Some Results on the Maker Breaker Triangle Game

A Thesis

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by

Pritam Acharya

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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Supervisor: Arindam Khan
Co-supervisor: Mohit Garg

Certificate

This is to certify that this dissertation entitled **Some Results on the Maker Breaker Triangle Game** towards the partial fulfilment of the BS-MS dual degree program at the Indian Institute of Science Education and Research Pune, represents work carried out by Pritam Acharya at the Department of Computer Science and Automation, Indian Institute of Science under the supervision of Dr. Arindam Khan, Associate Professor, CSA, IISc with Dr. Mohit Garg, Post-Doctoral Scholar, CSA, IISc as a co-supervisor, during the academic year 2024–2025.

Arindam Khan

Mohit Garg

Committee:

Dr. Arindam Khan

Dr. Mohit Garg

Dr. Anup Biswas

Declaration

I hereby declare that the matter embodied in the report entitled **Some Results on the Maker Breaker Triangle Game** are the results of the work carried out by me at the Department of Computer Science and Automation, Indian Institute of Science, under the supervision of Dr. Arindam Khan, Assoc. Professor, CSA, IISc, with Dr. Mohit Garg, Post-Doctoral Scholar, CSA, IISc, as a co-supervisor, and the same has not been submitted elsewhere for any other degree

P. Acharya

Pritam Acharya

IISER Roll No. – 20201198

Date: 25th March 2025

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କର୍ମେ ଯିଏ ନର କର୍ମ ଏକା ତାର ଜୀବନ ର ମାନଦଣ୍ଡ।

~ଉକ୍ଳମଣି ଗୋପବନ୍ଧୁ ଦାସ

[A human life is not just years, months, days, or time's embrace,
But deeds alone define its worth—its measure, its true grace.
~ Utkalamani Gopabandhu Das]

This thesis is dedicated to my parents and the indomitable human spirit.

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Abstract

Positional games are a broad class of games with complete information. Given their practical and 'real-world' origins, they have attracted constant attention from mathematicians working in combinatorics over the years. In this thesis, we study the Maker-Breaker triangle game played on a complete graph with n vertices. Two players, Maker and Breaker, alternate turns claiming unclaimed edges: Maker selects one edge per turn, while Breaker claims q edges. Maker wins the game if the edges he has claimed form a triangle; otherwise, Breaker wins. The objective is to figure out the 'threshold' $q_0(n)$ such that the Maker wins if $q < q_0$ and the Breaker wins if $q > q_0$ under 'perfect play'.

Finding out sharp bounds on the threshold q_0 remains a challenging open problem. Chvátal and Erdös, 1978, showed that $\sqrt{2n} \leqslant q_0(n) \leqslant 2\sqrt{n}$. Balogh and Samotij, 2011, improved the upper bound to $q_0 \leqslant 1.958\sqrt{n}$ by providing a randomized strategy for Breaker's win. Recently, Glazik and Srivastav, 2022, further improved the upper bound to $q_0(n) \leqslant 1.633\sqrt{n}$ by providing a completely deterministic strategy for the Breaker and using a potential method based analysis. In this thesis, we survey all these results.

For such Maker-Breaker games, Erdős had a remarkable insight: the asymptotic threshold value under perfect play should be the same if the players played completely randomly. This insight has since motivated extensive research into games where both players make moves uniformly at random, as well as those where one player moves uniformly at random while the other employs strategic, rational decision-making to optimize their chances of winning.

While the above-mentioned paradigm holds true for the triangle game, and the threshold for the RandomMaker vs CleverBreaker game was also resolved asymptotically, the CleverMaker vs. RandomBreaker triangle game was never studied explicitly. In this thesis, we address this gap in the literature, and, using standard probability techniques, show that in a CleverMaker vs. RandomBreaker game, the following is true.

I. If $q = o(n^2)$, then the CleverMaker has a strategy that wins with high probability.

II. For all positive constants c, if $q = cn^2$, then the RandomBreaker wins with a positive constant probability.

Contents

Part I

Introduction & Survey

Chapter 1

Introduction

Mathematics is an exact science, and combinatorics even more so. This exactness is why we can derive many combinatorics problems from seemingly simple occurrences around us. This abundance of 'down-to-earth' examples (as per Hefetz et al., 2014) is why this branch of mathematics has been so enticing for a large number of mathematicians and non-mathematicians who take a keen interest in puzzles and problem-solving. The general motivation of mathematics is to abstract things and examine them with general tools, and we aim to do the same with combinatorics.

The reader must have come across several combinatorial games in their lifetime. Examples are 'tic-tac-toe,' chess, hex, and nim. Although seemingly simple, any attempt at generalizing and developing a theory for such games has been largely unsuccessful or is currently in progress. The major reason behind the same as mentioned by Beck, 2008 in his seminal compendium of work relating to such games "Combinatorial Games: Tic-Tac-Toe Theory" is the rise of *combinatorial chaos*. This refers to the way these games behave unpredictably without any pattern or order to them. But, as nature would have it, these games form a large part of the practical application that mathematics has on the world. Hence, it is imperative that we make an effort to understand and tame this chaos.

Let us take one of the examples we mentioned above and elaborate on it. Tic-Tac-Toe is a simple game played between two players. The most common example probably known to every human is that of the 3×3 board and each player playing alternatively with zeroes and crosses occupying one of the 9 cells till one player successfully claims a 'line' of cells, i.e., 3 of a vertical line formed by 3 cells together, 3 of a horizontal line formed by 3 cells together, and 2 of diagonal lines. If either player fails to do so, then the game ends in a draw. Simple to analyze via a case-by-case approach. However, things become trickier as we look at the same game in higher dimensions. The $3 \times 3 \times 3$ game is a trivial first-player win because when the first player occupies a second element, they would have two possible 'lines' to close and this won't be possible since the second player is allowed to claim only 1 element per move. The $4 \times 4 \times 4$ game is a difficult computer-assisted first-player win. Although we can back-track the game tree and analyze it, since the game tree scales exponentially, it becomes very difficult. Just take the example of a $5 \times 5 \times 5$ board it requires backtracking of 3^{125} steps (as per Beck, 2008) which is unfeasible. Note – to find a general definition of 'line' in the n^d version of the game, we refer the reader to pp. 3 of the book by Hefetz et al., 2014.

Positional Games we are interested in (for the sake of this thesis) involve two players. The players make moves that consist of alternately occupying elements on the *board* of the game, which is a set X assumed to be finite. We declare a collection of a set of elements $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ such that $A_1, A_2, \ldots, A_n \subseteq 2^X$ as the 'winning set'. The players will take turns playing. In each turn, the player will make a move. In game theoretic terminology, this is known as a complete information game since the information regarding both the players is available to both the players simultaneously (unlike, say, a game like Poker where the opponent does not know about the cards that the player has). Classical game theory, as developed by Neumann, doesn't deal much with complete information games. As per Beck, 2008, "Most board games are a challenge for mathematics: to analyze a position, one has to examine the available options, and then the further options available after selecting any option, and so on. This leads to combinatorial chaos, where brute force study is impractical."

1.1 Maker Breaker Positional Games

Definition 1. (Taken from Definition 2.1.1. in the book by Hefetz et al., 2014) Let X be a finite set and $\mathcal{F} \subseteq 2^X$ a family of subsets. In a *Maker-Breaker* game over the hypergraph $(X, \mathcal{F})$,

- the set X is called the *board*; the elements of $\mathcal{F} \subseteq 2^X$ are the *winning sets*;
- the players are called *Maker* and *Breaker*;
- during a particular play, the players alternately occupy elements of X; as a default, we set Maker to start unless specified otherwise in any section;
- the winner is

– Maker if they occupy a winning set completely by the end of the game,
– Breaker if he occupies an element in every winning set.

Sometimes, we might have to assume that the Breaker moves first. In these cases, we refer the reader to Proposition 2.1.6, pp. 15 in the book by Hefetz et al., 2014. This proposition states that if either Maker or Breaker has a winning strategy for a given $\mathcal{F}$ as the second player, they also have a winning strategy if they start the game. A **winning strategy** is a set of moves for a given player that mathematically guarantees the victory of the said player. Chvátal and Erdös, 1978 noted that if the Maker and Breaker occupy only 1 element per move, we will have an obvious Maker win. So, they decided to introduce the concept of *bias.*

Definition 2. The $(a : b)$ biased Maker-Breaker game is a Maker-Breaker game in which the Maker is allowed to claim a free/unclaimed elements per move, and the Breaker is allowed to claim b free/unclaimed elements per move. a and b are called as the *bias* of the Maker and Breaker respectively.

In our context, a move consists of claiming a previously unclaimed object (vertex or edge) on the graph. The subgraph induced by the items collected by the Maker is known as the Maker graph; we define the Breaker graph similarly. If the Maker graph contains a copy (i.e., subgraph) of any of the sets in the winning set $\mathcal{F}$, then we declare the Maker as the winner. A comprehensive resource on positional games is the compendium by Beck, 2008.

There are several variations of a Maker-Breaker game. One possibility is to vary the definition of the winning set. A few popular examples include the Hamiltonicity game, where the winning set is a Hamiltonian cycle. Another is the Perfect Matching game, in which the winning set is a perfect match. Similarly, the connectivity game declares the set of all spanning trees of G (in our case K_n) as the winning sets. Another possibility of variation is to introduce a 'Random' player. Naturally, in any board game, since both players can see each other's moves, it's a complete information game. However, we can set any player to play completely random moves. Suppose we have a 'Random Maker' wherein the Maker only claims available edges uniformly at random, whereas the Breaker can be strategic in their play. We can also consider the case when both players play randomly; in fact, in their paper Chvátal and Erdös, 1978, this technique has yielded results that are very close to the optimal results because of what is popularly known as the 'probabilistic intuition.' In Table 1.1, we have compiled a list of relevant papers and results that may deal with these variations. These are popularly termed as **half-random** games in Groschwitz and Szabó, 2016. The authors then define the quantity below and study it. We have mentioned this to make the interpretation of Table 1.1 easier.

Definition 3. $q_0(n)$ is the *sharp threshold bias* for a ((1 : b) (and symmetrically for a (a : 1)) half-random game is a function such that for $\varepsilon > 0$ it satisfies:

1. Breaker wins the $(1 : (1+\varepsilon)q_0(n))$-biased game with probability tending to 1 against any strategy of the Maker.
2. Maker wins has a strategy to make the Breaker lose the $(1 : (1-\varepsilon)q_0(n))$ biased Maker-Breaker game.

1.2 Maker Breaker Triangle Game

We will now define the exact game we look into, i.e., the triangle game. This was first studied by Chvátal and Erdös, 1978 along with a few other important graph positional games. The major search was for a quantity called the *threshold bias* for the Breaker. We define both in the following. We give a survey of the best-known results to date in the same game and look at a few questions about a variation of the same game.

Definition 4. A $(1 : q)$ Maker-Breaker triangle game denoted by $\mathbb{G}(K_3; n, q)$ is a positional game played on a complete graph K_n of size n. Where the winning set is defined to be triangle (K_3) in K_n. The Maker wins the game if they occupy a K_3 before all the $\binom{n}{2}$ edges of K_n have been occupied. The Breaker wins the game if they prevent the Maker from occupying a triangle after all $\binom{n}{2}$ edges of K_n have been occupied.

Definition 5. The *threshold bias* of $\mathbb{G}(K_3; n, q)$ is the value $q_0(n)$ such that the game is a Maker's win for $q \leqslant q_0(n)$ and a Breaker's win for $q > q_0(n)$.

Definition 6. The following quantities are used concerning $\mathbb{G}(K_3; n, q)$ throughout the thesis unless explicitly specified otherwise in any section.

1. Maker Graph – The subgraph of K_n consisting of all the edges claimed by the Maker after t rounds.
2. Breaker Graph – The subgraph of K_n consisting of all the edges claimed by the Breaker after t rounds.
3. Maker Degree ($\deg_M(v)$) – The degree of v in the Maker graph.
4. Breaker Degree ($\deg_B(v)$) – The degree of v in the Breaker graph.
5. $\deg_M^t(v)$ – Degree of v in the Maker graph after round t.
6. $\deg_B^t(v)$ – Degree of v in the Breaker graph after round t.

1.3 Scope of the Thesis & Contributions

The first few chapters consist of an exposition of three papers by Chvátal and Erdös, 1978, Balogh and Samotij, 2011, and Glazik and Srivastav, 2022. I also explain an overall idea presented in the paper by Bednarska and Łuczak, 2000 in Chapter 2. No claim to originality is made in the material presented in Chapters 2, 3, and 4. The aim is to present the mathematical richness behind the strategies adopted in each chapter above. As a result, we focused more on the explanation rather than repeating the calculations presented in the papers. The reason behind choosing the three papers to present in this thesis was because each covers techniques that can be applied to a wide variety of the variations in the Maker-Breaker games as mentioned in Section 1.1.

Chvátal and Erdös, 1978 is the paper that started the study of this field. It presents a simple strategy and threshold bound, however the questions that the paper poses have been of interest to several mathematicians. Hefetz et al., 2014 in their book credit the start of the study of Maker-Breaker games to this paper. The main theorem is Theorem 2.0.1 which has been explained in a lucid manner. The figures added are new. This paper also presents an entirely deterministic strategy. The analysis is also rather simple. Chvátal and Erdös, 1978 also studied the Connectivity, Hamiltonicity, and a general class of games called as 'Box games'.

Balogh and Samotij, 2011 use a half-deterministic and half-random strategy to establish the improvement on the threshold bias as presented in Beck, 2008. As I have explained in Sub Section 3.2.1, the strategy divides itself into several cases, which are further divided into sub-cases, and then eventually, the analysis focuses on proving a single favorable outcome. This analysis is done with the help of concentration bounds and relevant graph-theoretic properties, which emerge due to the Maker and Breaker's specified strategy.

Similarly, in Section 4.2, we have lucidly explained the strategy presented in the paper by Glazik and Srivastav, 2022. Contrary to the previous best-known result, this paper follows an entirely deterministic strategy and introduces the tool of 'potential function', which may be used in other positional games. The construction of this potential function, however, is very retroactive and done precisely to fit into the larger result that the authors had in mind. A generalization of this idea could be useful in other problems of similar flavor.

Results presented in Chapter 5 are new to my knowledge. We consider the 'CleverMaker' and 'RandomBreaker' settings of the triangle game and present a few results on the threshold bias. We apply probabilistic techniques and obtain the results presented.

The table below summarizes the best-known results for a few games and game settings based on the literature review conducted.

Maker	**Breaker**	**Results**	**Source**
R	C	$(a:1)$ biased M-B games. Sharp threshold bias $a = \ln\big(\ln(n)\big)$. for Connectivity, Hamiltonicity game	Groschwitz and Szabó, 2016
C	R	$(1:b)$ biased M-B games. Sharp threshold bias $b_{\mathcal{PM}} = n$, $b_{\mathcal{H}} = b_{\mathcal{C}} = n/2$	Groschwitz and Szabó, 2017
R	C	$(1:q)$ biased M-B triangle game. $q \leqslant c_0 n^{1/m(G)}$ – Maker wins.	Bednarska and Łuczak, 2000
C	C	$(1:q)$ biased M-B triangle game. $q_0 = 1.633\sqrt{n}$ as the Breaker threshold bias.	Glazik and Srivastav, 2022
C	R	$(1:q)$ biased M-B triangle game. $q = o(n^2)$ gives Maker win with high probability. $q = cn^2$ gives Breaker win with constant positive probability for fixed $c > 0$.	Results presented in Chapter 5

Table 1.1: A table summarizing all results known so far. In the second row, $b_{\mathcal{PM}}, b_{\mathcal{C}},$ and $b_{\mathcal{H}}$ denote the Perfect Matching, Hamiltonicity, and Connectivity Maker-Breaker games respectively. In the third row, the paper proves the existence of appropriate constants c_0 and C_0 and a quantity $m(G)$ according to Definition 18; depending on the structure we are interested in (G is K_3 in our case). More has been elaborated on in Chapter 2. R denotes Random, where the player claims each edge uniformly at random, and C denotes clever. Which means the player may play according to a strategy. Note that being clever does not mean the same as playing only non-randomly. It may be the case that the clever player decides that playing randomly is the best strategy to yield positive results and thus choose to play randomly.

Chapter 2

Erdős & Chvátal's Work

Chvátal and Erdös, 1978 introduced and studied the Triangle Game. We are interested in bounding the quantity of q, to arrive at a winning strategy for the Maker and Breaker. A 'winning strategy' is a prescription of moves that either player can follow to arrive at a winning scenario, irrespective of the opponent's moves. Chvátal & Erdős proved the following bounds on the bias, which leads to a winning strategy for the Maker and the Breaker, respectively.

