in Sm_S$ even if $\mathcal{F}$ is a representable presheaf.

Definition 3.1.7. *Let $\mathcal{F}$ be a simplicial presheaf on Sm_S and let $\tilde{\mathcal{F}}$ be the fibrant replacement in $Spc_S^{\mathbb{A}^1}$. $\mathcal{F}$ is said to be $\mathbb{A}^1$- naive if the canonical map $Sing^{\mathbb{A}^1}\mathcal{F}(X) \to \tilde{\mathcal{F}}(X)$ is a weak equivalence for all $X \in Sm_S^{aff}$.*

Lemma 3.1.8. *Let $\mathcal{F}$ be a Zariski-local simplicial presheaf on Sm_S. If $\mathcal{F}$ is $\mathbb{A}^1$ invariant on affines then it is $\mathbb{A}^1$-invariant for all schemes.*

Proof. Consider the map $\mathcal{F} \to \mathcal{F}(\_ \times \mathbb{A}^1)$. This is a map between Zariski local simplicial presheaves such that $i^*(\mathcal{F}) \to i^*(\mathcal{F}(\_ \times \mathbb{A}^1))$ is a weak equivalence between Zariski local objects. Hence $\mathcal{F} \to \mathcal{F}(\_ \times \mathbb{A}^1)$ is a weak equivalence by the argument in the proof of Lemma 3.4. □

Theorem 3.1.9. *Let $\mathcal{F}$ be a simplicial presheaf on Sm_S. Let us suppose that:*

- *$\mathcal{F}$ satisfies affine Nisnevich excision*
- *$\pi_0(\mathcal{F})$ is $\mathbb{A}^1$-invariant on affine schemes.*

Then $R_{Zar}Sing^{\mathbb{A}^1}\mathcal{F}$ is Nisnevich-local and $\mathbb{A}^1$-invariant and the canonical map

$$\pi_0\mathcal{F}(X) \to [X, \mathcal{F}]_{\mathbb{A}^1}$$

is an isomorphism for $X \in Sm_S^{aff}$

Proof. $Sing^{\mathbb{A}^1}\mathcal{F}$ satisfies affine Nisnevich excision by Lemma 3.1.6. Hence $R_{Zar}Sing^{\mathbb{A}^1}\mathcal{F}$ is Nisnevich local and the map $Sing^{\mathbb{A}^1}\mathcal{F}(X) \to R_{Zar}Sing^{\mathbb{A}^1}\mathcal{F}(X)$ is a weak equivalence for $X \in Sm_S^{aff}$ by theorem 3.1.5. Hence we have that $R_{zar}Sing^{\mathbb{A}^1}\mathcal{F}$ is $\mathbb{A}^1$-invariant on affines because $Sing^{\mathbb{A}^1}\mathcal{F}$ is $\mathbb{A}^1$-invariant. So by applying the previous lemma, we conclude that $R_{zar}Sing^{\mathbb{A}^1}\mathcal{F}$ is $\mathbb{A}^1$-invariant. As $\pi_0(\mathcal{F})$ is $\mathbb{A}^1$-invariant,

$$\pi_0 Sing^{\mathbb{A}^1}\mathcal{F}(X) \cong \pi_0\mathcal{F}(X) \cong [X, \mathcal{F}]_{\mathbb{A}^1}$$

for all $X \in Sm_S^{aff}$. □

Now we give a criteria for a sheaf to be $\mathbb{A}^1$-naive in terms of the $Sing^{\mathbb{A}^1}$ functor.

Theorem 3.1.10. *A simplicial presheaf $\mathcal{F}$ is $\mathbb{A}^1$-naive iff $Sing^{\mathbb{A}^1}(\mathcal{F})$ satisfies affine Nisnevich-excision.*

Proof. Suppose $\mathcal{F}$ is $\mathbb{A}^1$-naive then $Sing^{\mathbb{A}^1}(\mathcal{F})$ is weak equivalent to $\tilde{\mathcal{F}}$ when restricted to affine schemes. Hence it is Nisnevich-local(As, $\tilde{\mathcal{F}}$ is the Nisnevich-local, $\mathbb{A}^1$-invariant replacement of $\mathcal{F}$. Hence $Sing^{\mathbb{A}^1}(\mathcal{F})$ has affine Nisnevich excision by theorem 2.1.7

For the converse, let's assume that $Sing^{\mathbb{A}^1}\mathcal{F}$ satisfies affine Nisnevich excision. This implies that the canonical map $Sing^{\mathbb{A}^1}\mathcal{F}(X) \to R_{Zar}Sing^{\mathbb{A}^1}\mathcal{F}(X)$ is a weak equivalence for all $X \in Sm_S^{aff}$ by Theorem 3.1.6 and $R_{Zar}Sing^{\mathbb{A}^1}\mathcal{F}$ is Nisnevich-local. Also, $R_{Zar}Sing^{\mathbb{A}^1}\mathcal{F}$ is $\mathbb{A}^1$-invariant by Theorem 3.1.9. This implies that, $R_{Zar}Sing^{\mathbb{A}^1}\mathcal{F} \simeq \tilde{\mathcal{F}}$ and hence $\mathcal{F}$ is $\mathbb{A}^1$-naive. □

We are now ready to prove the theorem mentioned at the beginning.

Theorem 3.1.11. *Suppose $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is a homotopy fiber sequence of pointed simplicial presheaves over Sm_S satisfying the following conditions:*

1. *$\mathcal{G}$ and $\mathcal{H}$ satisfy Nisnevich excision.*
2. *$\pi_0 G$ and $\pi_0 H$ are $\mathbb{A}^1$-invariant on affine schemes.*

Then, $\mathcal{F}$ is $\mathbb{A}^1$-naive.

Proof. By the theorem 2.2.4,

$$Sing^{\mathbb{A}^1}\mathcal{F}(X) \to Sing^{\mathbb{A}^1}\mathcal{G}(X) \to Sing^{\mathbb{A}^1}\mathcal{H}(X)$$

is a homotopy fiber sequence for $X \in Sm_S^{aff}$. As both $\mathcal{G}$ and $\mathcal{H}$ satisfy affine Nisnevich excision, both $Sing^{\mathbb{A}^1}\mathcal{G}$ and $Sing^{\mathbb{A}^1}\mathcal{H}$ also satisfy affine Nisnevich excision by lemma 3.1.6. As a consequence of the commutativity of homotopy limits (look at the remark after lemma 3.1.6) $Sing^{\mathbb{A}^1}\mathcal{F}$ also satisfies affine Nisnevich excision. Hence, $\mathcal{F}$ is $\mathbb{A}^1$-naive by the previous theorem. □

The following result is not need for the rest of this section but we state it because it is interesting and is closely related to the previous results. See [[3], Theorem 2.2.5] for a proof.

Theorem 3.1.12. *Let $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be a homotopy fiber sequence of pointed simplicial presheaves on Sm_S. If:*

- *$\mathcal{H}$ satisfies affine Nisnevich excision*
- *$\pi_0(\mathcal{H})$ is $\mathbb{A}^1$-invariant on affine schemes.*

Then $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an homotopy fiber sequence in $Spc_S^{\mathbb{A}^1}$.

3.1.3 Application to G-torsors

We will specialize the general representability result in the case of G-torsors for some group G.