Theorem 2.0.1. *For a given (n, q)-triangle game for every $q \leqslant \sqrt{n}/2$ the Maker has a winning strategy. Otherwise, for every $q \geqslant 2\sqrt{n}$ the Breaker has a winning strategy.*

Proof. We present the heuristics of the proof in the interest of space. Considering the Maker's case, assume $q \leqslant \sqrt{n}/2$. Let $v \in V(K_n)$ be an arbitrary vertex in K_n. Maker's strategy is as follows. For all positive integers i, the Maker's i-th move will be motivated to close a triangle in the Maker graph. To do so, they will claim a free edge (i.e., a previously unclaimed edge) if this edge closes the triangle then Maker wins otherwise they claim a random free edge vv_i such that $v_i \in V(K_n)$. If no such edge exists then Maker forfeits the game allowing the Breaker to win. The proof is by contradiction. We assume by following this strategy the Maker does not win. Then, the maximum number of turns the Breaker will need to completely claim all edges incident to a vertex is $d = d(n) \geqslant \left\lfloor \frac{n-1}{q-1} \right\rfloor$, thus there will be an integer $1 \leqslant i \leqslant d$ such that in the i-th move the Maker has claimed uv_i and the Breaker has also claimed all edges incident to v_i after round i and before round d. However, a simple calculation will reveal that the total number of edges Breaker can claim in d rounds is smaller than total number of available edges in d rounds, i.e., $\binom{d}{2} > qd$. Hence, our assumption is false.

We focus on the Breaker now. Assume $q \geqslant 2\sqrt{n}$ the strategy that Breaker can follow in order to win is as follows. Whenever the Maker claims an edge u_iv_i in their i-th move the Breaker will claim $\lfloor q/2 \rfloor$ edges at u_i and $\lceil q/2 \rceil$ edges at v_i. In case the given number of edges are not free at the given vertex then Breaker claims arbitrary free edges in order to complete its quota of q edges in a single move. The Breaker also claims those edges first which pose an immediate threat, i.e., edges u_iz such that v_iz was previously claimed by the Maker, since otherwise in the next move the Maker will complete a K_3 and win the game. The key observation lies in noticing that the maximum degree a vertex can have in the Maker graph is $\lfloor q/2 \rfloor + 1$ because everytime the Maker claims a vertex x the Breaker will claim atleast $\lfloor q/2 \rfloor$ edges incident to x. Hence, the Maker's degree of x is at most $1 + \frac{n-1}{\lfloor q/2 \rfloor + 1} \leqslant \lfloor \frac{q}{2} \rfloor + 1$. Building on this observation we can proceed by contradiction and assume the Maker has won the game, hence it has claimed a triangle successfully however, in doing so we can demonstrate that there will be a round where the Maker degree of a vertex will exceed $\lfloor q/2 \rfloor + 2$ atleast contradicting our observation. Thus, completing the proof. □

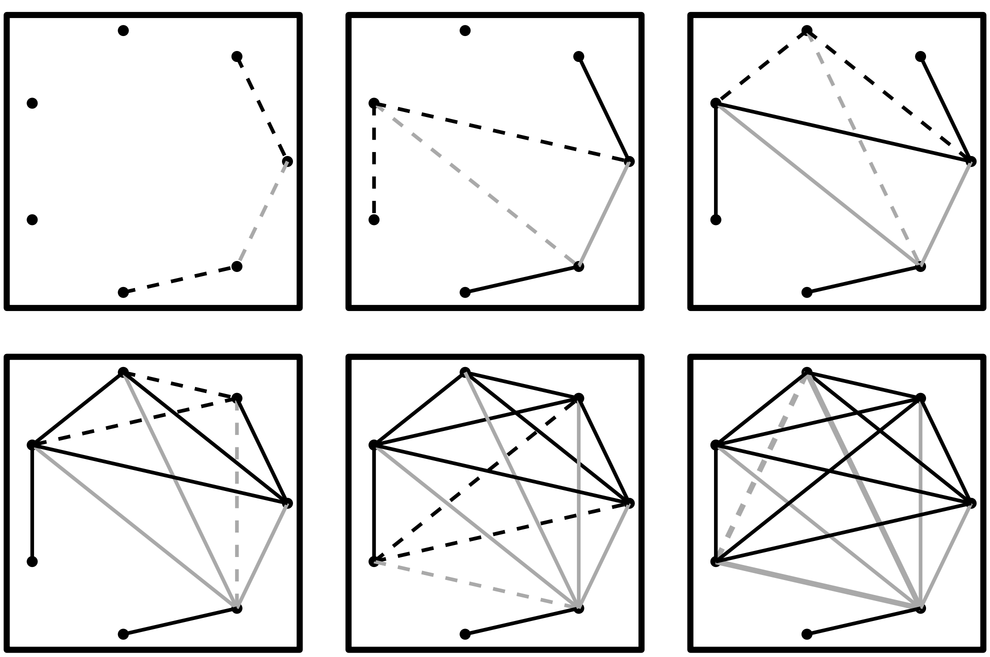

Figure 2.1: An example play of the Maker Breaker triangle game. This is the $\mathbb{G}(K_3; 7, 2)$ game. Each gray line represents the edge the Maker claims, and each black line represents the Breaker's edges. In the final step, the Maker manages to claim a K_3 as highlighted in the fat gray edges and hence wins the game. Figure idea sourced from Glazik and Srivastav, 2022.

Chapter 3

A Randomized Breaker Strategy

This chapter is based on the results by Balogh and Samotij, 2011. No claim to originality is being made. We add exposition to the results so that following the paper becomes easier. The strategy presented by the authors here is a randomized one. The authors improvise on the deterministic strategy presented by Chvátal and Erdös, 1978. The basic intuition is that as the Breaker randomizes some of its moves, it claims the edges that would have otherwise been part of a potential two-path in the future. This improvisation must be done after closing the two paths that might emerge after the Maker's moves. As we will see, the first set of moves prescribed in the strategy also keeps this in view. The techniques used in analysis involve defining a random variable, which denotes the number of edges the Breaker ends up claiming till a particular round, and then showing that there can never be more than q two paths in the Maker's graph and hence, the Breaker always wins. We will follow the paper's notation closely. Let $\mathbb{G}(K_3; n, q)$ denote an instance of the triangle game being played on a complete graph with n vertices K_n and the bias of the Breaker is q, and the Maker is allowed to claim 1 edge per move.

3.1 Concentration Bound

The following concentration bound on the hypergeometric distribution will be used extensively in this paper. Consider a hypergeometric distribution, with $m, n, N \in \mathbb{Z}^+$. A random variable has a hypergeometric distribution with parameters m, n, N if it describes the number of successes in a sequence of n draws from a set of size N with m marked elements, without replacement. An alternative way to understand it is to consider the following example. Suppose you have N people with each individual possessing a binary marker $\{0.1\}$. Let the number of individuals possessing the marker 1 be m. Now suppose we sample n people uniformly without replacement from the population. Then, the hypergeometric distribution informs us about the behavior of the random variable, which tracks the possibility of k positive individuals among the n sampled people. Mathematically,

$$\mathbb{P}\left[X=k\right]=\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

We will use Hoeffding's bounds as mentioned in Section 6 of the paper Hoeffding, 1994, in particular, a result that bounds the probability of the random variable lying in the tail.

Lemma 1. Let $m, n, N \in \mathbb{Z}^+$ with $m, n \leqslant N$ and $m \leqslant \frac{N}{2}$. X be a random variable following the hypergeometric distribution with parameters N, m, n and let $\mu = \mathbb{E}[X] = \frac{mn}{N}$. Now, if $t \geqslant 0$, then

$$\mathbb{P}(X \leqslant \mu - t) \leqslant \exp\left(-\frac{t^2}{2\mu}\right).$$

3.2 The Main Strategy

We begin with a set of definitions. This is similar to what we have seen in Chapter 2 and will be seen in Chapter 4. G_M^t and G_B^t denote the Maker and Breaker graph till round t respectively. If the superscript is omitted, they denote the edge-disjoint subgraphs created by the Maker G_M and the Breaker G_B.. The Maker and Breaker degrees are denoted by $\deg_M^t(v)$ and $\deg_B^t(v)$ respectively. The neighborhood of a vertex v in G_B^t is denoted by $N_B^t(v)$ and in G_M^t is denoted by $N_M^t(v)$. Define the below constants

$$q=\left(2-\frac{1}{24}\right)\sqrt{n}, \quad \alpha=\frac{\sqrt{n}}{7}, \quad \beta=\frac{3\sqrt{n}}{4}, \quad \text{and} \quad \Delta=\frac{2n-\alpha\beta}{q-\alpha}\approx 1.043\sqrt{n}.$$

Definition 7. A vertex $v \in V(K_n)$ is called *large* at time t if $\deg_M^t(v) > \beta$. Otherwise, v is called *small.*

Definition 8. For a large vertex v at time t we let

$$\alpha_v^t=\frac{\deg_M^t(v)-\beta}{\Delta-\beta}\alpha$$

As we will see in the strategy, α_v^t ends up being an integral quantity in the distribution of the number of edges the Breaker has to claim at each vertex, constituting a Maker edge in the given round. It helps us ensure a 'balanced' distribution of Breakers across the vertices. We will see in this exposition the maximum degree any vertex can have in the Maker graph as long as the Breaker follows our prescribed strategy is Δ. So, the ratio in the quantity α_v^t is "how far the Maker is from β at round t" divided by "how far the Maker can go from β" and this ratio is multiplied by α. And due to the definition of each of the constants, we can get to our desired bound. This section will present a strategy and the relevant analysis proving the following theorem.

Theorem 3.2.1. *For Breaker bias $q \geqslant \left(2 - \frac{1}{24}\right)\sqrt{n}$ we have a winning strategy for the Breaker in $\mathbb{G}(K_3; n, q)$ for all large enough n.*

Lemma 2. For all large v with $\deg_M(v) \leqslant \Delta$, we have $\alpha_v \in [0, \alpha]$ and

$$\frac{q}{2} + \alpha_v - \deg_M(v) = \frac{q}{2} + \alpha - \Delta + \left(1 - \frac{\alpha_v}{\alpha}\right)(\Delta - \alpha - \beta) \in \left[0, \frac{q}{2} - \beta\right]. \tag{3.1}$$

Proof. The proof is simple calculations adhering to the assumptions made. We refer the reader to Claim 7 in Section 4 of the paper by Balogh and Samotij, 2011. □

3.2.1 Breaker's Strategy

As is the recurring theme across $\mathbf{G}(K_3; n, q)$ Maker-Breaker games the Breaker's immediate aim would to close all two paths which might emerge in the Maker graph. So, if the Maker claims an edge $e_M = \{u, v\}$ then the Breaker would have a two-fold goal. First, they would want that $N_M^t(v)$ and $N_M^t(u)$ are cliques in the Breaker graph so that there is no question of the Maker closing a two path in their subsequent moves. Second, the Breaker would also aim to keep the Maker degree of any vertex below Δ. We will now understand the exact steps the Breaker takes to achieve this result. Suppose for round $t + 1$ the Maker claims the e_M we mentioned above. Now, additionally suppose $\deg_M^t(u), \deg_M^t(v) \leqslant \Delta$ we will see three cases. Each considering (w.l.o.g) the possible combinations of *large* and *small* that the vertices making up the Maker edge can be.

- **Case 1** – u and v both are small. In this case, there is no immediate danger to the Breaker so it will focus on closing all potential two paths by claiming all edges that are

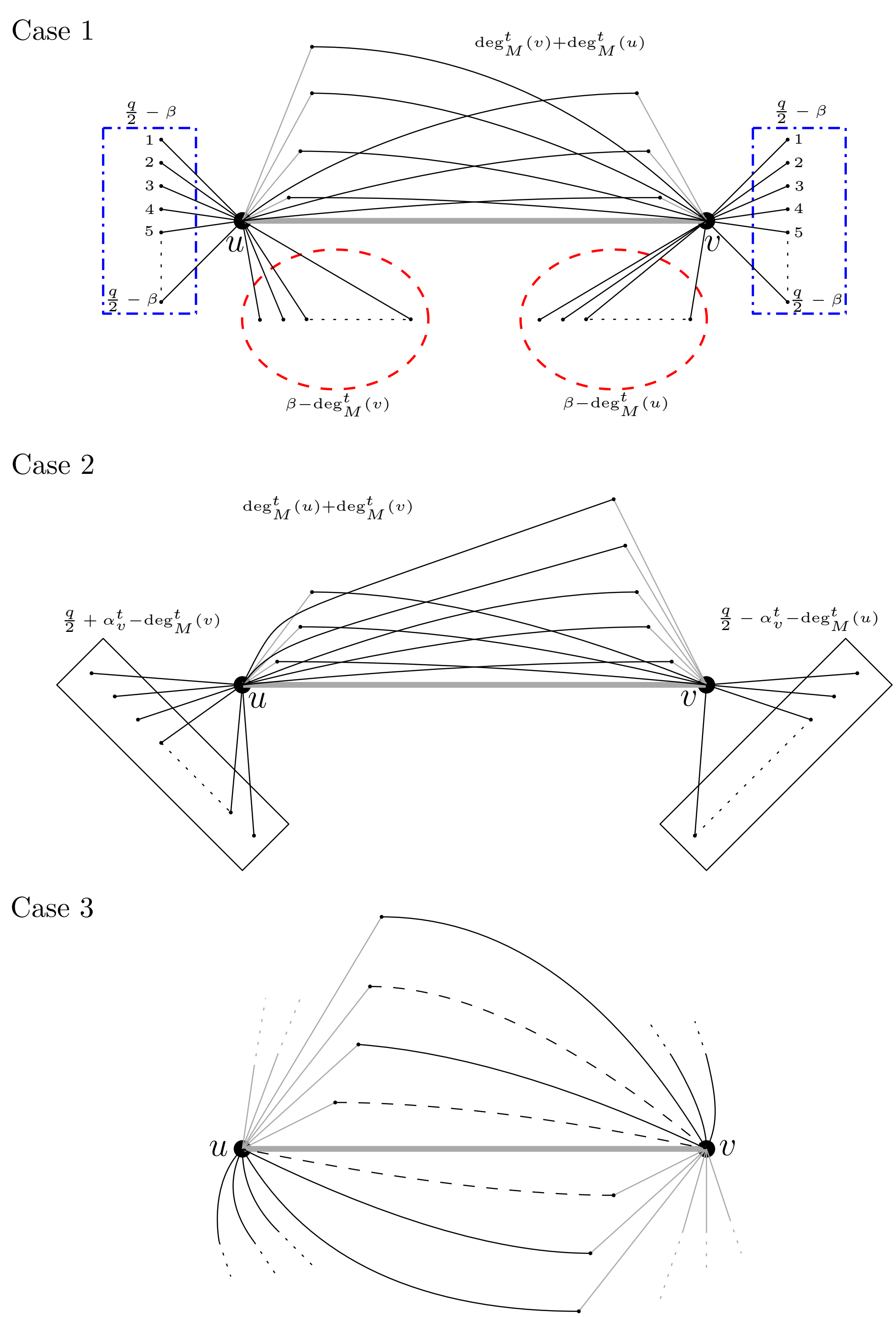

Figure 3.1: An illustration of the Breaker's strategy. In all the cases, the solid dark gray lines represent the edges in the Maker graph after the round t. The solid black lines represent the edges claimed by the Breaker. The size of each set is mentioned wherever applicable. In Case 3, the *dashed black* lines represent the closing edges which have been claimed by the Breaker as the random edges in Case 1.

unclaimed between u and $N^t_M(v)$ and v and $N^t_M(u)$. If the number of available edges at u (or v) is less than $q/2$ then the Breaker comfortable claims all these edges. In the other case, if the number of available edges is more than $q/2$, then the Breaker picks a vertex u or v then picks $q/2 - \beta$ unclaimed edges at either of the vertices at random. We can perhaps think of it like the Breaker first deciding on a random ordering of the vertices either u or v is connected to and then claiming the edges connecting both in the ordering decided. More formally, for a vertex $x \in \{u, v\}$, the Breaker decides on a random ordering of the set of vertices V, and then connects (by claiming the corresponding edges) x with the first $q/2 - \beta$ vertices on the decided vertices. The reason behind such thinking is that the random moves are not fully random. This is because they depend on the edge the Maker claims (which is a deterministic move) and the set of edges the Breaker claims in the first part of his set of moves to close the two paths. So, in Section 2, Proposition 2.1 of Groschwitz and Szabó, 2017. We can see that any random permutation is equivalent to the random method that one might naturally think of from the Breaker's perspective. After these set of moves the Breaker will claim arbitrarily $\beta - \deg^t_M(u)$ edges at v and $\beta - \deg^t_M(v)$ edges at u. So, in total the Breaker would claim $\deg^t_M(u) + (q/2 - \beta) + (\beta - \deg^t_M(u))$ edges at v and $\deg^t_M(v) + (q/2 - \beta) + (\beta - \deg^t_M(v))$ edges at u.

- **Case 2** – u is small and v is large. The Breaker would naturally want to claim more edges in the small vertex. Quantitatively, the Breaker tries to claim $q/2 + \alpha^t_v$ edges at u and $q/2 - \alpha^t_v$ edges at v. In total the Breaker claims q edges. There are two parts to this as well. First, the Breaker claims all the potential two paths, i.e., the edges between u and $N^t_M(v)$ and the edges between v and $N^t_M(u)$. This is possible because of the bounds on α^t_v because of Lemma 2. Observe that $q/2 + \alpha^t_v \geqslant \deg^t_M(v)$ and $q/2 - \alpha^t_v \geqslant q/2 - \alpha \geqslant \beta \geqslant \deg^t_M(u)$. Now, for the second part, the Breaker would claim the remaining $q/2 + \alpha^t_v - \deg^t_M(v)$ edges randomly at u this again is possible because of $q/2 + \alpha^t_v - \deg^t_M(v) \leqslant q/2 - \beta$. Similarly the Breaker claims $q/2 - \alpha^t_v - \deg^t_M(u)$ edges at v randomly.

- **Case 3** – u and v both are large. This is the case where we will require the concentration bounds which we introduced. Since both the vertices are large ideally the Maker would like to connect both to produce enough two-paths such that the Breaker exhausts its quota of edges per move and is still unable to close all two-paths off. Quantitatively, there is a possibility that $\deg^t_M(u) + \deg^t_M(v) \geqslant q$. So, while the Breaker closes all two paths by claiming all edges between $u - N^t_M(v)$ and $v - N^t_M(u)$ it may not be sufficient. Let us look at what the ideal case would be. Define $e^t_b(u, v)$ as the set of edges between $u - N^t_M(v)$ and $v - N^t_M(u)$ that have been claimed by the Breaker in the rounds preceding t. More formally, $e^t_B(u, v)$ are those set of edges which are in G^t_B between $u - N^t_M(v)$ and $v - N^t_M(u)$. So, the Breaker would only need to claim $\deg^t_M(u) + \deg^t_M(v) - e^t_B(u, v)$ edges to close all possible two paths. The situation which will be favourable to us will be

$$q \geqslant \deg^t_M(u) + \deg^t_M(v) - e^t_B(u, v). \tag{3.2}$$

If Eq. 3.2 holds then Breaker can claim all edges in a manner such that they claim $\min\{q - \deg^t_M(u), q/2\}$ at u and at least $\min\{q - \deg^t_M(v), q/2\}$ edges at v. We can also see than in the end the number of edges claimed at any large vertex is after all less than what we wanted to claim in Case 2. That is, observe that due to Lemma 2 and for any $x \in \{u, v\}$ we have $\min\{q - \deg^t_M(u), q/2\} \geqslant q/2 - \alpha^t_x$.

Thus, it is evident that if the Breaker plays by the above strategy, the Maker will never be able to close any new two-path that might emerge. However, we must prove that Eq. 3.2 holds. This is also where the majority of the analysis comes in. Meanwhile, we prove that under this strategy the Maker degree of any vertex will not cross Δ.

Lemma 3. Any vertex in G_M can have a maximum degree less than Δ.

Proof. We will proceed by contradiction. Assume there is some node v with degree equal to Δ in the Maker graph. Now, under our strategy, whenever the Maker claims an edge at the same node (under our assumption) the Breaker would respond with at least $q/2$ edges or $q/2 - (\deg_M(v) - \beta)\alpha/(\Delta - \alpha)$. Hence, simple calculations show that $\deg_B(v) \geqslant \frac{\Delta q}{2} - \sum_{\deg_M(v)=\beta+1}^{\Delta-1} \frac{\deg_M(v)-\beta}{\Delta-\beta}\alpha \geqslant \frac{\Delta q}{2} - \frac{(\Delta-\beta)\alpha}{2} = \frac{\Delta(q-\alpha)+\beta\alpha}{2} = n$ which is a contradiction since the maximum possible degree any vertex can have is $n - 1$. □

We need to show that Eq. 3.2 holds. Recall how in our strategy we made the Breaker claim more edges at the lower degree vertex in Case 2. This is because we want to ensure as the Maker keeps playing and claiming more edges, the Breaker keeps up with the vertices which are gaining more edges in the Maker graph. In order to satisfy Eq. 3.2 we need $e_B(u, v)$ to be high enough. For this to happen we need the random moves the Breaker plays to claim more edges at u or v. However, this might not happen, or at least we cannot mathematically guarantee that this happens. Additionally, if the Maker joins u or v to nodes with high Maker degree, then the Breaker would be forced to use a large part of its allowed q edges to close the potential two paths which may arise in the future (notice that α_v^t is linearly dependent on $\deg_M(v)$). Thus, with our formulation of the Breaker's strategy, whenever they claim more edges at the smaller degree vertex, it compensates for the lack in the random part of the strategy at hand. We quantify the above ideas in the lemma given below where we claim the existence of a constant which guarantees us that the Maker degree never exceeds a quantity which is smaller than Δ. Given our choice of this constant we can regulate how small this quantity get which upper bounds the Maker degree of a vertex.

Lemma 4. Let there be a vertex v. Let v become large in round t. Then there exists a $\lambda'_v \in [0,1]$ such that $\deg_M(v) < \frac{(2n-(3-2\lambda'_v)\alpha\beta)}{q-\alpha}$ (notice how this is a modification of our definition of Δ. Additionally, the Breaker will claim at least $\beta\left(\frac{q}{2}+\alpha-\Delta+\lambda'_v(\Delta-\alpha-\beta)\right)$ random edges at v at the end of round t. λ'_v depends only on what happened in the first t rounds.

Proof. We prove the existence of λ'_v using information only up to round t and hence, it depends only on what happened till then. As per the strategy, for every edge claimed by the Maker involving a vertex v, the Breaker would claim a total of $q/2-\lambda'\alpha$ edges at v where $\lambda' \in [0,1]$ is some constant. Now, out of these total edges claimed, the Breaker will claim $q/2+\alpha-\Delta+(1+\lambda')(\delta-\alpha-\beta)$ random edges. Notice that if u, the neighboring vertex of$v$, is small then $\lambda'=0$ otherwise $\lambda'=\frac{\alpha_u}{\alpha}$. Set λ'_v to be the average value of $1-\lambda'$ over the first β edges that the Maker claims at v. Clearly, $\lambda'_v \in [0,1]$ since it is the average of quantities in $[0,1]$. We can the fact that the Maker degree is bounded above by Δ and do some calculations to obtain that the Maker degree never exceeds $\frac{(2n-(3-2\lambda'_v)\alpha\beta)}{q-\alpha}$. □

Given the existence of λ'_v for a large vertex, the following constant is defined and used later.

$$\lambda_v = \frac{q/2+\alpha-\Delta+\lambda'_v(\Delta-\alpha-\beta)}{q/2-\beta} \tag{3.3}$$

3.3 Analysis

The validity of the strategy proposed by Balogh and Samotij, 2011 depends on the following major theorem [Statement directly taken from Balogh and Samotij, 2011]. This analysis will prove that Eq. 3.2 holds, and as a result, the strategy presented will work. This will show that Theorem 3.2.1 holds as well.

Theorem 3.3.1. *For every t and every pair of vertices u and v such that $uv \notin G_M^t \cup G_B^t$, then we have the following. Suppose u and v are large at the beginning of round $t+1$ and let λ_u, λ_v be as Eq. 3.3. Then for every $\varepsilon > 0$ regardless of the Maker's strategy,*

$$e_B^t(u,v) \geqslant \lambda_u \lambda_v \beta^2 \frac{\left(\frac{q}{2} - \beta\right)}{n} - \varepsilon\sqrt{n}. \tag{3.4}$$

with probability $1 - \exp\left(-\Omega_\varepsilon(\sqrt{n})\right)$

Proof. We will define a few sets and random variables on which the analysis will be done. Let $E(u,v)$ denote the set of edges between u, v and $N_M^t(v)$ and $N_M^t(u)$ respectively. This will denote a total of $\deg_M^t(u) + \deg_M^t(v)$ edges. Let $E_B^t(u,v)$ be the *random variable* denote the set of Breaker edges in $E(u,v)$ towards the end of round t. Let $F_B^t(u,v)$ be the *random variable* denoting all the edges in $E(u,v)$ assuming that the Breaker only claims random edges in $E(u,v)$. Notice that in the conditions specified for the theorem to hold we have assumed that $uv \notin G_B^t \cup G_M^t$ so we can say that the Maker has claimed no edges in $E(u,v)$ because otherwise the Breaker would have claimed uv in order to close any potential two paths. Additionally, in case 1 of our strategy then Breaker randomly orders the vertices neighboring u, v and then claims the edges. So, we can extend this argument and see that $F_B^t(u,v) \subseteq E_B^t(u,v)$. Since the Maker's moves are entirely deterministic they can always leverage the entire information till a given round say s, and play accordingly. However, this luxury is not available with the Breaker. The earlier the Breaker makes a random move or decision regarding a vertex the more information Maker has to play their next move. Recall that the Breaker claims $q/2 - \beta$ random edges. Now, for a vertex u in the first $\lambda_u\beta$ rounds the Breaker would claim $\lambda_u\beta(q/2-\beta)$ edges randomly. And similarly for v it'd be $\lambda_v\beta(q/2-\beta)$. Next notice that claiming an edge whose one endpoint is incident at an edge claimed by the Breaker is useless for the Maker, since it will never lead to a two path and subsequently to a triangle. Hence, the Maker would avoid claiming such edges. In our context we can say that whenever the Maker claims an edge at u we can safely assume that the other end point is not incident at to an edge at v in G_B^t. Similarly for v we can make such claims. So, following the definition of $e_B^t(u,v)$ and the above logic we can safely say that the Maker will only contribute to a decrease in $e_B^t(u,v)$.

Estimating $e_B^t(u,v)$. Define $r = q/2 - \beta$ and $p = r/n$. Since we are only focusing on the edges in $E(u,v)$ the authors only consider those rounds in which the Breaker claims edges incident on u or v. For a suitable choice of λ_u and λ_v we can assume that this claiming of edges on u, v happens in the first $(\lambda_u + \lambda_v)\beta$ rounds. Thereafter the Breaker will cease to claim random edges at u, v. This is because we are focussing on Case 1 of the Breaker's strategy where the limelight is on the vertices u, v. In any case, the entire strategy is focussing on particular vertices. Even when the Breaker is claiming random edges it does so only at vertices u or v. Hence, as per our requirement of t in the theorem statement, we may additionally assume that $t = (\lambda_u + \lambda_v)\beta$. We will collect all definitions made and define a few more constants below,

Definition 9. For every $x \in \{u, v\}$ and s with $0 \leqslant s \leqslant t$,

1. $r = \frac{q}{2} - \beta$
2. $p = \frac{r}{n}$
3. $t = (\lambda_v + \lambda_u)\beta$
4. $d_x(s) = \deg_M^s(x)$
5. $\delta_x(s) = d_x(s) - d_x(s-1)$.
6. For all $e \in E(u, v)$ let A_e be the event that Breaker claims e by the end of round t

Notice how under our assumptions $d_x(t) = \lambda_x\beta$. We will now lower bound the probability of the event A_e. This will be done using calculations under our assumptions and then used to calculate the expected value of the number of edges in $E(u, v)$ which are claimed by the Breaker. Suppose in this case, $e = uw$ for some w. Now, as per our theorem $uv \notin G_M^t \cup G_B^t$. The Maker would simply not be able to close the triangle because of the conditions of our theorem. So, if the Breaker is claiming $e - uw$ at round t it means that the Maker must have claimed vw for some $w \in V$ in a previous round, i.e., for some $s_e \leqslant t$. So, the Breaker would have claimed $r(d_u(t) - d_u(s_e))$ random edges between s_e and t. Recall that $r = q/2 - \beta$. This gives us,

$$\mathbb{P}(A_e) \geqslant \frac{r(d_u(t) - d_u(s_e))}{n-1} \geqslant p(d_u(t) - d_u(s_e)).$$

Symmetrically, if $e = vw$ then $\mathbb{P}(A_e) \geqslant p(d_v(t) - d_v(s_e))$. Let us take a step back and understand which part of the strategy this analysis relates to. Since we wish to understand how many random edges the Breaker can claim within the set $E(u, v)$ it is important to recall that in the strategy the Breaker isn't just arbitrarily claiming random edges but they are doing so in response to the Maker's move. Since the Maker is deterministic the Breaker can only claim random edges following the random ordering as mentioned in Case 1 after the Maker claims certain edges which share neighbors in $E(u,v)$. Thus in estimating $e^t_B(u,v)$ we have to track the Maker's move keeping in mind the set $E(u,v)$ that we are interested in. This is exactly what the authors are doing here.

Now notice that once an edge has been claimed it cannot be "re-claimed" by any player. If the Breaker claims $uw \in E(u,v)$ in round s_{uw} then the Maker would have claimed vw in the same round as the Breaker moves as a 'response' to the Maker's moves. So, we can conclude that $\delta_u(s_{uw}) = 0$ and $\delta_v(s_{uw}) = 1$. Similarly, if we focus on $vw \in E(u,v)$ then $\delta_v(s_{vw}) = 0$ and $\delta_u(s_{vw}) = 1$. We can now estimate the **expected value** of $e^t_B(u,v)$. We take an indicator function which takes value 1 every time e has been claimed by the Breaker and 0 otherwise. $e^t_B(u,v)$ will be the sum of such indicator random variables over all e. Also, note that $d_v(s)\delta_u(s) = d_v(s-1)\delta_u(s)$.

$$
\begin{aligned}
\mathbb{E}\left[e^t_B(u,v)\right] &= \sum_{e\in E(u,v)} \mathbb{P}(A_e) \\
&\geqslant p\cdot\sum_{s=1}^{t}\left[(d_u(t)-d_u(t))\delta_v(s)+(d_v(t)-d_v(s))\delta_u(s)\right] \\
&= p\cdot\sum_{s=1}^{t}[(d_u(t)-d_u(t))\delta_v(s)+(d_v(t)-d_v(s-1))\delta_u(s)] \\
&= p\cdot\left[d_u(t)\sum_{s=1}^{t}\delta_v(s)+d_v(t)\sum_{s=1}^{t}\delta_u(s)-\sum_{s=1}^{t}(d_v(s)d_u(s)-d_v(s-1)d_u(s-1))\right] \\
&= pd_u(t)d_v(t) \\
&= \lambda_u\lambda_v\beta^2(q/2-\beta)/n
\end{aligned}
$$

We will now see the concentration of the random variable $e^t_B(u,v)$. Using the hypergeometric concentration which we mentioned in the previous section we'll see that the the value of the random variable is concentrated around the lower bound on its expectation. For a small positive constant $\varepsilon > 0$ and a round s before t we consider the event $B(s,\varepsilon)$ which is,

$$e^t_B(u,v) \geqslant pd_u(s)d_v(s) - \varepsilon\sqrt{n}$$

So, the event B denotes the value of the number of closing path edges in the Breaker graph to be greater than a quantity which is smaller than what we established in the calculations beforehand. We will now have the following lemma. This lemma states that for two rounds s' and s which are 'close enough' depending on the parameter ε and n, p but $s' > s$ if we have that the value of $e_B^s(u,v)$ was greater than the bound mentioned above then the the value of $e_B^{s'}(u,v)$ would also be greater than the corresponding quantity for the given bound with a large probability. What this means is that number of random edges the Breaker claims between u, v will remain the same across the moves the Maker chooses to make. Mathematically,

Lemma 5. Let $\varepsilon, \varepsilon'$ be positive reals and let s, s' be integers which satisfy $0 \leqslant s \leqslant s' \leqslant t$. If $s' - s \leqslant \frac{\varepsilon\sqrt{n}}{\Delta p}$ and $(s'-s)^2 \leqslant \frac{\varepsilon'\sqrt{n}}{2p}$, then

$$\mathbb{P}\left(B(s', \varepsilon + \varepsilon') | B(s, \varepsilon)\right) \geqslant 1 - \exp\bigl(-c\sqrt{n}\bigr), \tag{3.5}$$

where constant $c > 0$ depends only on ε'.