Definition 3.1.13. *Let (C, τ) be a small Grothendieck site and let G be τ-sheaf of groups on C. Let $X \in C$. A* ***G-torsor over X*** *is a triple $(\mathcal{E}, \pi, a)$ where $\mathcal{E}$ is a τ-sheaf on C, with a right action $a : \mathcal{E} \times G \to \mathcal{E}$ of G and a map $\pi : \mathcal{E} \to X$ which is G-equivariant for trivial action on X such that:*

1. *(free-action) the morphism $\mathcal{E} \times G \to \mathcal{E} \times_X \mathcal{E}$ coming from projection and the map a, is an isomorphism.*
2. *(local triviality) The collection of maps $U \to X$ such that $\mathcal{E} \times_X U \to U$ has a section is a τ-covering sieve of X (We call a map τ-locally split if it satisfies this condition)*

The collection of G-torsors over $X \in C$ can be assembled into a category $\mathbf{Tors}_\tau(G)$ fibered in groupoids over C (Refer to [18] for the definition). Let $B\mathbf{Tors}_\tau(G)$ be the simplicial presheaf such that $B\mathbf{Tors}_\tau(G)(X)$ is the nerve of the groupoid of sections of $\mathbf{Tors}_\tau(G)$ over C/X. You can think of this as the groupoid of G-torsors over X but the definition has functoriality built into it.

The category $\mathbf{Tors}_\tau$ is a stack for topology τ and this implies that $B\mathbf{Tors}_\tau(G)$ satisfies τ-descent by [[18], Theorem 3.9]

Let us denote BG as the pointed simpicial presheaf such that $BG_n := G^{\times n}$ with the face map

$$d_i(g_0, ..., g_{n-1}) = \begin{cases} (g_0, ..., g_{i-1}g_i, ...g_{n-1}) & \text{if } 0 < i \leq n-1 \\ (g_1, ..., g_{n-1}) & \text{if } i = 0 \\ (g_0, ..., g_{n-2}) & \text{if } i = n \end{cases}.$$

and the degeneracy maps

$$s_i(g_0, ..., g_{n-1}) = \begin{cases} (g_0, ..., g_{i-1}, e, g_i...g_{n-1}) & \text{if } 0 < i \leq n-1 \\ (e, g_0, ..., g_{n-1}) & \text{if } i = 0 \\ (g_0, ..., g_{n-1}, e) & \text{if } i = n \end{cases}.$$

BG is the quotient of an objectwise contractible simplicial presheaf EG, $BG = G/EG$, with G acting freely on BG. This defines a G-torsor $EG \to BG$, which is classified by a map $BG \to B\mathbf{Tors}_\tau(G)$. The simplicial presheaf EG is given by $EG_n := G^{\times n+1}$, with face maps $d_i(g_0, ..., g_n)$ the projection omitting g_i, and the degeneracy $s_j(g_0, ...g_n) = (g_0, ..., g_{i-1}, g_i, g_i, g_{i+1}, ..., g_n)$. The G action on EG_n is $g \cdot (g_0, . . ., g_n) = (g \cdot g_0, . . ., g \cdot g_n)$. Let $B_\tau G$ be the τ-local replacement of BG. As $B\mathbf{Tors}_\tau(G)$ is τ-local, we get a map

$$B_\tau G \to B\mathbf{Tors}_\tau(G)$$

. The next lemma shows that the above map is infact a weak equivalence.

Lemma 3.1.14. *Let (C, τ) be a small site and G be a τ-sheaf of groups on C. Then:*

1. *The map $B_\tau G \to B\,\boldsymbol{Tors}_\tau(G)$ is a weak equivalence.*
2. *There is a natural isomorphism $H^1_\tau(\_, G) \cong \pi_0(B_\tau G)(\_)$*
3. *There is a canonical weak equivalence $\boldsymbol{R}\Omega B_\tau G \simeq G$. Here, $\boldsymbol{R}\Omega$ is the right derived functor of the loop space functor.*

Proof. Refer to [[4],lemma 2.3.2] □

There is a relationship between the first non-abelian cohomology group and groupoid of G-torsors. We advise the reader to refer [[17], III. 4] for an explanation.

Let G be an S-group scheme and let X be an S-scheme. By a G-torsor over X we will mean the same as Definition 3.13 with C being the category of S-schemes with τ being the fppf topology. If G is affine over S, any G-torsor over X is automatically representable by an S-scheme. [See [17], III, theorem 4.3].

But if we restrict to the case where X and G are in Sm_S, then taking C to be Sm_S and τ to be the etale topology, we get an equivalent definition of torsors. This is because of the following lemma.

Lemma 3.1.15. *Let G be an affine S-group scheme. Let $\pi : \mathcal{E} \to X$ be a torsor over X with $X \in Sch_S$. If $G \to S$ is finitely presented, flat , or smooth, then so is $\pi : \mathcal{E} \to X$*

Proof. It follows from the fact that the properties are preserved under base change and are fppf-local on target. [See [4], lemma 2.3.3] □

Now we state and prove the affine representability for Nisnevich locally trivial G-torsors under the assumption that $H^1_{Nis}(\_, G)$ is $\mathbb{A}^1$-invariant on Sm_S^{aff}

Theorem 3.1.16. *Let G is a finitely presented smooth S-group scheme. If $H^1_{Nis}(\_, G)$ is $\mathbb{A}^1$-invariant on Sm_S^{aff}, then*

- *The simplicial presheaf $R_{Zar}Sing^{\mathbb{A}^1}B_{Nis}G$ is Nisnevich-local and $\mathbb{A}^1$-invariant*
- *For every affine $X \in Sm_S^{aff}$, the canonical map $H^1_{Nis}(\_, G) \to [X, BG]_{\mathbb{A}^1}$ is bijective and functorial with respect to X.*

Proof. The statements follows from Theorem 3.1.9 because $B_{Nis}G$ is Nisnevich local, hence satisfying the affine Nisnevich excision by Theorem 2.1.7 and $\pi_0(B_{Nis}G) \cong H^1_{Nis}(\_, G)$ by Lemma 3.1.14 and hence is $\mathbb{A}^1$-invariant. Also, $BG \to B_{Nis}G$ is a weak equivalence in $Spc_S^{\mathbb{A}^1}$, which means that $[X, BG]_{\mathbb{A}^1} \cong [X, B_{Nis}G]_{\mathbb{A}^1}$.

□

3.2 A formalism for homotopy invariance

We have proved the representability result under the assumption that $H^1_{Nis}(\_, G)$ is $\mathbb{A}^1$-invariant. In this section, we prove that the above assumption holds under some hypothesis on the group scheme G.

3.2.1 Recollections on group schemes

In this section, we recollect a few definitions about group schemes and state a condition under which the group schemes are *linear*. We will only consider group schemes over spec of a ring R. Many of our the theorems for group schemes over $Spec(R)$ become very simple when R is a field. In general, there is a tradeoff between the assumptions on the group scheme and the base ring we work with.

Definition 3.2.1. *We write $GL_{n,R}$ for the general linear group scheme over R and $\mathbb{G}_{n,R}$ for $GL_{1,R}$. A* ***linear R-group scheme*** *is a group scheme over R with a finitely presented closed immersion group homomorphism $G \to GL_{n,R}$.*

We refer the reader to [[4]] and [19] for the definition of reductive, split and isotropic R-group schemes.

The following theorem gives a criteria for an R-group scheme to be linear. Refer to [[4], Proposition 3.1.3] for the theorem and references for a proof.