Proof. We will sketch the proof, for the calculations and details we refer the readers to the paper by Balogh and Samotij, 2011. Let us first understand the conditioning. For t' such that $0 \leqslant t' \leqslant t$ let, $e_u(t')$ denote the number of edges in $G_B^{t'}$ between u and $N_M^t(v)$ and $e_v(t')$ denote the number of edges in $G_B^{t'}$ between v and $N_M^t(u)$. In words, the variables represent the number of closing edges the Breaker has managed to claim till round t' out of the total closing edges that it will need to claim till round t. Fix constants e_u and e_v such that $e_u + e_v \geqslant pd_u(s)d_v(s) - \varepsilon\sqrt{n}$. The event we condition on will be $e_u(s) = e_u$ and $e_v(s) = e_v$, i.e.; the beaker has fulfilled the required lower bound till the s-th round. Now, there will be at $d_v(s) - e_u$ edges not adjacent to u in G_B^s after the s-th round. Since the Breaker claims r edges in response to the Maker's moves, we can see that there will be at most $n - rd_u(s)$ unclaimed edges at u. Starting from round $s+1$ till round s' the Breaker claims $r(d_u(s') - d_v(s'))$ total edges. We reframe the above quantities into the formulation of Lemma 1. So, we have $N = n - rd_u(s)$ out of which $m = d_v(s) - e_u$ vertices are marked by the Maker since they might form two paths and the Breaker is claiming $n = r(d_u(s') - d_u(s))$ edges in total from round $s+1$ till s'. So, we can use the results in Lemma 1 for Hypergeometric(N, m, n) as shown above for the random variable $e_u(s') - e_u$. Our random variable of interest is lower bounded by a random variable with Hypergeometric(N, m, n) distribution. Hence, we have with probability at least $1 - \exp\left(\frac{-(\varepsilon')^2\sqrt{n}}{32}\right)$ the following holds. Notice how the constant c pops up as a function of ε' in the above as mentioned in Lemma 5. Additionally, we have taken the complement of the set as mentioned in Lemma 1. So, we have

$$e_u(s') - e_u \geqslant \frac{(d_v(s) - e_u)r(d_u(s') - d_u(s))}{n - rd_u(s)} - \frac{\varepsilon'\sqrt{n}}{4}. \tag{3.6}$$

Now, if $e_u \leqslant pd_v(s)d_u(s)$ which is possible since $e_u + e_v \geqslant pd_u(s) + d_v(s)$ and both $e_u, e_v \geqslant 0$ we will have from Eq. 3.6 that $\frac{d_v(s)-e_u}{n-rd_u(s)} \geqslant \frac{d_v(s)}{n}$ as a result

$$e_u(s') - e_u \geqslant pd_v(s)(d_u(s') - d_v(s)) - \frac{\varepsilon'\sqrt{n}}{4}$$

Similar arguments apply to the random variable $e_v(s') - e_v$. We have shown that with large enough probability, the difference in the number of edges in $G_B^{s'}$ from the same amount in G_B^s is lower bounded by $pd_v(s)(d_u(s') - d_v(s)) - \frac{\varepsilon'\sqrt{n}}{4}$ and similarly for the adjacent vertex v. Now notice that we had initially set out to find a bound on $e_B^{s'}(u,v)$ and this is nothing but the sum of the random variables $e_v(s')$ and $e_u(s')$. So, taking a sum of these variables will give us $e_B^{s'}(u,v)$. Before we proceed, notice that for a vertex $x \in \{u,v\}$, we have $d_x(s') - d_x(s) \leqslant d_u(s') + d_v(s') - d_u(s) - d_v(s) = s' - s$ since the Maker can claim only one edge per move and we are calculating an upper bound. Due to the definition of $e_u(t')$ and $e_v(t')$, we can see that these events will be independent because $N_M^t(u) \cap N_M^t(v) = \emptyset$ because otherwise if there was a common vertex in neighboring set of this means that the Maker has successfully created a two path till round t is played. Since the Maker is completely deterministic, it can close these two paths by claiming u, v before round t is played and ends the game. However, we have assumed in Theorem 3.3.1 that $uv \notin G_M^t \cup G_B^t$. Hence, this cannot be the situation. This implies $N_M^t(u) \cap N_M^t(v) = \emptyset$. Which, in turn implies that $e_u(t')$ and $e_v(t')$ are independent. Condition to $e_B^s(u,v) = e_u + e_v$ such that $pd_u(s)d_v(s) - \varepsilon\sqrt{n} \leqslant e_u + e_v \leqslant pd_u(s)d_v(s)$. Recall the assumption that $p(s - s') \leqslant \frac{\varepsilon'\sqrt{n}}{2}$ Now, using basic rules of probability, we see that for the intersection of both the events, we have that with probability at least $1 - 2\exp\left(\frac{-(\varepsilon')^2\sqrt{n}}{32}\right)$,

$$\begin{aligned} e_b^{s'}(u,v) &\geqslant e_u + e_v + p\left[d_v(s)(d_u(s') - d_u(s)) + d_u(s)(d_v(s') - d_v(s))\right] - \frac{\varepsilon'\sqrt{n}}{2} \\ &\geqslant pd_u(s')d_v(s') - (\varepsilon + \varepsilon')\sqrt{n} \end{aligned}$$

If we condition on $e_B^s(u,v) \geqslant pd_u(s)d_v(s)$ then with probability 1 we'll get $e_B^{s'}(u,v) \geqslant pd_u(s')d_v(s') - \varepsilon\sqrt{n}$ which completes the proof of Claim 5. □

We now enter the final leg of the analysis where we conclude Theorem 3.3.1. Define

$K := \frac{64}{\varepsilon}$. Notice how this K will be greater than $\frac{16\Delta^2 p}{\varepsilon\sqrt{n}}$, mathematically,

$$K \geqslant \frac{16\Delta^2 p}{\varepsilon\sqrt{n}} \geqslant \max\left\{\frac{4t^2 p}{\varepsilon\sqrt{n}}, \frac{2t\Delta p}{\varepsilon\sqrt{n}}\right\} \tag{3.7}$$

The motivation behind defining this K is to subsequently use it to parameterize the rounds properly so that our conditions of $s' - s \leqslant \frac{\varepsilon\sqrt{n}}{\Delta p}$ and $(s'-s)^2 \leqslant \frac{\varepsilon'\sqrt{n}}{2p}$ from Lemma 5 is satisfied allowing us to use its results. To do so we use K in the following manner,

$$s_k := \frac{kt}{K}, \quad \varepsilon' := \frac{\varepsilon}{2K}.$$

Notice how $s_{k+1} - s_k \leqslant \frac{\varepsilon\sqrt{n}}{2\Delta p}$ and $(s_{k+1} - s_k)^2 \leqslant \frac{\varepsilon'\sqrt{n}}{2p}$ holds true for each k. We can define events on the basis of s_k and K following our previous notation $B\left(s_k, \frac{K+k}{2K}\cdot\varepsilon\right)$ and $B\left(s_{k+1}, \frac{K+k+1}{2K}\cdot\varepsilon\right)$ and so on for successive events. Now, using these definitions, we apply Lemma 5 to get that,

$$\mathbb{P}\left(B\left(s_{k+1}, \frac{K+k+1}{2K}\cdot\varepsilon\right) \,\middle|\, B\left(s_k, \frac{K+k}{2K}\cdot\varepsilon\right)\right) \geqslant 1 - \exp\big(-c\sqrt{n}\big).$$

The very beginning of the game can be represented as $B(s_0, \varepsilon/2)$ and the probability of this event is 1. Using the above conditioning, we can conclude that Eq. 3.4, i.e., the event $B(t,\varepsilon)$ holds with probability $1 - \exp(-\Omega_\varepsilon(\sqrt{n}))$. □

We are not done yet, we still need to check if Eq. 3.2 is satisfied or not. To do so we expand the terms on the right. Using the upper bound on the Maker degree of u, upper bound on the Maker degree of v, and the lower bound on $e^t_B(u,v)$ obtained via Theorem 3.3.1 we get,

$$q - \varepsilon\sqrt{n} > \frac{2n - (3-2\lambda'_u)\alpha\beta}{q-\alpha} + \frac{2n - (3-2\lambda'_v)\alpha\beta}{q-\alpha} - \lambda_u\lambda_v\beta^2(q/2-\beta)/n. \tag{3.8}$$

Now, recall that this needs to be satisfied in Case 3 of the strategy at hand. There are additional conditions on u, v, i.e., they need to be large. There will only be polynomially bounded number of vertices which are large, i.e., with $\deg^t_M(v) > \beta = \frac{3\sqrt{n}}{4}$ so we can just enumerate all possible triples (u,v,t) in polynomial number of steps and then check if there exists a positive ε such that for all $\lambda'_u, \lambda'_v \in [0,1]$ Eq. 3.8 holds true. This can be done computationally.

In fact due to the definitions the authors have made, we needn't even go through the trouble of checking polynomial number of possibilities. Notice that the entire right hand side in Eq. 3.8 is linear and symmetric in λ'_u, λ'_v. So, we can just check if Eq. 3.8 is satisfied for the following pairs of (λ'_u, λ'_v): $(0,0), (0,1)$ and $(1,1)$. Given the choice of constants made by the authors it is always the case that inequality in Eq. 3.8 is satisfied. Hence, whenever we have Case 3 we can be sure that due to the random part of the Breaker's strategy there will always be enough edges in the Breaker graph which will mitigate the danger posed by Maker edges which have become large.

Chapter 4

A Deterministic Breaker Strategy

This chapter is based on the paper by Glazik and Srivastav, 2022 and the results contained in it. We focus on lucidly explaining the results. No claim to originality is made.

In this paper, the authors obtain an improvement on the upper bound on the threshold bias ($q_0(n)$) from the previously known bound of $q_0(n) \leqslant (2 - \frac{1}{24})\sqrt{n}$. Additionally, the strategy presented for the Breaker is deterministic in nature as opposed to the randomized strategy given by Balogh and Samotij, 2011, which leads to the aforementioned $q_0(n)$. The deterministic idea gives us a winning strategy for the Breaker for all $q \geqslant q_0(n)$ where $q_0(n) = \sqrt{(\frac{8}{3} + o(1))n}$. Like previous strategies, in order to prevent the game from immediately ending, we will first close any potential 'two-paths' and then define a function which will help us quantitatively define how 'dangerous' a vertex is. By 'dangerous', we mean that if the Maker claims the vertex v, which is defined to be more 'dangerous' than the vertex u, then the chances of the Maker winning the game increase significantly. So naturally, the Breaker would want to claim these 'dangerous' vertices before they end up in the Maker graph. This will be quantified by the *potential function*, which we will define in the next section. This notion of danger is quite natural since as the number of Maker edges incident on the vertex v increases, the propensity of the Maker to claim that vertex increases since it might eventually end up completing a triangle in the Maker graph.

4.1 Motivation & Basic Ideas

One of the basic motivations of the Breaker would be to prevent the appearance of two-paths, i.e., a graph structure containing three vertices and two edges, in the Maker graph. Note that every triangle contains two-paths. In case this structure appears in the Maker graph then the Breaker's first move would be to close the same. By closing the two-path, we mean that the Breaker would claim the edge which, if claimed, ends up forming a triangle in the Maker graph of which the said two-path is a part. Hence, the Breaker would be on the constant lookout for any new two-paths the Maker manages to form in the Maker graph. This part forms the essential part of the strategy that the Breaker adopts in order to achieve victory under the deterministic threshold bias. In addition to the above, as mentioned in the previous section, the major improvement occurs after the Breaker starts to claim vertices according to the 'danger' they pose.

In this paper, the authors highlight an important relationship between triangles and stars. The motivation of the Breaker is closely related to preventing the Maker from building $q/2$-stars. This is because stars and triangles (in graphs) are closely related. Let us do a worst case analysis. Suppose the Maker successfully claims a $q/2$-star and the Breaker has not claimed any edges incident on the nodes of the star (the ones connected to the center). Then, the Breaker has to potentially claim $\binom{q/2}{2}$ edges, and a simple calculation reveals that this quantity is greater than q and since the Breaker can claim only up to q edges per move we (for $q \geqslant 8$) are in a situation in which the Maker is guaranteed to win. Naturally, since we are focusing on calculating the threshold bias this is an unfavorable situation. Another aspect is that when the Maker manages to build two $q/2$-stars and connect the center of both by claiming an edge then the (in the worst case) the total number of two-paths that end up in the Maker graph exceeds q and thus makes it impossible for the Breaker to win under any strategy.

4.1.1 Balance of a Vertex

In this section, motivated by our exposition previously, we define a quantity that helps us in quantifying a certain important aspect of the game. Suppose you are in the middle of an iteration of the triangle game. Let, $v \in V$ with a degree in the Maker graph that is slightly less than what it takes to end up forming a $q/2$-star. In mathematical terms, $\deg_M(v) < \frac{q(1-\delta)}{2}$. Now, suppose that the Maker wants to complete the $q/2$-star at v so it decides to completely focus on v from this moment onwards. So, the Maker decides to claim edges incident on v till all the edges on v are claimed either by the Maker or by the Breaker. Along similar lines, the Breaker's aim besides the essential aspect of closing all two-paths will be to maintain $\deg_M(v) < \frac{q(1-\delta)}{2}$.

We will now define some important quantities. The notation used here is the same as the paper. Suppose both players are playing dynamically and rationally (i.e., with a logical strategy and to win) and at the moment when the Maker started concentrating on the vertex v the following quantities are being calculated. In order to achieve the aim of $\deg_M < \frac{q(1-\delta)}{2}$ the Breaker will need to claim some edges incident on v. Let, this quantity be B_v. Now, suppose the Maker achieves its goal of $\deg_M > \left\lceil \frac{q(1-\delta)}{2} \right\rceil$, say it takes the Maker T rounds to do so. Now, if the Breaker is allowed to claim b edges per round then the total number of edges the Breaker can claim at v before the Maker achieves its target is $B_{\text{total}} = Tb$. Now, following our exposition in the previous section we can say that in order not lose the game immediately the Breaker will have to claim some edges to close new Maker-paths. Let C be the amount of edges the Breaker claims in order to do so. In essence, we are left with $A := B_{\text{total}} - C$ edges in which the Breaker can be creative and strategic in order to prevent the Maker from achieving its goal. We can also think of A as the total available edges.

Now, observe the ratio $\frac{B_v}{A}$, this quantity represents the fraction of available edges which are necessary for the Breaker to achieve its goal of keeping $\deg_M < \frac{q(1-\delta)}{2}$. In some sense, it's a measure of the 'danger' that the vertex v poses for the Breaker. If we manage to quantify this using suitable calculations and approximations, then we will be in a position to analyze the game in a comprehensive manner. We draw a few insights from the ratio. Observe that if $\frac{B_v}{A} > 1$ then it implies that the number of edges the Breaker needs to claim to fulfill its goal is larger than the number of available edges. Hence, the lower the ratio the better it is for the Breaker. We mention a notation which we will use in a short while. Recall that, for $f, g : \mathbb{N} \to \mathbb{R}$ we write $f \sim g$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. We now arrive at the crux of this section and make the below claim. Let the balance of a vertex be denoted by $\text{bal}(v)$ so we have that:

Lemma 6. $\text{bal}(v) \sim \frac{B_v}{A'}$ for some $A' \leqslant A$.

Proof. In order to curb the Maker from building a $\frac{q(1-\delta)}{2}$-star the Breaker must claim at least $n - \frac{q(1-\delta)}{2}$ edges incident at v. Hence the Breaker has to still claim $B_v = n - \frac{q(1-\delta)}{2} - \deg_B(v) \leqslant n - \deg_B(v)$. Similarly, since the Maker is allowed to claim one edge per move and only cares about v we can calculate the number of rounds the Maker may require to breach the degree condition. This gives us $T = \frac{q(1-\delta)}{2} - \deg_M(v)$. Since Breaker can claim q edges per move, then $B_{\text{total}} = q\left(\frac{q(1-\delta)}{2} - \deg_M(v)\right)$. We will now calculate C. Since C is a dynamic quantity whose exact value depends on the moves made by the Maker, and how many closing edges have been claimed by the Breaker already, we will upper bound C with C' which we calculate after assuming that all closing edges are unclaimed. So the quantity can be calculated as

$$\begin{aligned} C' &:= \sum_{i=\deg_M(v)}^{\lceil \frac{q(1-\delta)}{2} \rceil - 1} i \\ &= \frac{(\lceil q(1-\delta)/2 \rceil - 1)\cdot \lceil q(1-\delta)/2 \rceil}{2} - \frac{(\deg_M(v)+1)\deg_M(v)}{2} \\ &\sim \frac{q^2(1-\delta)^2}{8} - \frac{\deg_M(v)^2}{2}. \end{aligned}$$

Now, we define A' finally as $A' := B_{\text{total}} - C' \leqslant A$. we use this to calculate the ratio and get

$$\begin{aligned} \frac{B_v}{A'} &= \frac{B_v}{B_{\text{total}} - C'} \\ &= \frac{n - \deg_B(v)}{\frac{q^2(1-\delta)}{2} - q\deg_M(v) - \left(\frac{q^2(1-\delta)^2}{8} - \frac{\deg_M(v)^2}{2}\right)} \\ &= \frac{8(n - \deg_B(v))}{q^2(1-\delta)(3+\delta) - 4\deg_M(v)(2q - \deg_M(v))} = \text{bal}(v). \end{aligned}$$

The above calculation concludes the proof of our Lemma. □

4.1.2 The Potential Function

We begin by defining a few constants, these constants have been defined by the authors in a retroactive manner, i.e., they have been defined to fit into the calculations exactly. If we try to carry out the same calculations by going out of the defined range for any constant then we run into trouble in the form of infinities or undefined quantities. Let $\varepsilon^* > 0$ be a positive constant and $\beta = \frac{8}{3} + \varepsilon^*$. We consider $q = \sqrt{\beta n}$. We know if $\beta \geqslant 4$ then there is a winning strategy given by Chvátal and Erdös, 1978. So, we assume $\beta \leqslant 4$ wherever necessary. We fix a constant $\delta \in \left(0, 1 - \frac{8}{3\beta}\right)$.

Definition 10. For every vertex $v \in V$ we define the *balance* of v as

$$\text{bal}(v) = \frac{8(n - \deg_B(v))}{q^2(1-\delta)(3+\delta) - 4\deg_M(v)(2q - \deg_M(v))}$$

At the beginning of the game, the Maker and Breaker degree of v is 0. Now, using the above formula we get that p_0 is the balance of every vertex at the beginning of the game, where

$$p_0 = \frac{8n}{q^2(1-\delta)(3+\delta)} = \frac{8}{\beta(1-\delta)(3+\delta)}.$$

We make the following basic observation.

Lemma 7. It holds that $\frac{8}{3\beta} < p_0 < \frac{8}{3\beta(1-\delta)} < 1$.

Proof. Using $\delta \in \left(0, 1 - \frac{8}{3\beta}\right)$ the third inequality follows immediately. Similarly a simple substitution will give us the second inequality as well. Now, observe that $(1-\delta)(3+\delta) = 3 - 2\delta - \delta^2 < 3$, so we get that $p_0 = \frac{8}{\beta(1-\delta)(3+\delta)} > \frac{8}{3\beta}$. □

We are yet to define the potential function for a vertex. However, we must define a few other terms before we can proceed to the potential function. As the game proceeds we will observe that the Breaker will not be able to maintain the starting potential across all vertices, some will observe an increase in potential. Depending on the Maker's moves some vertices will get more Breaker edges while other will get less. This *deficit* of a given vertex will also be defined below and used to calculate the potential.

We calculate the, at any point of the game, the balanced Breaker degree of a node. The balanced Breaker degree is simply the Breaker degree that is necessary to achieve $\text{bal}(v) = p_0$.

Definition 11. The *balanced Breaker degree* of a node v is

$$\deg^*(v) := n - p_0 \left(\frac{q^2(1-\delta)(3+\delta)}{8} - \deg_M(v) \left(q - \frac{\deg_M(v)}{2} \right) \right).$$

Building on this, we define the deficit of v as

Definition 12. The *deficit* of v is defined as $d(v) := \deg^*(v) - \deg_B(v)$.

We are now ready to define the potential of a node.

Definition 13. Let $\mu := 1 + \frac{6\beta \ln(n)}{\delta q}$. Define the potential of v as

$$\text{pot} := \begin{cases} 0 & \text{if } \deg_M(v) + \deg_B(v) = n-1 \\ \mu^{d(v)/q} & \text{else} \end{cases}$$

For an unclaimed edge $e = \{u, w\}$ we define the *potential of* e as $\text{pot}(e) := \text{pot}(u) + \text{pot}(w)$. For every round t we define $\text{pot}_t(v)$ and $\text{pot}_t(e)$ as the potential of the node v and edge e directly *after* round t. The *total potential* of a round t is defined as $\text{POT}_t := \sum_{v \in V} \text{pot}_t(v)$. The total *starting potential* is defined as $\text{POT}_0 := \sum_{v \in V} \text{pot}_0(v)$.

We can easily do a small sanity check and see that

Lemma 8. The total starting potential fulfills $\text{POT}_0 = n$.

Proof. For all $v \in V$ we have $\deg_M(v) = \deg_B(v) = 0$. Substituting the same in the formula for balanced Breaker degree we get, $\deg^*(v) = 0$. So, $\text{pot}(v) = \mu^{d(v)/q} = \mu^{0/q} = 1$, so summing over n nodes we get $\text{POT}_0 = n$. □

Since we have clarified that the 'potential' function quantifies the amount of 'danger' a node poses to the Breaker, it is only natural that the Breaker aims to keep the total potential as low as possible. In fact we show next that keeping the potential of each node below $2n$ ensures the Maker from raising the degree of a node above $\frac{q}{2}$

Lemma 9. If n is sufficiently large, for every round t and every node $v \in V$ we have:

$$0 < \text{pot}_t(v) \leqslant 2n \Rightarrow \deg_M^t(v) \neq \left\lceil \frac{q}{2} \right\rceil - 1.$$

Proof. The proof is by contradiction. For a round t and $v \in V$ such that $\text{pot}_t(v) > 0$ and $\deg_M^t(v) = \lceil q/2 \rceil$ we will show that the potential will be more than $2n$, i.e., $\text{pot}_t(v) > 2n$. The idea is to make suitable substitutions and use the formulae of Definitions 10, 11, 12, and 13. We can calculate $d_t(v)$ (the deficit of v after round t) by substituting $\deg_M^t(v) \geqslant \frac{q}{2} - 1$ and using the formula for balanced-Breaker degree. This will give us $d_t(v) = \frac{2\delta n}{3}$. Now, using this result we can see that

$$\begin{aligned}
\text{pot}_t(v) &= \mu^{\frac{d_t(v)}{q}} \geqslant \mu^{\frac{2\delta n}{3q}} \\
&= \left(1 + \frac{6\beta \ln(n)}{\delta q}\right)^{\frac{2\delta n}{3q}} \\
&= \left(1 + \frac{6\beta \ln(n)}{\delta q}\right)^{\left(\frac{\delta q}{6\beta \ln(n)}+1\right)\left(\frac{\delta q}{6\beta \ln(n)}+1\right)^{-1}\frac{2\delta n}{3q}} \\
&\geqslant e^{\alpha},
\end{aligned}$$

where

$$\alpha = \left(\frac{\delta q}{6\beta \ln(n)} + 1\right)^{-1} \frac{2\delta n}{3q} = \left(\frac{\delta q \mu}{6\beta \ln(n)}\right)^{-1} \frac{2\delta n}{3q} = \frac{4\beta n \ln(n)}{q^2 \mu} > 2\ln(n),$$

where we have used the definition of μ and the fact that for large enough n we will have $\mu < 2$. Using this we get $\text{pot}_t(v) \geqslant e^{\alpha} > n^2$ and for $n \geqslant 2$ this is at least $2n$. □

We are now done with the pedagogy of all the machinery we will use to prove the effectiveness of the strategy, which we will define in the next section.

4.2 The Strategy

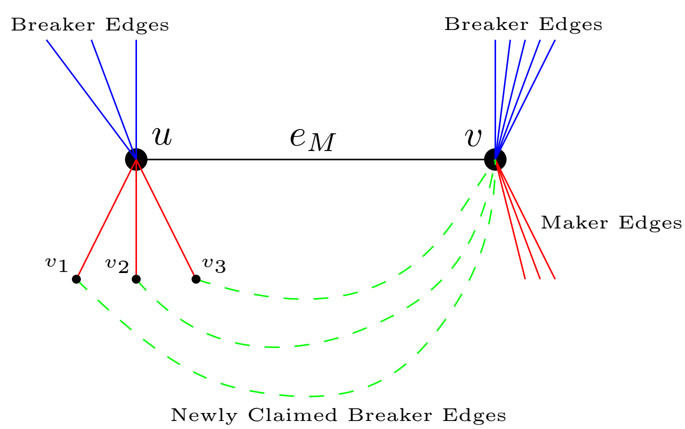

Figure 4.1: e_M is the edge claimed by the Maker in the round t. The red (edges below e_M) edges denote the edges in the Maker graph. The blue edges (edges above e_M) denote the Breaker edges. The green (dashed) edges denote the edges the Breaker has to claim in Part 1 of their strategy.

In this section, we mention the strategy adopted by the Breaker. Suppose the Maker manages to capture path $\{a, b, c\}$ of length 2 where $a, b, c \in V$. Now, the Breaker must claim $\{a, c\}$ in order to prevent the Maker from winning immediately in the next round unless $\{a, c\}$ is already a part of the Breaker graph. It is natural for the Maker to think that they will win if they manage to reach a stage where they have constructed more than q paths of length 2. So, the Breaker will exhaust their quota of q edges and still not be able to close all 2-paths. Also, note that by claiming an edge $\{a, b\}$ the Maker is constructing $\deg_M^t(a) + \deg_M^t(b)$ new two paths.

The Strategy: In an arbitrary round t. Let $e_M = \{u, v\}$ be the edge claimed by the

Maker in this round. The Breaker has two phases to their strategy.

- **Part 1: closing edges.** The Breaker will claim $\deg_M^{t-1}(u)$ edges incident in v and $\deg_M^{t-1}(v)$ edges incident in u this will help in closing all the emerging two-paths that the Maker could have taken advantage of. This part is highlighted in Figure 4.1. Notice the green dashed edges. If any such edge is already claimed then the Breaker will claim an arbitrary edge incident in u or v. If all the edges incident in u or v are claimed then the Breaker claims arbitrary unclaimed edges instead till it fulfils claiming $\deg_M^{t-1}(u) + \deg_M^{t-1}(v)$ edges in total. For the dashed edges highlighted in Figure 4.1 v is called the **head** and v_1, v_2, v_3 are called the **tail** of their respective edges. Similarly, we will have u as the head and $\{u_i\}$'s as the tail of their respective edges.
- **Part 2: free edges.** This is a crucial part of the Breaker's strategy since this is where they $\text{pot}(v)$ function is used extensively. Due to this, we have achieved major improvement. If, after part 1, the Breaker is left with edges to claim (which will always be the case, as we will see), they will follow an iterative process to claim further edges. They will claim an edge e such that $\text{pot}(e) \geqslant \text{pot}(e')$ for all available e, e'. This iteration will go on till they have claimed q edges. Edges claimed in this part are called *free edges.* Let $f(t)$ denote the number of free edges in round t. Observe that $f(t) = q - (\deg_M^{t-1}(u) + \deg_M^{t-1}(v))$.

Definition 14. In a round t if $e_M = \{u, v\}$ and all the edges incident to v, u are claimed then we call this round t as an *isolation round.*

Lemma 10. We can assume that the game contains no isolation rounds.

Proof. The crux of the argument is that if, after claiming e_M, the Maker knows that there will be no edges left incident on $\{u, v\}$, then it becomes an irrational move to make. Since the Breaker would have claimed and blocked all possible triangles containing e_M., a rational Maker will always want to avoid the isolation round. □

Will we always have free edges available? What if we reach a stage where there are no free edges? To answer these questions we make the following observations which ties the total potential with the number of available free edges. Lemma 11 tells us that the Breaker will have at least 2 free edges as long as they manage to keep total potential below $2n$ (which will always be the case as we will see in the following sections).

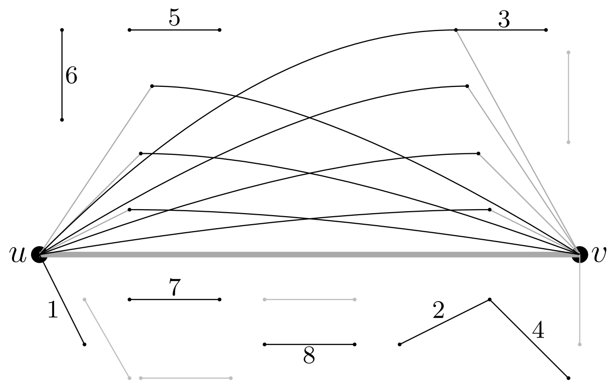

Figure 4.2: A snapshot of the execution of the strategy for a given round t. The thick, solid, dark gray line denotes the edge the Maker claims in round t. The solid dark gray lines indicate the edges in the Maker graph incident on u and v. Solid black edges indicate the edges claimed by the Breaker. The curved edges are claimed in Part 1 of the strategy. The scattered solid black edges are claimed in Part 2. An important aspect is that after a free edge has been claimed, the potentials are recomputed before claiming another free edge. The numbering on the free edges denotes the sequence in which they have been claimed. So, the potential of edge labeled 1 must be higher than the edge labeled 2. The solid light gray edges with gray nodes denote the edges that are part of the game but not claimed so far.

Lemma 11. For every round t with $f(t) \leqslant 1$ there exists a round $t' < t$ with $\mathrm{POT}_{t'} > 2n$

Proof. Let us start by assuming $f(t) \leqslant 1$ this gives us $\deg_M^{t-1}(u) + \deg_M^{t-1}(v) \geqslant q - 1$. So for some $x \in \{u, v\}$ we have that $\deg_M^{t-1}(x) \geqslant \lceil \frac{q-1}{2} \rceil \geqslant \lceil \frac{q}{2} \rceil - 1$ and as we know once the Maker claims an edge incident on $v \in V$ the Maker's degree can only increase. Hence, there must have been a round $t' < t$ such that $\deg_M^{t'}(x) = \lceil \frac{q}{2} \rceil - 1$ along with $\mathrm{pot}_{t'}(w) > 0$. Now applying Lemma 9 to get that $\mathrm{pot}_{t'}(x) > 2n$. Hence, $\mathrm{POT}_{t'} > 2n$. □

To elucidate the $q/2$-star argument further, assume the Maker has managed to construct two stars of size $q/2 - 1$; then, if the edge connecting the center of these two stars is claimed by the Maker, then the total number of 'two-paths' that emerge is $q - 2$ which the Breaker can close and also has some room for improvement. Now, if the Maker claims two $q/2$-stars then the connecting edge will give q potential two-paths. Now, the Breaker is left with no room for improvement. As the size of the star breaches $q/2$ the connecting edge will give rise to more than q potential two paths which the Breaker will not be able to close and hence lose the game.

4.3 Main Results and Schematic

Theorem 4.3.1. *For every round s it holds that $POT_s < 2n$.*

Theorem 4.3.1 forms the primary result of this paper, and its analysis and proof are very involved and fragmented into several smaller lemmas and theorems. We will first see how we can use Theorem 4.3.1 to demonstrate the Breaker's strategy's effectiveness and then analyze it. We also present a schematic flow of the proof of the above theorem, which will help the reader keep track. We have the below Theorem 4.3.2. Refer to Figure 4.3 for a schematic idea of how the authors prove Theorem 4.3.1.

Theorem 4.3.2. *If the Breaker follows the strategy then at the end of the game there does not exist any Maker vertex with degree of at least $\frac{q}{2}$ and thus Breaker wins the game.*

Proof. Assume $\exists\ v \in V$ such that $\deg_M(v) \geqslant \frac{q}{2}$ towards the end of the game. Then, let the Maker claim the $\lceil q/2 \rceil$-th edge incident in v in the round t. So, following similar ideas as in Lemma 11 we have $\deg_M^{t-1}(v) = \lceil q/2 \rceil - 1$. But, from Lemma 9 we know is potential of v is less than or equal to $2n$ then it cannot have $\deg_M^{t-1}(v) = \lceil q/2 \rceil - 1$. We also know from Theorem 4.3.1 that $\mathrm{pot}_{t-1}(v) \leqslant \mathrm{POT}_{t-1} < 2n$. We also know $pot_{t-1}(v) > 0$. So, we have a contradiction. Hence, with every $e_M = \{u, v\}$ the Maker does not create more than q two-paths. The Breaker is able to close the same since it does not exhaust their quota of q edges. As a result, the Breaker wins the game. □

Now we will look at how the authors have proved Theorem 4.3.1. A schematic version of the proof has been traced in Figure 4.3 for the reader's convenience. We divide all the rounds into two categories (to be defined later) namely, *critical* and *non-critical.* In a broad sense, since we have established that the Breaker needs to maintain a low potential, or in better words, given a long enough time period the total potential of the game should have a decreasing trend. However, it's imperative that in some rounds the total potential of the game is going to increase and no matter what set of edges the Breaker claims, the potential difference between the round just concluded and the previous round is going to be significant and positive, therefore causing an increase in total potential. Thus, the round which just concluded (and thereby caused an increase in the total potential) is called a *critical round.* The authors will present several lemmas and theorems (refer to Figure 4.3) in order to handle this increase. The basic idea is that a constant number of *critical rounds* cannot significantly affect the total potential of the game and hence, as has been demonstrated, that the total potential of the game stays below $2n$ (Theorem 4.3.1).

4.3.1 Change in Potential

In this subsection, we do some calculations using the tools we introduced before in order to use the results later. This subsection contains a lot of lemmas which are calculation heavy and not mathematically rich, so in this thesis, I will only sketch the proofs instead of rewriting them for simplicity. For the detailed proofs I refer the reader to Section 3.2 of the paper by Glazik and Srivastav, 2022.

Lemma 12. Consider a vertex $u \in V$ and let $\deg^{*\prime}(u), \deg'_M(u)$ and $d'(u)$ be the balanced Breaker-degree, Maker-degree, and deficit of u after the Maker claims an additional edge incident in u. Then,

$$d'(u) - d(u) = \deg^{*\prime}(u) - \deg^*(u) \leqslant p_o(q - \deg_M(u))$$

The proof of Lemma 12 is quite simple. Just notice that the Maker claiming an edge does not affect $\deg_B(u)$ and $\deg'_M(u) = \deg_M(u) + 1$. Therefore, using the formula defined in Definition 11 we can calculate $\deg^{*\prime}(u) - \deg^*(u)$ which will end up proving Lemma 12. The next lemma calculates the change in potential of a node caused by a single edge.