Theorem 3.2.2. *Let G be a reductive R-group scheme. Then G is a linear R-group scheme if one of the following assumption holds:*

1. *R is Noetherian and regular*
2. *G is split*

Notice that if R is a field then G is linear by the above theorem.

3.2.2 The local-to-global principle

In this section we will provide an analogue of Quillen's local to global principle for torsors under linear R-groups schemes. The main result is the multivariable analogue of [[25], Theorem 1], which was formulated by Quillen to prove the Serre's conjecture(Quillen-Suslin theorem). Refer to [20] for an account of Serre's conjecture and [[20], Chapter V] for Quillen patching.

The important result we will need from this section is that for a reductive R-group scheme G, local to global principle holds when R is regular Noetherian or G is split.

The following is a generalization of the lemma [[20], Lemma 1]. Refer to [[4], section 3.2]

Lemma 3.2.3. *Let G be a linear R-group scheme for a commutative ring R. Let $g \in R$ and let $f(t) \in G(R_g[t])$ such that $f(0) = 1 \in G(R_g)$. There exists an integer $s \geq 0$ such that for any $c, d \in R$ with $c - d \in f^s R$, we have $h(t) \in G(R[t])$ with $h(0) = 1$ and such that $h_g(t) = f(ct)f(dt)^{-1} \in G(R_g[t])$*

Proof. See [[4], Lemma 3.2.1] □

Lemma 3.2.4. *Let G be a linear R-group scheme for a commutative ring R. Given $g_0, g_1 \in R$ such that $g_0R+g_1R = R$, and $f(t) \in G_{f_0f_1}[t]$ with $f(0) = 1$, then there exists $h_i(t) = G(R_{f_i}[t])$ for $i = 0, 1$ with $h_i(0) = 1$ such that $f(t) = h_0(t)(h_1(t))^{-1}$*

Proof. See [[4], Lemma 3.2.2] □

Let G be a linear R-group scheme for a commutative ring R.For a commutative R-algebra A, by a G-torsor over A we mean a G-torsor over *specA*. By assumptions our torsors are fppf-locally trivial.

We say that a G-torsor over $A[t_1, ..., t_n]$ is *extended from* A if it is pulled back from a G-torsor over A.

Lemma 3.2.5. *Consider a commutative ring R. Let $\mathcal{P}$ be a G-torsor over $R[t]$. The set $Q(\mathcal{P})$ consisting of $g \in R$ such that $\mathcal{P}|_{SpecR_g[t]}$ is extended from R_g forms an ideal of R*

Theorem 3.2.6. *(Local-to-global principle) Let G be a linear R-group scheme for a commutative ring R. If $\mathcal{P}$ is a G-torsor over $R[t_1, ..., t_n]$, then*

- A_n *The set $Q(\mathcal{P})$ consisting of $g \in R$ such that $\mathcal{P}|_{SpecR_g[t_1,...t_n]}$ is extended R_g is an ideal in R.*
- B_n *If $\mathcal{P}|_{SpecR_{\mathrm{m}}[t_1,...,t_n]}$ is extended for every maximal ideal $\mathrm{m} \subset R$, then $\mathcal{P}$ is extended*

Proof. The statement (A_1) holds by previous lemma. We now show that $(A_n) \implies (B_n)$. It is enough to check that if $\mathcal{P}$ satisfies the hypothesis of (B_n), then $Q(\mathcal{P})$ is a unit ideal.

Let $\mathcal{P}_0$ be the pull back of $\mathcal{P}$ along the zero section map $SpecR \to SpecR[t_1, ..., t_n]$ and let $\mathcal{P}^{'}$ be the pullback along the map $SpecR[t_1, ...t_n] \to SpecR$.

By the hypothesis in (B_n), $\mathcal{P}|_{SpecR_m[t_1,...t_n]}$ is extended for every maximal ideal $m \subset R$. Hence we know that there exists an isomorphism $\phi : \mathcal{P}|_{SpecR_m[t_1,...t_n]} \to \mathcal{P}^{'}|_{SpecR_m[t_1,...t_n]}$. As G-torsors over affine base are finitely presented by lemma 3.1.15, there exist $f \in R - m$ such that ϕ is the localization of an isomorphism of G-torsors over $SpecR_g[t_1, ...t_n]$. Hence it follows that $g \in Q(\mathcal{P} - m)$ and that $Q(\mathcal{P})$ is not contained in any maximal ideal. So $Q(\mathcal{P}) = R$

Now we show that $(A_1) \implies (A_n)$. We use induction on n. Let's assume that (A_n) is true and hence B_n is true by the previous argument. Consider $Q(\mathcal{P})$ for (A_n). It is easy to check that $R.Q(\mathcal{P}) \subset Q(\mathcal{P})$ so we need to show that for $g_1, g_2 \in Q(\mathcal{P})$, $g_1 + g_2 \in Q(\mathcal{P})$. Let us write $g = g_1 + g_2$. Let $\mathcal{P}|_{t_n}$ be the restriction of $\mathcal{P}$ along the map of schemes induced by the quotient map $R[t_1, ...t_n] \to R[t_1, ...t_{n-1}]$. We apply (A_1) on the map $R[t_1, ...t_n] \to R[t_1, ...t_{n-1}][t_n]$ and conclude that $\mathcal{P}_g$ is extended from $(\mathcal{P}|_{t_n})_g$. Now it is enough to prove that $(\mathcal{P}|_{t_n})_g$ is extended from R_g. As (B_{n-1}) holds, it suffices to show that $(\mathcal{P}|_{t_n})_g$ is extended for every maximal ideal in R_g. Let p be the preimage of m under the localization map $R \to R_g$. Since $g \notin \mathrm{p}_g$, it follows that one of g_0 and g_1 is not in p. Let's assume that $g_0 \notin$ p. But by assumption, $\mathcal{P}_{g_0}$ is extended from $(\mathcal{P}_0)_{g_0}$. Hence, the restriction of $(\mathcal{P}|_{t_n})_g$ to m is extended from $(\mathcal{P}_0)_{\mathrm{p}}$, which proves our claim.

□

By Theorem 3.2.2, A reductive R-group scheme G is linear when R is regular Noetherian or if G is split. Hence local-to-global principle hold in those cases by the previous theorem.

3.2.3 A formalism for homotopy invariance

Lemma 3.2.7. *Let k be an infinite base field. Suppose $\mathbf{F}$ be a functor from the category of k-algebras to the category of pointed sets satisfying the following conditions:*

A1 $\mathbf{F}$ *commutes with filtered colimit of rings with flat transition morphisms.*

A2 *For every field extension L/k the map*

$$\mathbf{F}(L[t_1, ..., t_n] \to \mathbf{F}(L(t_1, ..., t_n))$$

has trivial kernel for all $n \geq 0$.

A3 *For any smooth k-algebra A, any etale A-algebra B, and for any $g \in A$ such that $A/gA \cong B/gB$ the map*

$$ker(\mathbf{F}(A) \to \mathbf{F}(A_g)) \to ker(\mathbf{F}(B) \to \mathbf{F}(B_g))$$

is a surjection.

Let B be the localization of a smooth k-algebra at a maximal ideal and $K_B = Frac(B)$. then the map

$$\mathbf{F}(B[t_1, ..., t_n] \to \mathbf{F}(K_B(t_1, ..., t_n))$$

has a trivial kernel for all $n \geq 0$

Proof. See [[4], Proposition 3.3.4] □

Now we prove the $\mathbb{A}^1$-invariance of the functor $H^1_{Nis}(\_, G)$ for affine schemes.

Theorem 3.2.8. *Let G be a isotropic reductive k-group scheme(See [[4], 3.3.5] for the definition) for an infinite field k. For any smooth k-algebra A, the map*

$$H^1_{Nis}(SpecA, G) \to H^1_{Nis}(SpecA[t_1, ..., t_n], G)$$

is a bijection for all $n \geq 0$.

Proof. Our aim is to show that every Nisnevich locally trivial G-torsor $\mathcal{P}$ over $A[t_1, ..., t_n]$ is extended from A. By Theorem 3.2.2 and local-to global principle, it is enough to show that, for every maximal ideal $\mathrm{m} \in A$, the G-torsor $\mathcal{P}_\mathrm{m}$ over $A_\mathrm{m}[t_1, ..., t_n]$ is extended from A_m.

See [[4], theorem 3.3.7] which shows that the functor $A \mapsto H^1_{Nis}(SpecA, G)$ from k-algebras to pointed sets satifies the axioms **A1-A3**. Hence by theorem 3.2.7, it suffices to show that $\mathcal{P}_\mathrm{m}$ is trivial over $Frac(A_\mathrm{m})(t_1, ..., t_n)$. This follows from the fact that a field has no non trivial Nisnevich covering sieve. □

Chapter 4

The triangulated category of motives

In this section we describe Voevodsky's construction of triangulated category of motives and motivic cohomology. The main idea of the construction is the introduction of the category of finite correspondences. We compute motivic cohomology as the cohomology of certain sheaves with transfer.

4.1 Finite correspondences

In this section we construct an additive category Cor_k of finite correspondences over a field k. The idea of finite correspondences is motivated from Grothendieck's smooth correspondences. Instead of considering rational equivalence classes, we consider the cycles themselves to define the morphism in the category Cor_k. Before we proceed to define the category of finite correspondence, we will recall a few basic ideas about algebraic cycles.

4.1.1 Algebraic cycles

We recall the definition of algebraic cycles on the category Sch_k of schemes over a field k. Our main aim would be to define push forward of cycles. We refer to [24] for this subsection.

Definition 4.1.1. *Let $X \in Sch_k$. An **algebraic cycle**(of dimension n) on X is an element of the free abelian group $Z_n(X)$ generated by the closed integral subschemes W of X with $dim_k W = n$. So an algebraic cycle is of the form $\Sigma_i n_i W_i$ where W_i is a closed integral subscheme of X with dimension n. We write $Z_*(X)$ for the graded group $\oplus_n Z_n(X)$ and an element of this group is called the algebraic cycle on X*

If X is locally equi-dimensional, we can define $Z^n(X)$ the free abelian group generated by the closed integral subschemes of X with codimension n.

We define the *support* of an algebraic cycle $\Sigma_i n_i W_i$ to be the union $\cup_i |W_i|$ and denote it by $|W|$.

We can define push forward of cycles along proper maps. Let $f : X \to Y$ be a proper morphism. Then for an integral closed subscheme W of X, $f(W)$ is an irreducible closed subset of Y and an integral closed subscheme on Y with the reduced induced subscheme structure. Also, we have that $k(W)$ is a finitely generated field extension of $k(f(W))$ because of the map at the level of stalks (Here,$k(W)$ is the stalk at the generic point of W, which is always a field). We have the push forward $f_*(W) \in Z_n(X)$ by

$$f_*(W) = \begin{cases} 0 & \text{if } dim_k W > dim_k f(W) \\ [k(W) : k(f(W))] \cdot f(W) & \text{if } dim_k W = dim_k f(W) \end{cases}.$$

This gives a push forward map on cycles $f_* : Z_n(X) \to Z_n(Y)$ by linearity.

4.1.2 Finite correspondences and presheaves with transfer

We will follow [[2]] for this section and follow the conventions in it. We consider the category of smooth separated schemes Sm/k. An algebraic cycle on a scheme X is the formal $\mathbb{Z}$-linear combination of irreducible closed subsets of X. For every irreducible closed subset W, we have an integral closed subscheme $\tilde{W}$ determined by the reduced induced subscheme structure on W.

Definition 4.1.2. *Let X be a smooth connected scheme over k and Y be any separated scheme over K. An elementary correspondence from X to Y is an irreducible closed subset W of $X \times Y$ such that its associated integral subscheme is finite and surjective over X.If X is non-connected then we define the elementary correspondence from X to Y to be an elementary correspondence from a connected component of X to Y.*

The group of finite correspondences $Cor(X, Y)$ from X to Y is defined as the free abelian group generated by the elementary correspondences from X to Y.

Remark. For a morphism $f : X \to Y$ in Sm/k, the graph Γ_f is an elementary correspondence from X to Y as $\Gamma_f \to X$ is an isomorphism and $\Gamma_f \to Y$ is closed because Y is separated.

The category of finite correspondence Cor_k is the category with objects as smooth schemes over k and with morphims as $Cor(X, Y)$ for $X, Y \in Sm/k$. We now define compositions in this category.

Given elementary correspondences $V \in Cor(X, Y)$ and $W \in Cor(Y, Z)$, we can form the intersection product $[T] = (V \times Z) \cdot (X \times W)$ in $X \times Y \times Z$(See [24] for the definition of intersection product). The composition $W \circ V$ of V and W is the push forward of $[T]$ along the map $p : X \times Y \times Z \to X \times Z$. The push forward $p_*[T]$ is finite over $X \times Z$. This is a finite correspondance from X to Z by [[2], lemma 1.4]

It follows that Cor_k is an additive category with a zero object ϕ and disjoint union as a coproduct. By the remark above, we have a faithful functor $Sm/k \to Cor_k$ defined by identity on objects and takes a map $f : X \to Y$ in Sm_k to its graph Γ_f.

We have a tensor product on Cor_k which is defined as the product of underlying schemes

$$X \otimes Y := X \times Y$$

. If we have elementary correspondences W_1 from X_1 to Y_1 and W_2 from X_2 to Y_2, then the cycle associated to $V \times W$ gives a finite correspondence from $X_1 \otimes Y_1$ to $X_2 \otimes Y_2$. This makes Cor_k into a symmetric monoidal category

Definition 4.1.3. *A* ***Presheaf with transfer*** *is a contravariant additive functor $F : Cor_k^{op} \to$* ***Ab*** *from the category of finite correspondences to the category of Abelian groups. The category of presheaves with transfer* ***PST****(k) is a category with objects as presheaves with transfer and morphisms as natural transformation.*

The following is a lemma a special case of a result for functor categories. Refer to [[6] 1.6.4] and [[6] exercises 2.3.7 and 2.3.8]

Theorem 4.1.4. *The category* ***PST****(k) is an abelian category with enough injectives and projectives.*

By the Yoneda lemma, we have that the following functor

$$Sm_k \to Cor_k \to \mathbf{PST}(k)$$

. We denote this functor as $\mathbb{Z}_{tr} : Sm_k \to Cor_k$ and $\mathbb{Z}_tr(X)$ is the presheaf with transfer represented by X such that $\mathbb{Z}_{tr}(X)(U) = Cor_k(U, X)$. We write $\mathbb{Z}$ for $\mathbb{Z}_{tr}(Spec(k))$ and hence we have a map $\mathbb{Z}_{tr}(X) \to \mathbb{Z}$ for every $X \in Sm_k$