Lemma 13. (i) Let e_M be the single edge claimed by the Maker in any round. Then, e_M increases the potential of a vertex by *at most* a factor of μ and causes a total potential increase of at most $(\mu - 1)\text{pot}(e_M)$.

(ii) Let e_B be the single edge claimed by the Breaker in any round. Then, e_B causes a total potential decrease of *at least* $(1 - \mu^{-1/q})\text{pot}(e_B)$.

Proof. **(i)** If $e_M = \{u, v\}$ then for any vertex $w \in V$ if $w \neq \{u, w\}$ then $\text{pot}(w)$ does not change. If, $w \in \{u, v\}$ and if e_M is the last edge claimed then $\text{pot}(w) = 0$ by Definition 13. In all other cases using Lemma 12, Lemma 7, and Definition 13, we get $\frac{\text{pot}'(w)}{\text{pot}(w)} \leqslant \mu$, where $\text{pot}'(w)$ is the potential of w just after Maker claims e_M. The change can be calculated by $(\text{pot}'(v) + \text{pot}'(u)) - (\text{pot}(u) + \text{pot}(v)) \leqslant (\mu - 1)\text{pot}(e_M)$ as e_M only affects the potential of $\{u, v\}$.

(ii) Using similar ideas as above, let $e_B = \{u, v\}$. Note that e_B will only affect u and v (decreasing their potential). The total potential decrease can be calculated by $(\text{pot}(u) + \text{pot}(v)) - (\text{pot}'(v) + \text{pot}'(u))$. Using Definition 13 we get $\text{pot}(w) - \text{pot}'(w) \geqslant (1 - \mu^{-1/q})\text{pot}(w)$ for $w \in \{u, v\}$. We correct a small typographical error in the paper here. The ratio of the potentials is $\frac{\text{pot}'(w)}{\text{pot}(w)} = \mu^{(d'(w) - d'(w))/q} = \mu^{-1/q}$.

□

We will now segment the increase and decrease in the potential caused by Maker and Breaker edges to analyze each part separately and bound them using the tools that we introduced previously. To do the same we define the segments below. Recall that every round starts with a Maker move. Let $e_M = \{u, v\}$ be the edges the Maker claims. This is followed by q Breaker moves.

Definition 15. For every vertex $w \in V$,

1. $I_t(w)$ denote its potential increase.
2. $D_t(w)$ denote its potential decrease. There are four ways for potential to decrease by a Breaker move.
   (a) $D_t^{\text{free}}(w)$ – Decrease caused by w being part of a free edge.
   (b) $D_t^{\text{head}}(w)$ – Decrease caused by w being the head of a closing edge.
   (c) $D_t^{\text{tail}}(w)$ – Decrease caused by w being the tail of a closing edge.
   (d) $D_t^{0}(w)$ – Decrease cause if w is part of the last unclaimed edge incident in w (claimed by either the Maker or Breaker). The authors consider this case separately. If the last incident edge in w is claimed then it contributes to both $D_t^{\text{free}}(w)$ - the amount of potential change caused by calculating from Definition 13. Additionally, it will also contribute to $D_t^{0}(w)$ - the total potential decrease after subtracting the calculated contribution to $D_t^{\text{free}}(w)$.
   (e) We split $D_t^{\text{head}}(w) = D_t^{+}(w) + D_t^{-}(w)$.
   (f) $D_t^{+}(W) := \max\{D_t^{\text{head}}(w) - I_t(W), 0\}$. Note that if the Maker claims an edge connecting two high Maker-degree vertices then as the Breaker executes the closing-path part of its strategy, it may so happen that $D_t^{\text{head}}(w) > I_t(w)$. Otherwise, $D_t^{+}(w) = 0$.
   (g) $D_t^{-}(w) := \min\{I_t(w), D_t^{\text{head}}\}$. Note that if $D_t^{+} = 0$ then $D_t^{-}(w) = D_t^{\text{head}}(w)$.

Define $I_t := \sum_{v\in V} I_t(v)$. For every round t we can now write the total potential change in the above terms as

$$\mathrm{POT}_t - \mathrm{POT}_{t-1} = I_t - D_t = I_t - (D_t^{\mathrm{free}} + D_t^{+} + D_t^{-} + D_t^{\mathrm{tail}} + D_t^{0}).$$

We are now ready to quantify the change in potential for each round t. We will mention the below lemma without proof because its mostly calculations. We refer the reader to Section 3.2, Lemma 12 in the paper by Glazik and Srivastav, 2022 for the complete proof.

Lemma 14. For any arbitrary round t let the Maker claim the edge e_M. Then,

(i) for every $w \in V$ it holds $I_t(w) - D_t^{-}(w) \leqslant (\mu^{p_0 f(t)/q} - 1)\mathrm{pot}_{t-1}(w)$.

(ii) $I_t - D_t^{-} \leqslant (\mu^{p_o f(t)/q} - 1)\mathrm{pot}_{t-1}(e_M)$.

We can now quantitatively characterize what a 'critical round' means. We know that $\mu \to 1$ as $n \to \infty$ using Lemma 7 and the fact that all of the proofs we have done so far are with the assumption that n is very large we get that $\mu p_0 < 1$. Let us characterize the potential change.

Definition 16. Let us fix a constant $\eta \in (0, 1 - \mu p_0)$. For every round t we define $\Delta_t := I_t - D_t^{-} - (1-\eta)D_t^{\mathrm{free}}$ and $r_t := D_t^{+} + D_t^{\mathrm{tail}} + \eta D_t^{\mathrm{free}} + D_t^{0}$. We call the round t *critical*, if $\Delta_t > 0$ and *non-critical* otherwise.

Notice that we can now use Δ_t and r_t to rewrite the change in potential as $\mathrm{POT}_t - \mathrm{POT}_{t-1} = \Delta_t - r_t$. In every round, there is either a positive decrease or no decrease, so we have $r_t \geqslant 0$. Now, as we had qualitatively defined a critical round to be the one where any Breaker move cannot suffice to decrease the increase in potential caused by the Maker, we finally have, using the above formulation, that if $\mathrm{POT}_t > \mathrm{POT}_{t-1}$ then t is critical. Using some basic calculus identities and properties, we have the following lemma.

Lemma 15. For all $x \in \mathbb{R}$ with $x \geqslant 1$ it holds that $x(1 - \mu^{-1/q}) \geqslant 1 - \mu^{-x/q}$.

We will now bound the potential of each edge in a critical round, which remains unclaimed after the round. This will help us in bounding the potential of an edge that is being claimed after a fixed critical round (say t_0) with the potential of the vertex, which is part of the Maker edge claimed in the round t_0. In essence; it conveys that given a long enough time period, the rise in potential caused by the critical round will be mitigated by the Breaker's strategy. We will see a series of lemmas that put the above idea into mathematical terms. For this section, we begin with the following.

Lemma 16. If t is a critical round with $f(t) \geqslant 2$ and e_M is the edge chosen by the Maker in this round t then for every unclaimed edge e after round t it holds that:

$$\mathrm{pot}_t(v) < \frac{\mu p_0}{1-\eta}\mathrm{pot}_{t-1}(e_M).$$

Proof. For $e_M = \{u, v\}$ we have from Lemma 14(ii) that,

$$\begin{aligned} I_t - D_t^- &\leqslant (\mu^{p_0 f(t)/q} - 1)\mathrm{pot}_{t-1}(e_M) \\ &= \mu^{p_0 f(t)/q}(1 - \mu^{-p_0 f(t)/q})\mathrm{pot}_{t-1}(e_M) \\ &\leqslant \mu(1 - \mu^{-p_0 f(t)/q})\mathrm{pot}_{t-1}(e_M). \end{aligned}$$

Where the last inequality followed from the fact that $p_0 < 1$, and $f(t) < q$ (by definition). Now we will use Lemma 15 with $x := p_0 f(t)$. Observe that $x > 8f(t)/3\beta > 8f(t)/13 \geqslant 16/12 > 1$ so Lemma 15 is applicable. Applying the same, we get $I_t - D_t^- \leqslant \mu p_0 f(t)(1 - \mu^{-1/q})\mathrm{pot}_{t-1}(e_M)$. Now since by our assumptions t is a critical round, we get $0 < \Delta_t < \mu p_0 f(t)(1 - \mu^{-1/q})\mathrm{pot}_{t-1}(e_M) - (1-\eta)D_t^{\mathrm{free}}$. Which gives us,

$$(1-\eta)D_t^{\mathrm{free}} < \mu p_0 f(t)(1 - \mu^{-1/q})\mathrm{pot}_{t-1}(e_M). \tag{4.1}$$

Now, given our strategy, the Breaker only claims those edges iteratively which have the highest potential. So suppose an edge e is a free edge which is unclaimed after round t so we can safely say that all the free edges which were claimed in round t have a potential which is greater than that of e. According to Lemma 14 we know every free edge causes a potential decrease of at least $(1-\mu^{-1/q})\mathrm{pot}_t(e)$ so we have that $D_t^{\mathrm{free}} \geqslant f(t)\mathrm{pot}_t(e)(1-\mu^{-1/q})$. Combining this with Eq. 4.1 we get,

$$f(t)\mathrm{pot}_t(e)(1-\mu^{-1/q}) \leqslant (1-\eta)D_t^{\mathrm{free}} < \mu p_0 f(t)(1-\mu^{-1/q})\mathrm{pot}_{t-1}(e_M).$$

Which gives us $\mathrm{pot}_t(e) < \frac{\mu p_0}{(1-\eta)}\mathrm{pot}_{t-1}(e_M)$. □

4.3.2 Bounds on Total Potential

In this section we will prove the crux of the argument presented in the paper by Glazik and Srivastav, 2022. While we have seen that the potential may fluctuate and even increase in some rounds (note: we mean the total potential). The guarantee that the Breaker wins is provided by the proofs contained in this subsection. This guarantee is dependent on the total potential (and thereby the potential of a vertex) never going beyond $2n$., which will, in turn, prove that no vertex will ever cross $\deg_M^t(v) > q/2$ for all rounds t. So, we will see that whenever the potential increases beyond n there will be a round shortly after this breach which will bring the total potential down to be almost as large as it was before the round of the breach. We sketch the proofs in this section mostly since they are mostly calculations, and we elucidate any mathematical richness wherever necessary.

We define a few constants before we start. It's important to note that these constants have been retroactively defined to arrive at the results. They hold very little mathematical significance. Let $\gamma \in (0,1)$, and $\varepsilon > 0$ with

$$\frac{1-\eta}{(1+\varepsilon)\mu p_0} > 1. \tag{4.2}$$

Note that this gives us $\log(1-\eta) - \log(1+\varepsilon) - \log(\mu p_0) > 0$.

Similarly, $1-\gamma < 1 \Rightarrow \log(1-\gamma) \leqslant 0 \Rightarrow 1 - \log(1-\gamma) > 1$. Now, we define c as follows,

$$c := \left\lceil \frac{1-\log(1-\gamma)}{\log(1-\eta) - \log(1+\varepsilon) - \log(\mu p_0)} \right\rceil$$

and observe that $c > 0$ due to the observations we made previously. Notice that c is a function of n via μ but as we know $1 < \mu < 2$ as $n \to \infty$. We will now define a series of rounds characterized by the potential change they entail. which will help us in proving Theorem 4.3.3.

Definition 17. 1. t_0 – Let this be a round with $\mathrm{POT}_{t_0} > n$, $\mathrm{POT}_{t_0-1} \leqslant n$ and $\mathrm{POT}_t < 2n \ \forall\ t < t_0$. Let $e_0 = \{u, v\}$ be the edge claimed by the Maker in this round. WLOG, assume that $\mathrm{pot}_{t_0-1}(u) \geqslant \mathrm{pot}_{t_0-1}(v)$.

2. t_1 – Let this be the first round after $t_0 - 1$ such that $\mathrm{pot}_{t_1}(u) \leqslant (1-\gamma)\mathrm{pot}_{t_0-1}(u)$.
3. t_2 – Let this be the first round after t_0 with $\mathrm{pot}_{t_2}(w) \geqslant (1+\varepsilon)\mathrm{pot}_s(w)$ for some $w \in V$ and some s such that $t_0 \leqslant s < t_2$. That is, t_2 is the first round after t_0 in which the potential of a vertex has increased beyond what it was for all rounds played between t_0 and t_2 since $t_0 \leqslant s < t_2$
4. t_3 – Let this denote the c-th critical round after $t_0 - 1$.
5. $t^* := \min(t_1, t_2, t_3)$.
6. If a game ends before round t_i is reached then we will set $t_i := \infty$.

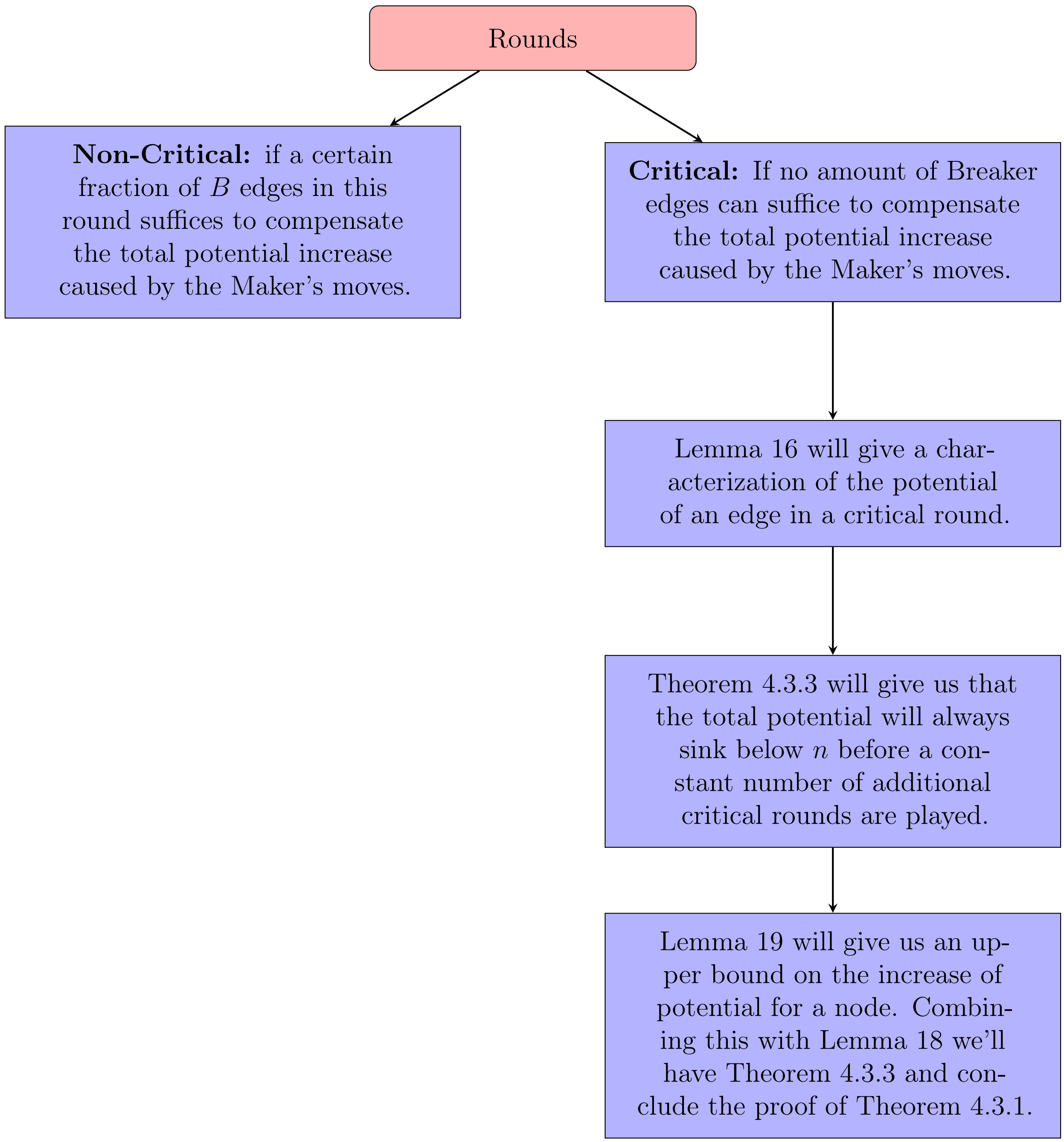

Figure 4.3: A schematic for the proof of Theorem 4.3.1.

We now have defined everything which will give us the following theorem saying that the potential will eventually decrease after the critical round t_0 has been played. This is followed by an observation which follows from the definitions we have made.

Theorem 4.3.3. *For sufficiently large n, if the game is not ended before round t^*, then $POT_{t^*} \leqslant POT_{t_0-1}$.*

Lemma 17. For sufficiently large n, for every t with $t_0 \leqslant t < t_2$ we have $\mathrm{POT}_t < 2n$.