For a pointed scheme (X, x), we define $\mathbb{Z}_{tr}(X, x)$ as the cokernel of the map $x_* : \mathbb{Z} \to \mathbb{Z}_{tr}(X)$ coming from the point $x : Spec(k) \to X$. The map x_* splits the map $\mathbb{Z}_{tr}(X) \to \mathbb{Z}$, we have $\mathbb{Z}_{tr}(X) = \mathbb{Z}_{tr}(X, x) \oplus \mathbb{Z}$

Let (X_i, x_i) be pointed schemes in Sm_k. We define $\mathbb{Z}_{tr}(X_1 \wedge ... \wedge X_n)$ as:

$$Coker(\oplus_i \mathbb{Z}_{tr}(X_1 \times ...\widehat{X_i}... \times X_n) \xrightarrow{id \times ...x_i ... \times id} \mathbb{Z}_{tr}(X_1 \times ... \times X_n))$$

We use the cosimplicial scheme $\Delta^\bullet$ to define a chain complex associated to a presheaf with transfer. Recall that $\Delta^n = Speck[x_0, ..., x_n]/(\Sigma_{i=0}^n x_i = 1)$

Let F be a presheaf with transfer. We associate a simplicial presheaf with transfer $C_\bullet F$ defined as $(C_\bullet F(U)_n = F(U \times \Delta^n)$. We define a chain complex $C_* F$ with $C_n F(X) = F(X \times \Delta^n)$ and the differentials given by taking alternating sum of the face maps in $\Delta^\bullet$.

For a simplicial abelian group A, we associate a normalized chain complex $A_\bullet^{DK}$ such that $A_n^{DK} = \cap_{i=0,...n} ker(d_i)$ with $d_i : A_n \to A_{n-1}$ and the differentials $\partial_i : A_n^{DK} \to A_{n-1}^{DK}$ given by the restriction of d_0. Hence, for a simplical presheaf $C_\bullet F$, we have the so-called normalized chain complex $C_\bullet^{DK} F$ which is quasi isomorphic to $C_* F$.

In the construction of the triangulated category of motives, homotopy invariant presheaves play a central role. Notice that this notion is similar to the notion of $\mathbb{A}^1$-invariant simplicial presheaves defined in the previous chapters.

Definition 4.1.5. *A presheaf F of abelian groups is said to be* ***homotopy invariant*** *if for every projection of the form $X \times \mathbb{A}^1 \to X$, the map $F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism.*

Definition 4.1.6. *We say that two finite correspondences $f, g \in Cor(X, Y)$ are $\mathbb{A}^1$-homotopic if there exists a correspondence $H \in Cor(X \times \mathbb{A}^1, Y)$ such that the restriction to $X \times 0$ and $X \times 1$ is f and g respectively. We say that a map $f : X \to Y$ is an $\mathbb{A}^1$-**homotopy equivalence** if there exists a map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are $\mathbb{A}^1$-homotopic to identity.*

Lemma 4.1.7. *Let $f : X \to Y$ is an $\mathbb{A}^1$-homotopy equivalence with an homotopy inverse*

$g : Y \to X$. *Then,* $f_* : C_*\mathbb{Z}_{tr}(X) \to C_*\mathbb{Z}_{tr}(Y)$ *is a chain homotopy equivalence with the chain homotopy inverse* $g_* : C_*\mathbb{Z}_{tr}(Y) \to C_*\mathbb{Z}_{tr}(X)$ *induced by* g.

Proof. Refer to [[2], lemma 2.26] □

The category **PST**(k) has a tensor product structure. Notice that if F and G are in **PST**(k), we can define $F \otimes G(X) = F(X) \otimes_{\mathbb{Z}} G(X)$. But this construction does not have transfers. Hence we need a more complicated construction.

Let F be a presheaf with transfer. Let $\mathbb{Z}_F$ be the set of pairs of the form $(X, s \in F(X))$ for $X \in Sm_k$. By Yoneda lemma, sections $s \in F(X)$ are in bijection with maps $\mathbb{Z}_{tr}(X) \to F$ in $\mathbf{PST}(k)$. There is a canonical surjection

$$\oplus_{(X,s)\in\mathbb{Z}_F}\mathbb{Z}_{tr}(X) \to F \qquad \ldots .(*)$$

. This is because of universality of coproduct and surjection comes from noticing that $F(Y)$ is in bijection with maps $\mathbb{Z}_{tr}(Y) \to F$.

We can iterate this construction on the kernel of the map (*) and get a resolution $\mathcal{L}_\bullet(F) \to F$. We define tensor product on representables as $\mathbb{Z}_{tr}(X) \otimes^{tr} \mathbb{Z}_{tr}(Y) = \mathbb{Z}_{tr}(X \times Y)$ and on presheaves with transfer F and G as $F \otimes^{tr} G = H_0(Tot(\mathcal{L}_\bullet(F) \otimes^{tr} \mathcal{L}_\bullet(G)))$

We define $\underline{Hom}$ presheaves to be:

$$\underline{Hom}(F, G)(X) = Hom_{PST(k)}(F \otimes^{tr} \mathbb{Z}_{tr}(X), G)$$

The functor $F \otimes^{tr} \_$ is the left adjoint of $\underline{Hom}(F, \_)$. [[2], lemma 8.3]

4.1.3 Tensor triangulated categories

Before we proceed, we recall a few basic definitions on triangulated categories. One basic example of a triangulated category is the derived category of an Abelian category. One main idea that we will briefly outline is that of localizations in triangulated category. The Voevodsky's category of effective motives will be the localization of some derived category.

We consider an additive category $\mathbf{C}$ with a notion of *shift*, which is an auto equivalence $\Sigma : \mathbf{C} \to \mathbf{C}$. We write $X[1] := \Sigma X$. An additive functor $F : \mathbf{C} \to \mathbf{D}$ between additive categories with shift is said to be *graded* if $F(X[1]) = F(X)[1]$.

Let $\mathbf{C}$ be an additive category with shifts. A triangle in $\mathbf{C}$ is a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

. A morphism between triangles is a commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow & & \downarrow & & f\downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

Definition 4.1.8. *A* ***triangulated category*** *is an additive category* $\boldsymbol{A}$ *with shifts, together with a collection* $\mathcal{D}$ *of triangles called the distinguished triangles in* $\boldsymbol{C}$ *such that they satisfy the following axioms:*

TC1 $\mathcal{D}$ *is closed under isomorphism of triangles*

TC2 Every $X \xrightarrow{f} Y$ *extends to a distinguished triangle* $X \xrightarrow{f} Y \to Z \to X[1]$ *and* $A \xrightarrow{id} A \to 0 \to A[1]$ *is a distinguished triangle.*

TC3 $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *is a distinguished triangle if and only if* $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ *is distinguished.*

TC4 Given a commutative diagram with distinguished triangles as rows

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow & & \downarrow & & & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

there exists morphism $h : Z \to Z'$ *such that*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow & & \downarrow & & f\downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

is morphism of triangles.

TR5 (Octahedral axiom) Given distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z' \xrightarrow{h} X[1]$, $Y \xrightarrow{u} Z \xrightarrow{v} X' \xrightarrow{w} Y[1]$ *and* $X \xrightarrow{u \circ f} Z \xrightarrow{a} Y' \xrightarrow{b} X[1]$, *there exists a distinguished triangle*

$$Z' \xrightarrow{k} Y' \xrightarrow{l} X' \xrightarrow{m} Z'[1]$$

such that $v = l \circ a$, $h = b \circ k$, $m = g[1] \circ w$, $w \circ l = f[1] \circ b$, $k \circ g = a \circ u$

Definition 4.1.9. *A* ***tensor triangulated category*** *is a triangulated category* $\boldsymbol{C}$ *with symmetric monoidal tensor product* $\otimes$ *with natural isomorphisms* $X \otimes Y[1] \xrightarrow{\cong} (X \otimes Y)[1] \xleftarrow{\cong} X[1] \otimes Y$ *for* $X, Y \in \boldsymbol{C}$ *such that the following condition is satisfied:*

TTC1 For any distinguished triangle $X \to Y \to Z \to X[1]$ *and any* $C \in \boldsymbol{C}$*, the following triangle is distinguished*

$$X \otimes C \to Y \otimes C \to Z \otimes C \to X[1] \otimes C$$

The idea of Verdier localization is a way to construct new triangulated categories by inverting a set of morphisms.

Let us consider a full additive subcategory $\mathbf{B}$ of a triangulated category $\mathbf{C}$. $\mathbf{B}$ is called a *thick* subcategory of $\mathbf{C}$ if

- $\mathbf{B}$ is closed under taking direct summands.
- Let $X \to Y \to Z \to X[1]$ is a distinguished triangle. Then if two out X, Y, Z are in $\mathbf{B}$, then so is the third.

Let $\mathbf{B}$ be a thick subcategory of a triangulated category $\mathbf{C}$. Let $\mathbf{W}$ be the collection of maps $f : X \to Y$ in $\mathbf{A}$ such that when it is completed to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, Z lies in $\mathbf{B}$. This forms a multiplication system of morphisms [[6],10.3.4]. We can form a localised category $\mathbf{C}[\mathbf{W}^{-1}] = \mathbf{C}/\mathbf{D}$ with the same objects as $\mathbf{C}$ and maps

$$Hom_{\mathbf{C}[\mathbf{W}^{-1}]}(X, Y) = \lim_{X' \to X \in S} Hom_{\mathbf{C}}(X', Y)$$

Refer to [[6], 10.3.7] for a general construction of localized categories.

Let $L_S : \mathbf{C} \to \mathbf{C}/\mathbf{D}$ be the canonical functor.

Theorem 4.1.10. *(Verdier) Let* $\boldsymbol{B}$ *be a thick subcategory of a triangulated category* $\boldsymbol{C}$*. Then* $\boldsymbol{C}/\boldsymbol{B}$ *is a triangulated category such that*

- T *is a distinguished triangle if and only if it is isomorphic to the image of a distinguished triangle in* $\boldsymbol{C}$ *under* L_W
- *The functor* L_W *is universal. If* $F : \boldsymbol{C} \to \boldsymbol{D}$ *is an exact functor such that* $F(B)$ *is isomorphic to* 0 *for all* $Bin\boldsymbol{B}$*, then* F *factors through* L_S *uniquely.*
- $\boldsymbol{B}$ *is the subcategory of objects in* $\boldsymbol{C}$ *that become isomorphic to zero in the localized category* $\boldsymbol{C}/\boldsymbol{B}$

Remark. Let $\mathbf{C}$ be a tensor triangulated category. If the thick subcategory $\mathbf{B}$ is such that $A \otimes B$ is in $\mathbf{B}$ whenever A or B is in $\mathbf{B}$ then, the localized category $\mathbf{C}/\mathbf{B}$ is also tensor triangulated with the tensor product inherited from $\mathbf{C}$.

4.2 The triangulated category of motives

In this section we will define motivic cohomology groups, which will be Zariski-hypercohomology with respect to some chain complexes described below.

Definition 4.2.1. *We define the motivic complex* $\mathbb{Z}(n)$ *for every* $n \geq 0$ *as the following complex of presheaves with transfer:*

$$\mathbb{Z}(n) := C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})[-n]$$

. *Here* $\mathbb{G}_m$ *is* $(\mathbb{A}^1 - 0, 1)$ *and the shifting convention* $[-n]$ *means* $(C[-n])_i = C_{n+i}$ *for a bounded above chain complex* $C_\bullet$ *for* $i > n$ *and* $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$ *if* $i = n$*. The differential* $d_{C[-n]} : (C[-n])_i \to (C[-n])_{i-1}$ *is given by* $(-1)^n d_C : C_{n+i} \to C_{n+i-1}$

Now we recall a few definitions related to hypercohomology. Details can be found in [[17], Appendix C].

Let us consider the global section functor, $\Gamma(X,) : Sh(X) \to \mathbf{Ab}$ from category of sheaves over a scheme X to the category of abelian groups. This functor is left exact. Let $C^\bullet$ be a bounded below cochain complex of sheaves. This admits a quasi isomorphism $I^\bullet$ such that

I^n's are injective. The hypercohomology groups are the hyper right derived functors of the global section functor.

$$\mathbb{H}^n(X, C^\bullet) = \mathbb{R}^n\Gamma(X, C^\bullet)$$

There are spectral sequences that connect ordinary sheaf cohomology to hypercohomology [[17], Appendix C].

$$E_1^{p,q} = H^q(X, C^p) \implies \mathbb{R}^{p+q}\Gamma(X, C^\bullet)$$

and

$$E_2^{p,q} = H^p(X, (H^q(C^\bullet)) \implies \mathbb{R}^{p+q}\Gamma(X, C^\bullet)$$

Definition 4.2.2. *Let $X \in Sm_k$. The $(p,q)^{th}$* ***motivic cohomology group*** *$H^{p,q}(X, \mathbb{Z})$ is the p^{th} Zariski-hypercohomology of the motivic complex $\mathbb{Z}(q)$*

$$H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q))$$

We say that a presheaf with transfer F is a **Nisnevich sheaf with transfer** if the underlying presheaf is a Nisnevich sheaf on Sm_k. We denote this by Shv^{tr}_{Nis}. We have a notion of tensor product $\otimes_{Nis}$ on this category which is defined as the sheafification of the tensor product on underlying presheaves.

$$F \otimes_{Nis} G = a(l(F) \otimes^{tr} l(G))$$

where, a is the sheafification functor and l is the forgetful functor. This makes Shv^{tr}_{Nis} into a symmetric monoidal category.
We have an internal hom on Shv^{tr}_{Nis} defined as follows.

$$\underline{Hom}(F, G)(X) = Hom_{Shv^tr_{Nis}}(F \otimes_{Nis} \mathbb{Z}_t r(X), G)$$

.

Consider the category of bounded above cochain complexes $Ch^-(\mathcal{A})$ for an abelian category $\mathcal{A}$. A map $f : X \to Y$ is called a quasi-isomorphism if it induces isomorphisms on cohomology groups

$$H^n(f) : H^n(X) \to H^n(Y)$$

for all $n \geq 0$. The derived category $D^-(\mathcal{A})$ of $\mathcal{A}$ is the localization of $Ch^-(\mathcal{A})$ with respect to quasi-isomorphisms. This category has a canonical triangulated structure.[[6], Chapter 10]

In the construction of the triangulated category of motives, we consider the derived category $D^-(Shv^{tr}_{Nis})$ of Nisnevich sheaves with transfer.
First notice that the category $Ch^-(\mathbf{PST(k)})$ has a tensor product defined as:

$$(F \otimes G)^n := \oplus_{i+j=n} F^i \otimes^{tr} G^j$$

.