This follows from the way t_2 has been defined, for every $v \in V$ we'll end up having that $\mathrm{pot}_t(v) \leqslant (1+\varepsilon)\mathrm{pot}_{t_0}(v)$. Hence, $\mathrm{POT}_t = \sum_{v\in V} \mathrm{pot}_t(v) \leqslant \sum_{v\in V}(1+\varepsilon)\mathrm{pot}_{t_0}(v) = (1+\varepsilon)\mathrm{POT}_{t_0}$. Now using the bounds on the increase and decrease in the potential in any round we have $\mathrm{POT}_{t_0} \leqslant \mu^{p_0 f(t_0)/q}\mathrm{POT}_{t_0-1} \leqslant \mu\mathrm{POT}_{t_0-1}$. So, $\mathrm{POT}_t \leqslant (1+\varepsilon)\mathrm{POT}_{t_0} \leqslant \mu(1+\varepsilon)\mathrm{POT}_{t_0-1} \leqslant \mu(1+\varepsilon)n < \frac{3}{2}n < 2n$ where the last inequality is because $1 < \mu < 2$ for large n.

Let us now understand how the authors have proceeded further in order to conclude Theorem 4.3.1. We will first refine the characterization of unclaimed edges. The result we obtained in Lemma 16 which gave us an upper bound on the potential of any unclaimed edge after round t can be refined further. For whatever follows, assume that the game does not end before round t^*. Now, if we focus our attention to the rounds between t_0 and t^* then we can bound the potential of any unclaimed edge in any round between this interval using $\mathrm{pot}_{t_0-1}(u)$ and the number of critical rounds that may happen between t_0 and t^*. This is captured below in Lemma 18.

Lemma 18. Let s be a round with $t_0 \leqslant s \leqslant t^*$ and $s < t_2$. Let $\mathrm{crit}(s) \in [c]$ be the number of critical rounds between t_0 and s inclusive of both. Then, for every edge e unclaimed after round s it holds that

$$\mathrm{pot}_s(e) < \left(\frac{(1+\varepsilon)\mu p_0}{1-\eta}\right)^{\mathrm{crit}(s)} 2\mathrm{pot}_{t_0-1}(u).$$

Proof. We only sketch the proof which proceeds by induction on $\mathrm{crit}(s)$. We first bound the potential of any edge unclaimed after round s using Lemma 16. This gives us the base case when $\mathrm{crit}(s) = 1$. Now, proceeding with the induction we can take the claim to be true for all s' with $\mathrm{crit}(s') = i$ and $i \in [c-1]$ (note that $[i] = \{1, 2, \ldots, i\}$). This forms our Induction Hypothesis. Now, for s with $\mathrm{crit}(s) = i + 1$ we can focus on the last critical round before the round s let it be s' and if s itself is critical then set $s = s'$. So, we have the number of critical rounds before s' to be i, i.e., $\mathrm{crit}(s' - 1) = i$ now we apply the fact that $t < t_2$, Lemma 16, and the induction hypothesis in that order to conclude the proof. ☐

Operating under the same framework as the above, we can also upper bound the total increase in potential in all the $t_0 \leqslant s \leqslant t^*$ where s is a critical round. The proof of the same is just an application of all the previous bounds we have obtained in Lemma 14, and Lemma 18. This gives us the following.

Lemma 19. For every $\xi > 0$, if n is sufficiently large, we'll have that

$$\sum_{t_0 \leqslant s \leqslant t^*, s \text{ critical}} I_s \leqslant 2c(\mu - 1)\text{pot}_{t_0-1}(u) < \xi \text{pot}_{t_0-1}(u).$$

We will now do a case analysis on all possible values t^* can take. We present a series of three lemmas which consider all the values of t_1, t_2, t_3 and show that the total potential in each case is less than the total potential after round $t_0 - 1$. In fact, one result will claim that t^* cannot be t_3.

Lemma 20. If $t_1 \leqslant t_2$ and $t_1 \leqslant t_3$, then $\text{POT}_{t_0-1} \geqslant \text{POT}_{t^*}$.

Lemma 21. If $t_2 < t_1$ and $t - 2 \leqslant t_3$ then $\text{POT}_{t_0-1} \geqslant \text{POT}_{t^*}$.

Lemma 22. We have that $t_3 \geqslant \min\{t_1, t_2\}$.

Now, finally we can prove Theorem 4.3.1. It's just a matter of collecting all the theorems and lemmas we have seen above.

Proof of Theorem 4.3.1. Let s be a round with $\text{POT}_t < 2n$ for all $t < s$. If, $\text{POT}_s < n$, there is nothing to show. So, assume $\text{POT}_s > n$. Now, we have seen due to Lemma 8 we always have a round t_0 such that $\text{POT}_{t_0} \geqslant n$. Let, t_0 be the maximal such round satisfying $t_0 \leqslant s$ and $\text{POT}_{t_0-1} \leqslant n$. We define t^* as per Definition 17. In case $s = t^*$ we have due to Theorem 4.3.3 that $\text{POT}_s = \text{POT}_{t^*} \leqslant \text{POT}_{t_0-1} \leqslant n$. So, now further assume $s < t^*$. This implies, $s < t_2$, so by Lemma 17 we have that $\text{POT}_s < 2n$.

Part II

Major Takeaways & Discussion

Chapter 5

New Results

We will examine the **random Breaker** and **clever Maker** triangle game. Previous results can be found in Groschwitz and Szabó, 2017, Lamaison, 2019, and Bednarska and Łuczak, 2000. We study the triangle game under the half-random setup. Where the Breaker is random, and the Maker is clever or deterministic. This puts the Breaker at a significant disadvantage and moves the required threshold far from the optimal one.

5.1 Previous Results

We mention a result in the paper by Bednarska and Łuczak, 2000. The techniques used to prove these results are useful. We also see how (and what) 'probabilistic intuition' is useful for a certain class of games.

5.1.1 Random Maker & Clever Breaker Result

In their seminal paper Bednarska and Łuczak, 2000 made a mathematical connection between positional games played on graphs and threshold properties of random graphs. As Beck, 2008 claims, significant connections exist between both fields. Bednarska and Łuczak, 2000 examine a slightly more general version of $\mathbb{G}(K_3; n, q)$ in which they replace K_3 to be the winning set with any graph G whose size is dependant on n and it contains at least 3 non-isolated vertices. In the paper, they show that a 'random strategy' is almost optimal for the Maker. In essence, they look into the **random Maker** and **clever Breaker** framework of the G-game. (For triangle game, $G = K_3$.) They define the following quantity.

Definition 18. Let X be a graph. Let $\mathrm{vert}(X)$ be the number of vertices X has and $\mathrm{edge}(X)$ denote the number of edges X has. Also assume $\mathrm{vert}(X) \geqslant 3$. For G (a graph), with at least three non-isolated vertices define,

$$m(G) = \max_{X \subseteq G} \frac{\mathrm{edge}(X) - 1}{\mathrm{vert}(X) - 2}.$$

Theorem 5.1.1. *Let G be a graph with at least 2 non-isolated vertices. Then $\exists\ c_0, C_0, n_0 \in \mathbb{R}_{>0}$ such that for every $n \geqslant n_0$ we have the following for the game $\mathbb{G}(G; n, q)$,*

(i) For $q \leqslant c_0 n^{1/m(G)}$ the Maker has a winning strategy.
(ii) For $q \geqslant C_0 n^{1/m(G)}$ the Breaker has a winning strategy.

We will mostly focus on Part (i) of Theorem 5.1.1. To prove this, the authors use results from random graphs. Rather than looking into the mathematical details of the proof, we focus on the heuristic. The authors consider a modification of $\mathbb{G}(G; n, q)$ let us call it $\mathbb{G}'(G; n, q)$ where G contains a cycle and at any time point of the game the Breaker has complete information about the Maker's moves and decides their next move rationally. At the same time, the Maker is oblivious to the Breaker's moves. The Maker plays according to *random strategy* in $\mathbb{G}'(G; n, q)$, which means that it occupies an edge that hasn't been claimed previously (by the Maker) uniformly at random. In case it occupies an edge that has been occupied the Breaker, then the move is *lost.* It is easy to see that if the Breaker has a strategy to win for $\mathbb{G}(G; n, q)$ where both are rational players, the same can be applied to $\mathbb{G}'(G; n, q)$. Thus, this discussion gives us the following theorem for $\mathbb{G}'(G; n, q)$. Which in turn extends to $\mathbb{G}(G; n, q)$.

Theorem 5.1.2. *Let G be a graph containing a cycle. Then $\exists\ c_0 > 0, n_0$ such that $\forall\ n > n_0$ and $q \leqslant c_0 n^{1/m(G)}$ the Maker can play randomly and win with a probability of at least $\frac{1}{3}$ against a clever Breaker in the game $\mathbb{G}'(G; n, q)$.*

The authors additionally claim that for large n, this probability (of the Maker winning) will tend to be one. This is because of the exponential concentrations applied to the random graph that the Maker will construct during their play. For more details, we refer the reader to Lemma 3 of Bednarska and Łuczak, 2000.

When we apply Theorem 5.1.2 to $\mathbb{G}(K_3; n, q)$ we get that $m(K_3) = 2$ and this results in a winning strategy for the random Maker for $q \leqslant c_0 n^{1/2}$ in which the Maker wins against a clever Breaker with a probability of at least $\frac{1}{3}$. It is also *conjectured* that the constants c_0, C_0 in Theorem 5.1.1 can be brought arbitrarily close to each other in which case we will have a winning strategy for the Maker if $q \leqslant (c - \varepsilon)m^{1/m(G)}$ and a winning strategy for the Breaker if $q \geqslant (c + \varepsilon)m^{1/m(G)}$ for some positive constant c.

5.1.2 Probabilistic Intuition

I will present two examples here, as in Section 3.5 of the book by Hefetz et al., 2014. In the seminal paper by Chvátal and Erdös, 1978 they examined various games apart from the Triangle Game. One of them was the Connectivity Game. Let the threshold bias for the connectivity game be denoted by b_C. They show that $b_C = o(n/\ln(n))$ with probability tending to 1. But where did the probability part come from? It's because of the approach the authors took to prove the claim. Chvátal and Erdös, 1978 analyzed the random connectivity game in which both players adopt a random strategy. As a result, at the end of the game, both players would build a random graph. In particular, the Maker would have a random $G(n, m)$, i.e., a graph consisting of n vertices with m edges sampled uniformly at random. Section 3.5 Hefetz et al., 2014 gives a few references to popular results from random graphs, which talk about the connectedness of a random graph. The problem then reduces to merely applying these results to the random graph constructed by the Maker. This will give us the required bound on b_C. Chvátal and Erdös, 1978 eventually concludes with the threshold bias of the random game being the same as the deterministic game with high probability (tending to 1). So, the Maker has a winning strategy in the deterministic game if the Random Maker wins the random connectivity game with high probability.

This phenomenon was called the 'probabilistic intuition'. It was first seen in the paper Chvátal and Erdös, 1978. This technique has since then motivated a lot of mathematicians to look into positional games, adopting a variation of the random/deterministic Maker and random/deterministic Breaker as can be seen in the papers by Groschwitz and Szabó, 2016. However, this is not a general phenomenon in mathematics, and there is no general setup or criteria under which this is valid.

In fact, $\mathbb{G}(K_3; n, q)$ is one example where the probabilistic intuition is off. By simple deterministic methods Chvátal and Erdös, 1978 had proved the threshold to be $O(\sqrt{n})$ on the other hand, the threshold number for the appearance of triangles in random graphs is $\Theta(n)$ which gives us a worse threshold bound for the deterministic case. Nonetheless, as discussed in Sub-Section 5.1.1 some probabilistic intuition can still be derived from half-random results.

Hence, this motivates us to examine the RandomBreaker vs. CleverMaker triangle game.

5.1.3 Some Terminology

We will mention some results and definitions from Groschwitz and Szabó, 2017. In the paper, the authors have assumed that the strategy goes on till the end, i.e., till all the free edges are occupied. Say the Maker ends up winning while there are free edges left over, and then they will continue to keep claiming edges along with the Breaker till all free edges have been claimed. We apply the same scenario in the following section. So, when we say that a 'player forfeits,' it means that they keep occupying arbitrary edges that are free till there are no unclaimed edges left instead of playing according to the strategy prescribed. A *play sequence* Λ of length i is defined as the list $(\Lambda_1, \ldots, \Lambda_i) \in E(K_n)^i$ of the first i edges that were occupied during the game by the Maker and Breaker, *in the same order as they were occupied in.*

Recall that in our previous discussion, we highlighted that in half-random games (where one player is clever/deterministic/strategic and the other player is random), the graph constructed by the random player is not truly random since it depends on the strategic moves made by the clever player. So, ideally, we need to circumvent this interaction of the random and deterministic aspects of the game. I will mention the basic heuristics applied by the authors. The 'random player' can pick any permutation of the edges of K_n and occupy edges based on this permutation, avoiding those edges claimed by the opponent. Since the permutation was picked randomly, this becomes a random strategy for the random player. This has been formalized in Proposition 2.1 in Groschwitz and Szabó, 2017, which we mention below without proof for convenience. This is called as the *permutation strategy* in the paper. Let S be the strategy of the clever player, $m \leqslant \binom{n}{2}$ be a constant, $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$ denote the play-sequence of m edges. The below theorem establishes an 'in-probability' equivalence of the permutation strategy picked at random and the strategy of picking each edge at random by the random player.

Theorem 5.1.3. *For every S, m, and Λ related to a $(r : c)$-game on $E(K_n)$ between a clever player and a random player, we have the following. The probability that Λ is an 'actual' play sequence in a half-random game equals the probability that Λ is the play sequence of the game when a random player plays according to the permutation strategy.*

Let the permutations be denoted by $\sigma \in S(E(K_n)))$. With the map $\sigma : \left[\binom{n}{2}\right] \to E(K_n)$. For a given $\sigma \in S(E(K_n))$ and m as defined above. We take a subgraph denoted by

$G_\sigma(m) \subseteq K_n$ where the edge set is all the edges in the length m play sequence that σ represents. Mathematically, $E(G_\sigma(m)) := \{\sigma(i) \mid 1 \leqslant i \leqslant m\}$. Since the random player picks the permutation σ uniformly at random the properties of the graph $G_\sigma(m)$ will be distributed like the random graph $G(n, m)$ i.e., a graph on n vertices with m edges where each edge is picked with some probability p. The below theorem establishes the relation between the graph constructed by the Breaker as part of it's move and a random graph. For the proof we direct readers to Proposition 2.2 in Groschwitz and Szabó, 2017.

Theorem 5.1.4. *For positive integers b, i such that $i \leqslant \binom{n}{2}/b + 1$ we have the following. For every monotone increasing property $\mathcal{P}$ and strategy S of* CleverMaker *for a $(1 : b)$ half-random game the probability that in a half-random game against S of* CleverMaker *the graph of* RandomBreaker *after the i-th round has property $\mathcal{P}$ is at most $\mathbb{P}\left[G(n, i(b+1) \text{ has property } \mathcal{P}\right]$.*

Definition 19. A *double move* or a *triple move* is when the CleverMaker identifies two or three edges it must occupy in the next two or three moves to win or gain a significant advantage in the game.

To execute a *double/triple* move, it's essential that the RandomBreaker does not occupy the edges that the CleverMaker has identified as per Definition 19. In the below lemma, taken from Lemma 3.2 of the paper by Groschwitz and Szabó, 2017, we can see that for appropriate upper bounds on q, the CleverMaker completes any required double or triple move with high probability. Let $\varepsilon > 0$ be a sufficiently small fixed constant.

Lemma 23. If the number of free edges is $\binom{n}{2} - (q+1)t \geqslant \frac{\varepsilon}{4}n^2$ then for a half random $(1 : q)$ game with $q \leqslant n$ the following holds. The probability that the CleverMaker *fails* to complete a double (resp. triple) move within the first t rounds is *at most* $\frac{4}{\varepsilon n}$ (resp. $\frac{12}{\varepsilon n}$).