We can define it similarly for $Ch^-(Shv^{tr}_{Nis})$.

$$F \otimes_{Nis} G = a_{Nis}(l(X) \otimes l(Y))$$

. This induces a *derived* tensor product at the level of the derived category which is computed as follows. Consider $F, G \in Ch^-(Shv^{tr}_{Nis})$. Consider left resolutions $\mathcal{L}_\bullet(F) \to F$ and $\mathcal{L}_\bullet(G) \to G$. Then the derived tensor product is the Nisnevich sheafification of $\mathcal{L}_\bullet(F) \otimes^{tr} \mathcal{L}_\bullet(G)$

Theorem 4.2.3. *The triangulated category $D^-(Shv^{tr}_{Nis})$ along with $\otimes^L_{Nis}$ is a tensor triangulated category.*

Proof. See [[2], 8.17 and 14.2] □

Now we will define the triangulated category of effective motives $\mathbf{DM}^-_{eff}(k)$.

Definition 4.2.4. *Let $\mathcal{E}_{\mathbb{A}^1}$ be the smallest thick subcategory of $D^-(Shv^{tr}_{Nis})$ which contains the cone of*

$$\mathbb{Z}_{tr}(X \otimes \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$$

for every projection $X \times \mathbb{A}^1 \to X$. We define the triangulated category of effective motives $\boldsymbol{DM}^-_{eff}(k)$ as

$$\boldsymbol{DM}^-_{eff}(k) := D^-(Shv^{tr}_{Nis})/\mathcal{E}_{\mathbb{A}^1}$$

the Verdier localization of the derived category $D^-(Shv^{tr}_{Nis})$ with respect to $\mathcal{E}_{\mathbb{A}^1}$.

The category $\mathbf{DM}^-_{eff}(k)$ is a tensor triangulated category (See [[2], Lecture 14]). We can write $\mathbf{DM}^-_{eff}(k) := D^-(Shv^{tr}_{Nis})[W^{-1}_{\mathcal{E}_{\mathbb{A}^1}}]$ where, $W_{\mathcal{E}_{\mathbb{A}^1}}$ are class of maps whose cone lies in $\mathcal{E}_{\mathbb{A}^1}$

Definition 4.2.5. *Let $X \in Sm_k$. The motive $M(X)$ associated to X is the image of X under the map $M : Sm_k \to \boldsymbol{DM}^-_{eff}(k)$ which is defined as the composition*

$$Sm_k \to \boldsymbol{PST}(k) \to Shv^{tr}_{Nis} \to Ch^-(Shv^{tr}_{Nis}) \to D^-(Shv^{tr}_{Nis}) \to \boldsymbol{DM}^-_{eff}(k)$$

The triangulated category of motives $\mathbf{DM}^{-}(k,\mathbb{Z})$ is obtained from $\mathbf{DM}^{-}_{eff}(k)$ by inverting the Tate twist operation $X \mapsto X \otimes^{L}_{Nis} \mathbb{Z}^{tr}(1)$.

Now we state a theorem, which states that the full subcategory of $\mathbb{A}^1$-local complexes in $D^{-}(Shv^{tr}_{Nis})$ is equivalent to $\mathbf{DM}^{-}_{eff}(k)$.

Definition 4.2.6. *An object A in $\boldsymbol{D}^{-} := D^{-}(Shv^{tr}_{Nis})$ is said to be $\mathbb{A}^1$-local if $Hom_{\boldsymbol{D}^{-}}(\_, A)$ takes maps in $W_{\mathcal{E}_{\mathbb{A}^1}}$ to isomorphisms. We write $\mathcal{L}$ for the category of $\mathbb{A}^1$-local objects.*

Proposition 4.2.7. *The category $\mathcal{L}$ of $\mathbb{A}^1$-local objects is equivalent to the full subcategory of complexes in $D^{-}(Shv^{tr}_{Nis})$ with homotopy invariant cohomology sheaves*

Proof. Refer to [[2], Proposition 14.8] □

The category $\mathcal{L}$ has a tensor product defined as follows. Let $A, B \in \mathcal{L}$, then $A \otimes_{\mathcal{L}} B = TotC_{*}(A \otimes^{\mathbb{L}}_{tr} B)$

Theorem 4.2.8. *The category $(\mathcal{L}, \otimes_{\mathcal{L}})$ is tensor triangulated and the canonical functor $\mathcal{L} \to \boldsymbol{DM}^{-}_{eff}(k)$ is an equivalence of tensor triangulated categories.*

Proof. See [[2], Theorem 14.1] □

Hence, $\mathbf{DM}^{-}_{eff}(k)$ is the full subcategory of $D^{-}(Shv^{tr}_{Nis})$ with homotopy invariant cohomology sheaves.

4.3 Modules over motivic cohomology

In this section we will give a brief overview of the relationship between modules over motivic Eilenberg-Maclane spectrum (the spectrum which represents motivic cohomology in the motivic stable homotopy category) and Voevodsky's big category of motives $\mathbf{DM}_k$. This section is based on [21].

One of the main results towards this relationship is the following monoidal Quillen equivalence.

$$MSS^{tr} \rightleftarrows ChSS^{tr}_{\mathbb{G}^{tr}_m[1]}$$

Here, MSS^{tr} is the motivic symmetric spectrum with transfer(Definition 4.3.3) and $ChSS^{tr}_{\mathbb{G}^{tr}_m[1]}$ is the symmetric $\mathbb{G}^{tr}_m[1]$ of unbounded chain complexes of presheaves with transfer (See Definition 4.3.4)

We define the category $\mathbf{M}^{tr}$ of motivic spaces with transfers as the simplicial objects in the category of presheaves with transfer. This category is symmetric monoidal with respect to the tensor product $\otimes^{tr}$ defined levelwise as the tensor product of presheaves with transfer. See Definition 4.8.

We have a forgetful functor $\mathcal{U} : \mathbf{M}^{tr} \to sPre(Sm/k)$, which has a left adjoint $(\_)^{tr}$ that is strict symmetric monoidal. Refer to [[21], Lemma 2.1]. This category has a model structure [[21], Def 2.5 and Theorem 2.6] makes it into a left proper, combinatorial, simplicial model category. Here is a theorem that characterizes the weak equivalence and fibrations.

Proposition 4.3.1. *A map between motivic spaces with transfer is a motivic weak equivalence(or motivic fibration) if and only if it a weak equivalence(or fibration) in $Spc_k^{\mathbb{A}^1}$.*

Proof. Refer to [[21] Lemma 2.7] □

One of our main objects of discussion will be motivic symmetric spectra with transfers. For the definition of T-specta and T-symmetric spectra See section 2.5.

Definition 4.3.2. *A motivic spectrum $\boldsymbol{MS}^{tr}$ with transfers E is a sequence $(E_0, E_1, ...)$ of motivic spaces with transfer with bonding maps $T^{tr} \otimes^{tr} E_n \to E_{n+1}$ for $n \geq 0$*

We have a Quillen adjunction

$$\mathbb{Z}^{tr}_{\mathbb{N}} : \mathbf{MS} \leftrightarrows \mathbf{MS}^{tr} : \mathcal{U}$$

where $\mathcal{U}$ is the forgetful functor. The left adjoint $\mathbb{Z}^{tr}_{\mathbb{N}}$ is obtained by applying the transfer functor $(\_)_{tr}$, which is strict symmetric monoidal[[21], Lemma 2.1].

Similarly we can define motivic symmetric spectra.

Definition 4.3.3. *A motivic symmetric spectrum with transfers is a motivic spectrum with transfer $E = ((E_0, E_1, ...),\ \Sigma_n : T^{tr} \otimes^{tr} E_n \to E_{n+1})$ together with an action of the symmetric group Σ_n on E_n such that the iterated bonding maps*

$$((T)^{tr})^{\otimes^{tr} p} \otimes E_q \to E_{p+q}$$

is $\Sigma_p \times \Sigma_q$-equivariant. With the evident notion of morphism, this defines the category of motivic symmetric spectra with transfer, $\mathbf{MSS}^{tr}$

We have a Quillen adjunction of the form

$$\mathbb{Z}_{\Sigma}^{tr} : \mathbf{MSS} \leftrightarrows \mathbf{MSS}^{tr} : \mathcal{U}_{\Sigma}$$

, where **MSS** is motivic symmetric spectra (See section 2.5).

Definition 4.3.4. *Consider $Ch^{tr} := Ch(\mathbf{PST}(k))$ the unbounded chain complexes of presheaves with transfer. We can construct a symmetric spectra by taking suspension with respect to $... \to \mathbb{G}_m^{tr} \to 0$. We denote this category as $\boldsymbol{ChSS}^{tr}_{\mathbb{G}_m[1]}$.*

Theorem 4.3.5. *There is a Zig-zag of monoidal Quillen equivalences between MSS^{tr} and symmetric $\mathbb{G}_m^{tr}[1]$-spectra of unbounded chain complexes of presheaves with transfer.*

Proof. The proof follows from constructing a chain of Quillen equivalences. Refer to the discussion after [[21], Theorem 2.9]. Each of these Quillen equivalences have been proved in [[21], section 2]. □

Consider $Ch^{tr\sim}$ be the category of unbounded chain complexes of Nisnevich sheaves with transfer. Sheafification yields a left Quillen equivalence $Ch^{tr} \to Ch^{tr\sim}$, which extends to a left Quillen functor $\mathbf{ChSS}^{tr}_{\mathbb{G}_m^{tr}}[1] \to \mathbf{ChSS}^{tr\sim}_{\mathbb{G}_m^{tr}}[1]$ by [[23], Theorem 9.3].

Because weak equivalences in $\mathbf{Ch}^{tr\sim}$ are quasi isomorphisms, the category $Ho(\mathbf{Ch}^{tr\sim})$ is equivalent to the derived category of Nisnevich sheaves with transfers. By Theorem 4.19, $\mathbf{DM}_{eff}$ is the full subcategory of $Ho(\mathbf{Ch}^{tr\sim})$ consisting of homotopy invariant homology sheaves (Notice that in our definition of Theorem 4.19, we had considered cochain complexes). Hence, $\mathbf{DM}^{eff}$ is equivalent to homotopy category $Ch^{tr\sim}_{mo}$ of chain complexes whose homology sheaves are homotopy invariant (Refer to [[21], section 2.3] for precise was of describing $Ch^{tr\sim}_{mo}$ as a localization of $Ho(Ch^{tr\sim}$. Also, refer to Proposition 4.2.7, Theorem 4.2.8 and Definition 4.2.5 for a comparison)

By the discussion in [[21], section 2.3], category $\mathbf{ChSS}^{tr\sim}_{\mathbb{G}_m[1]}$ is equivalent to Voevodsky's big category of motives $\mathbf{DM}_k$. Also, refer to [[15], Theorem 10.96 and example 10.97] for a more general construction.

We now describe the idea of motivic cohomology as a ring spectrum and define the category of modules over it.The definition of motivic cohomology spectrum used here is slightly different from Voevodsky's motivic Eilenberg MacLane spectrum[See [5]].

The motivic cohomology spectrum $M\mathbb{Z}$ is the motivic symmetric spectrum

$$M\mathbb{Z} := (\mathcal{U}(Speck_+^{tr}), \mathcal{U}(T)^{tr}, \mathcal{U}(T^{\wedge 2})^{tr} ....)$$

. The structure maps of are obtained

$$T \wedge \mathcal{U}(A)^{tr} \to \mathcal{U}(T)^{tr} \wedge \mathcal{U}(A)^{tr} \to \mathcal{U}(T^{tr} \otimes^{tr} A^{tr}) \to \mathcal{U}(T \wedge A^{tr})$$

Here, $\mathcal{U}$ is the forgetful functor $\mathbf{M}^{tr} \to sPre(Sm_k)$ and $A = T^{\wedge n}$. This definition is weakly equivalent to Voevodsky's by [[22], Section 4.2]. This is a commutative ring spectrum with multiplication map $M\mathbb{Z} \wedge M\mathbb{Z} \to M\mathbb{Z}$ determined by $\mathcal{U}(A^{tr}) \wedge \mathcal{U}(B^{tr}) \to \mathcal{U}(A^{tr} \otimes^{tr} B^{tr}) \to \mathcal{U}(A \wedge B)^{tr}$ and the unit map $\mathbb{S} \to M\mathbb{Z}$ is determined by $A \to \mathcal{U}(A^{tr})$.

Definition 4.3.6. *An $M\mathbb{Z}$ module is a motivic symmetric spectrum E with a action $M\mathbb{Z} \wedge E \to E$ which is compatible with the multiplication map and unit map for $M\mathbb{Z}$. We denote this category as $M\mathbb{Z} - mod$.*

This category is symmetric monoidal and the forgetful functor $M\mathbb{Z} - mod \to \mathbf{MSS}$ is a lax symmetric monoidal with a strict symmetric monoidal left adjoint, which is $M\mathbb{Z} \wedge \_ : \mathbf{MSS} \to M\mathbb{Z} - mod$. There is a functor $\psi : MSS^{tr} \to M\mathbb{Z} - mod$ defined as $E \mapsto M\mathbb{Z} \wedge \mathcal{U}_\Sigma(E)$ where, $\mathcal{U}_\Sigma(E)$ is the underlying motivic symmteric spectrum of a motivic symmetric spectrum with transfer E. This has a left adjoint ϕ such that this is a Quillen adjunction (See [[21] lemma 2.37]).

This adjunction is in fact a symmetric monoidal Quillen equivalence, which is the main result that we need.

Theorem 4.3.7. *Suppose k is a field of characteristic zero. Then there is a strict symmetric monoidal Quillen equivalence*

$$\phi : M\mathbb{Z} - mod \leftrightarrows MSS^{tr} : \psi$$

Proof. See [[21], Theorem 5.5] ∎

From the discussion above, $Ho(MSS^{tr})$ is equivalent to $\mathbf{DM}_k$. We have a Quillen adjunction of the form

$$\mathbb{Z}^{tr}_{\Sigma} : \mathbf{MSS} \leftrightarrows \mathbf{MSS}^{tr} : \mathcal{U}_{\Sigma}$$

which gives adjunctions at the level of homotopy categories

$$\mathbf{DM}_k \leftrightarrows \mathbf{SH}(k)$$

.This parallels the case in topology where we have an adjunction between the derived category of abelian groups and the stable homotopy category.

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