Proof. Recall that the Maker moves first, so in the sequence of a double move, they only need to occupy one edge, and in the case of a triple move, they need to occupy two edges. Let us denote by e_2, e_3 the edges the Maker wants to occupy in the planned double/triple move. We just saw that CleverMaker would comfortably occupy the first edge e_1 of the planned double/triple move. Since the number of available edges is at least $\frac{\varepsilon n^2}{4}$, so the probability that the RandomBreaker occupies e_2 of the double move started by the CleverMaker is *at most* $\frac{4}{\varepsilon n^2}$. Notice that the RandomBreaker has *at most* n chances to occupy e_2, and each Breaker move is independent of the previous one. Let C_i denote the event that the RandomBreaker claims e_2 in the i-th move of the total n moves it is allowed in a single turn. Since the claiming of each edge is an independent event, we have that $\mathbb{P}(\text{RandomBreaker occupies } e_2) = \sum_{i=1}^{n} \mathbb{P}(C_i)$ and since $\mathbb{P}(C_i) \leqslant \frac{4}{\varepsilon n^2}$ for all $1 \leqslant i \leqslant n$ as long as the number of free edges is at least $\frac{\varepsilon n^2}{4}$. Hence, $\mathbb{P}(\textsf{RandomBreaker}$ occupies $e_2) \leqslant \frac{4}{\varepsilon n}$. RandomBreaker occupying e_2 implies CleverMaker failing to complete the double move. Thus, the Lemma 23 follows. Similarly, we can show for a triple move the probability of the CleverMaker failing is at most $\frac{12}{\varepsilon n}$. □

5.2 New Results

The aim of RandomBreaker is to prevent CleverMaker from winning. To win the CleverMaker will focus on a single vertex v and aim to build a star. However, notice that since the Breaker is claiming edges uniformly at random, the CleverMaker can aim to quickly build a two-path involving v and then try and complete the triangle if the closing edge of the two-path hasn't been claimed yet. Similarly, if we allow the Breaker to claim cn^2 edges per move, there will only remain a constant number of rounds in the entire game, which will ensure that the Breaker wins with some constant probability. We will see the detailed analysis in the following sections.

5.2.1 Maker's Win

We begin by mentioning the Maker's strategy. The basic heuristic was mentioned above. Let us call the below strategy as S. S is as follows:

1. The Maker will begin with a *triple move* as mentioned in Definition 19. They will fix any triangle in the K_n and try to occupy its edges.
2. The Maker will try to pick only the three edges of the triangle decided.
3. If the Breaker picks up any of the edges of the fixed triangle before the Maker, then the Maker forfeits the game.

As per our discussion in Sub-Section 5.1.3, by forfeit, we mean that the CleverMaker and RandomBreaker will continue playing arbitrarily till all the edges have been claimed. Also, note that the CleverMaker may win following any step other than the above-mentioned. However, we will show that for an appropriate bound on the bias for the Breaker, the Maker will win with a high probability following the strategy mentioned in Sub-Section 5.2.1. The main theorem of this Sub-Section is the following.

Theorem 5.2.1. *For a half-random* $(1:q)$ *game between* CleverMaker *and* RandomBreaker *with* $q = o(n^2)$ *the Maker will win the* $\mathbb{G}(K_3; n, q)$ *with high probability by following the strategy* S.

Proof. Per our assumptions, $q = f(n) = o(n^2)$ which means that $\lim_{n\to\infty} \frac{f(n)}{n^2} = 0$. Since the Maker is focusing on a single triangle, let it be denoted by the triangle (M_1, M_2, M_3) in Figure 5.1. The Maker begins with the resolve of executing a triple move and claiming the complete triangle from the beginning of the game (i.e., from the first round). So, the Breaker will claim $3f(n)$ edges in the first three rounds. We will define three events.

1. B_1 – The Breaker claims M_1.
2. B_2 – The Breaker claims M_2.
3. B_3 – The Breaker claims M_3.

The Maker will forfeit according to the strategy if the Breaker picks any of the M_i edges. So, the unfavorable outcome for the Maker is denoted by $B_1 \cup B_2 \cup B_3$, i.e., when the Breaker occupies M_1 **or** M_2 **or** M_3. Let us try and obtain an upper bound for the probability $\mathbb{P}(B_1 \cup B_2 \cup B_3)$. Notice that the probability of the event that the Breaker picks an edge

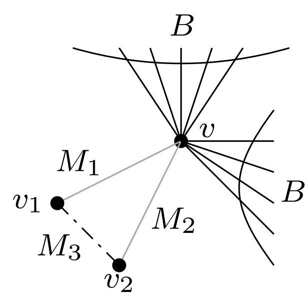

Figure 5.1: The CleverMaker vs. RandomBreaker scenario.

from a fixed triangle in the first three rounds where $q = f(n) = o(n^2)$ can be upper bounded as follows. First, let us upper bound $\mathbb{P}(B_1)$ which turns out to be

$$\mathbb{P}(B_1) \leqslant \frac{q}{\binom{n}{2} - 1} + \frac{q}{\binom{n}{2} - (q+2)} + \frac{q}{\binom{n}{2} - (2q+3)}.$$

In the above computation, we divide the total number of edges the Breaker occupies in a single move by the total available edges. Since the Maker moves first, we get the '-1' in the denominator in the first term and so on for the other terms. Now, observe that, from symmetry, we can use the same upper bound for $\mathbb{P}(B_2)$ and $\mathbb{P}(B_3)$. We will now use the union bound to calculate an upper bound on $\mathbb{P}(B_1 \cup B_2 \cup B_3)$.

$$\mathbb{P}(B_1 \cup B_2 \cup B_3) \leqslant \sum_{i=1}^{3} \mathbb{P}(B_i)$$

$$\leqslant 3 \cdot \left(\frac{q}{\binom{n}{2} - 1} + \frac{q}{\binom{n}{2} - (q+2)} + \frac{q}{\binom{n}{2} - (2q+3)} \right)$$
$$\leqslant 9 \cdot \left(\frac{q}{\binom{n}{2} - (2q+3)} \right)$$
$$\leqslant 9 \cdot \left(\frac{q}{\binom{n}{2} - 2q} \right)$$
$$\leqslant 9 \cdot \left(\frac{q}{n^2/3} \right)$$
$$= \frac{27q}{n^2} = o(1)$$

The second-last inequality is because $\frac{n^2}{2} - 2o(n^2) \geqslant \frac{n^2}{3}$, and the last equality is because $q = o(n^2)$. Hence, $\mathbb{P}(B_1 \cup B_2 \cup B_3) = o(1)$. This tells us that the probability the Breaker picks an edge from a fixed triangle in the first 3 rounds is very small, $o(1)$. Now, the complement of $(B_1 \cup B_2 \cup B_3)$ is that the Breaker does not pick any edge among (M_1, M_2, M_3) so, essentially allowing CleverMaker to win. Thus, the probability that the Maker wins is

$$\mathbb{P}\big(\textsf{CleverMaker} \text{ Wins by Claiming } (M_1, M_2, M_3)\big) = 1 - o(1).$$

Completing the proof of Theorem 5.2.1. □

5.2.2 Breaker's Win

We fix the value of the Breaker bias in this section and show that they can win with constant probability against any strategy of the Maker. The analysis involves successively occupying edges that might threaten the Breaker. Since the bias is high, we will show that the probability of 'risk mitigation' is constant. We begin with a small observation. In this sub-section $q = cn^2$ for an arbitrary fixed constant $c > 0$.

Lemma 24. If the Breaker claims q edges per move then the Maker can claim *at most* $\frac{\binom{n}{2}}{q}$ edges.

Proof. Since the total number of edges in K_n is $\binom{n}{2}$ and Maker is allowed to claim only 1 edge per move, the number of rounds till all edges are claimed by the Breaker is $\frac{\binom{n}{2}}{q}$ and hence the Maker can claim *at most* $\frac{\binom{n}{2}}{q}$ many edges. □

If we set $q = cn^2$ for an arbitrary fixed positive constant $c > 0$. Then, from Lemma 24 we have that the total number of rounds (and thereby, the number of edges the Maker claims) will be at most $t = \binom{n}{2}\Big/cn^2 = \Theta(1)$. The basic aim of the RandomBreaker would be to close all two-paths, as has been the case in the previous chapters as well. Suppose the CleverMaker focuses on a single vertex v and builds a star. The Maker can build a star of size at most t and this can give rise to at most $t \cdot (t-1) = \Theta(t^2)$ potential two-paths. For the RandomBreaker to 'succeed,' they must close these potential two paths. Given the high bias towards RandomBreaker, we will show that this will be the case with constant probability.

Define S_i to be the event that the RandomBreaker succeeds in the i-th step. Then the probability that the Breaker succeeds can be formulated as follows,

$$\begin{aligned}\mathbb{P}(\textsf{RandomBreaker wins}) &= \mathbb{P}(S_1 \cap S_2 \cap \cdots \cap S_t)\\ &= \mathbb{P}(S_1) \cdot \mathbb{P}(S_2 \mid S_1) \cdots \mathbb{P}(S_t \mid S_1, S_2, \ldots, S_{t-1}).\end{aligned}$$

Where $\mathbb{P}(S_i \mid S_1, S_2, \ldots, S_{i-1})$ represents the probability that the Breaker succeeds on the i-th move given that it has succeeded in the past $(i-1)$ moves. Let us calculate $\mathbb{P}(S_i \mid S_1, S_2, \ldots, S_{i-1})$.

Lemma 25. For $k \leqslant t^2$, $m < \binom{n}{2}$ we have $\mathbb{P}(S_i \mid S_1, S_2, \ldots, S_{i-1}) \geqslant \binom{m-k}{q-k}\Big/\binom{m}{q}$.

Proof. Let us first denote what k and m stand for. Suppose for a given round in the middle of play, the number of length two paths (i.e., the paths which pose danger to the Breaker) be k notice that per our previous discussion $k \leqslant t^2$. Further, let m denote the number of available edges hence, $m \leqslant \binom{n}{2}$. Naturally the Breaker would choose from the total number of available edges so the total number of ways to do so is $\binom{m}{q}$. Since $k \leqslant t^2$, $t = \Theta(1)$, and $q = cn^2$ as mentioned above so if we have k possible two paths then the RandomBreaker can adjust them in the q edges it claims (i.e., there exists an event where all the k dangerous edges have been claimed by the RandomBreaker). In the q-tuple of edges the RandomBreaker claims, fix k edges that are 'dangerous'. Then the remaining number of edges is $m-k$ and the remaining bandwidth with the RandomBreaker is $q-k$, so the total number of ways of claiming q edges in which these k dangerous edges are included is $\binom{m-k}{q-k}$. This essentially is how many ways we can fill the $q-k$ spots remaining from $m-k$ edges remaining. Hence, of the total possible $\binom{m}{q}$ moves the RandomBreaker makes, the possibility of making a move in which all the dangerous edges are claimed is $\binom{m-k}{q-k}\Big/\binom{m}{q}$. □

The bounds can be further simplified. Expanding both the binomial in the numerator and denominator, we obtain

$$\binom{m-k}{q-k} \bigg/ \binom{m}{q} = \frac{q(q-1)\cdots(1-(k-1))}{m(m-1)\cdots(m-(k-1))} = \prod_{i=0}^{k-1} \frac{q-i}{m-i}.$$

Notice that $\frac{q-i}{m-i}$ is decreasing as i increases, we can thus obtain the following bound,

$$\prod_{i=0}^{k-1} \frac{q-i}{m-i} \geqslant \left(\frac{q-(k-1)}{m-(k-1)}\right)^{k-1} \geqslant \left(\frac{q-(k-1)}{m-(k-1)}\right)^{k}$$

the last inequality follows because the term inside the parenthesis is less than 1. Now, $q-(k-1) = cn^2 - (k-1)$ and as $n \to \infty$ $(k-1) = O(1)$ so $q-(k-1) \approx cn^2$. Similarly, $m-(k-1) \leqslant \binom{n}{2} - (k-1) \approx n^2/2$ as $n \to \infty$. Thus, the ratio becomes

$$\left(\frac{q-(k-1)}{m-(k-1)}\right)^{k} \geqslant \left(\frac{cn^2}{n^2/2}\right)^{k} = (2c)^k.$$

Now, $k \leqslant t^2$ and $t = \Theta(1)$ so for $n \to \infty$ $k-1 = O(1)$. Hence finally we have,

$$\binom{m-k}{q-k} \bigg/ \binom{m}{q} \geqslant (2c)^k \geqslant (2c)^{t^2} = C(c,t)$$

Hence, we establish that $\mathbb{P}(S_i \mid S_1, S_2, \ldots, S_{i-1}) \geqslant (2c)^{t^2}$. We collect all the result and conclude with the following theorem.

Theorem 5.2.2. *For all positive constants c, if the bias of the* RandomBreaker *is $q = cn^2$, then the* RandomBreaker *wins against any strategy of the* CleverMaker *with positive constant probability.*

Proof. We have already established that

$$\begin{aligned}\mathbb{P}(\textsf{RandomBreaker}\text{ wins}) &= \mathbb{P}(S_1 \cap S_2 \cap \cdots \cap S_t)\\ &= \mathbb{P}(S_1) \cdot \mathbb{P}(S_2 \mid S_1) \cdots \mathbb{P}(S_t \mid S_1, S_2, \ldots, S_{t-1})\\ &\geqslant ((2c)^{t^2}) \cdot ((2c)^{t^2}) \cdots ((2c)^{t^2})\\ &= \left((2c)^{t^2}\right)^t = C(c,t)\end{aligned}$$

Hence by choosing an appropriate c we can fix the bias of the RandomBreaker to be such that it wins with a constant probability. □

5.3 Concluding Remarks

After having established Theorem 5.2.1 and Theorem 5.2.2 we close an important gap in the literature of half random triangle games. In addition to this, under strategy S as mentioned in Sub-Section 5.2.1 and using Lemma 23 we can build the following framework and establish the exponential decay bound on the random variable representing the possibility of a Breaker's win. So for the CleverMaker vs. RandomBreaker the following holds true as well.

Let the triangle fixed by the Maker as per S have the following edge set (e_1, e_2, e_3). Since the Maker moves first they will always occupy e_1. As per Lemma 23 the probability of claiming e_2, e_3 represented by $\mathbb{P}(e_2) = \mathbb{P}(e_3) = 4/\varepsilon n^2$. But, the Breaker gets n chances for e_2 (as $q < n$ in Lemma 23) and $2n$ chances for e_3. So the respective probabilities become $4/\varepsilon n$ and $8/\varepsilon n$. Breaker occupying e_2, e_3 is an unfavorable scenario for the Maker under strategy S. Let us define the following indicator random variables,

$$Y_1 = \begin{cases} 1 & \text{w.p. } \frac{4}{\varepsilon n} \\ 0 & \text{w.p. } 1 - \frac{4}{\varepsilon n} \end{cases} \qquad Y_2 = \begin{cases} 1 & \text{w.p. } \frac{8}{\varepsilon n} \\ 0 & \text{w.p. } 1 - \frac{8}{\varepsilon n} \end{cases}$$

where Y_1 is the indicator of e_2 being claimed and Y_2 is the indicator of e_3 being claimed. The sum $X = Y_1 + Y_2$ denotes either e_2 or e_3 being claimed, both of which are not favorable for the Maker. It's easy to see that the expected value of X is $\mathbb{E}[X] = (4/\varepsilon n) + (8/\varepsilon n) = 12/\varepsilon n$. Now, using Markov's Inequality which says that for a non-negative random variable X with finite expectation $\mathbb{E}[X] < \infty$, and for any $m > 0$, we have $\mathbb{P}(X \geqslant m) \leqslant \mathbb{E}[X]/m$. Since the random variable we defined (i.e., X) satisfies all the conditions and we know $X \geqslant 1$ is an unfavorable outcome, using the inequality we can obtain the bound $\mathbb{P}(X \geqslant 1) \leqslant \frac{12}{\varepsilon n}$. As $n \to \infty$ the probability of the event $(X \geqslant 1)$ becomes vanishingly small.

We can apply the Hoeffding's bound which states. For independent random variables $\{X_1, X_2, \ldots, X_n\}$ with $\mathbb{E}[X_i] = \mu_i, \mathbb{P}[a_i \leqslant X_i \leqslant b_i] = 1$ for constants a_i, b_i then, for any deviation $t > 0$,

$$\mathbb{P}\left[\left|\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right| \geqslant t\right] \leqslant 2\exp\left[\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right].$$

Per our set-up, (Y_1, Y_2) are independent since all of Breaker moves are independent of each other and done uniformly at random. Additionally, $\mathbb{P}[0 \leqslant Y_i \leqslant 1] = 1$ for $i = 1, 2$. Now, since $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$, we have $\sum_{i=1}^{2}(b_i - a_i)^2 = 2$. Also recall that $\sum_{i=1}^{2} Y_i = X$ and $\mathbb{E}[Y_1] + \mathbb{E}[Y_2] = \mathbb{E}[X] = 12/\varepsilon n$. Putting it all together according to the Hoeffding framework, we get, for any deviation $t > 0$,

$$\mathbb{P}\left[\left|X - \frac{12}{\varepsilon n}\right| > t\right] \leqslant 2\exp\bigl(-t^2\bigr)$$

Which tells us that the probability that X deviates from the mean is decaying exponentially fast. This tells us that X is concentrated sharply around $\mathbb{E}[X]$. It is quite natural that we get such concentrations since the bias for the Breaker as per Lemma 23 is very small at $q \leqslant n$. In Sub-Section 5.2.1 we prove that even if we allow the bias to go as high as $o(n^2)$ the Maker can still win with high probability. And similarly in Sub-Section 5.2.2 we establish that the Breaker can always establish a win with positive constant probability when $q = cn^2$ for all constants $c > 0$.

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