Generalized gravity theories

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

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March, 2025

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Certificate

This is to certify that this dissertation entitled Generalized gravity theories towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sayan Neogi at Indian Institute of Science Education and Research, Pune under the supervision of Sunil Mukhi, Professor, Department of Physics , during the academic year 2020-2025.

Sunil Mukhi

Committee:

Sunil Mukhi

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This thesis is dedicated to my parents

Declaration

I hereby declare that the matter embodied in the report entitled Generalized gravity theories are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Sunil Mukhi and the same has not been submitted elsewhere for any other degree.

Sayan Neogi

Acknowledgments

This thesis has been made possible by the culmination of support and guidance by several people who I have the great opportunity to interact with. First and foremost, I would like to thank my parents for being incredibly supportive of my decisions and enabling me to pursue my academic endeavours.

I would like to sincerely express my gratitude to my thesis supervisor – Prof. Sunil Mukhi. I am deeply thankful for his patience in guiding me through the project. In our several incredibly insightful discussions, I was greatly inspired from his thought-process in acquiring a complete understanding of the matter under consideration.

I would like to thank Prof. Arnab Rudra for his continued support and mentorship, starting from my first semester at IISER. I am also indebted to Mohammad Ali for the numerous hours of brainstorming sessions I had with him – I learnt a whole lot from these discussions and it really widened my perspective. I would also like to thank other graduate students in the Physics Department, namely, Dr. Saurabh Pant and Dhruva K.S for taking out the time to discuss whenever I had any doubts.

Finally, I would also like to acknowledge the amazing peers and friends I had the priviledge of having around me during my time at IISER.

Abstract

This thesis explores the existence of higher-derivative theories beyond the Lanczos-Lovelock models of gravity which are ghost-free in arbitrary backgrounds by having a second-order equation of motion (which we shall refer to as 'Lovelockian'). First, we review the instabilities associated with generic higher-derivative theories of gravity and show that the Lanczos-Lovelock models constitute the unique class of theories with polynomial contractions of the curvature tensor with a second-order equation of motion. Next, we investigate the case of Abelian gauge fields non-minimally coupled to gravity and demonstrate that there exists theories at arbitrary order in derivatives (and arbitrary form degree of the Abelian gauge field) which are Lovelockian. Furthermore, we analyze the flat-space expansion of theories involving covariant derivatives of the curvature tensor, thereby providing justification for a key assumption underlying an important result in the field. We conclude with a brief overview of the recently proposed Generalized Quasi-Topological theories of gravity – which propagate just the graviton mode in maximally symmetric backgrounds and discuss some of the remarkable properties of these theories.

Contents

Introduction

The Einstein-Hilbert action leads to second-order coupled differential equations famously known as the Einstein's field equations. Although it has been successfully verified in wide contexts, it is non-renormalizable – it takes an infinite number of counter-terms to absorb ultraviolet divergences. This suggests that the Einstein–Hilbert action is likely only an effective description of gravity valid at low energies. To alleviate this problem, early works [1–3] explored the scope of 'higher-derivative theories of gravity' in improving the ultraviolet behaviour of gravitational interactions. These theories extend the Einstein–Hilbert action by including additional terms involving higher powers of curvature invariants – such as $R^{\mu\nu}R_{\mu\nu}$, R^2 – and higher derivatives of the metric. Furthermore, higher-derivative terms arise from quantum corrections in any ultraviolet complete theory of quantum gravity such as string theory. Thus, the study of higher-derivative theories of gravity offers a promising avenue for bridging the gap between classical and quantum descriptions of gravity.

Since these theories have a higher-derivative equation of motion, they generically suffer from a number of instabilities and pathologies. These include the presence of ghost modes – additional modes whose kinetic terms have the wrong sign, Ostrogradsky instability [4] , ill-defined initial value problem and possible violations of causality [5, 6]. Hence, it becomes pertinent to investigate if there exists any 'consistent' theories which can avoid these problems.

As early as 1971, Lovelock [7] demonstrated that there exists a class of higher-derivative theories at each curvature order – which subsequently came to known as the Lanczos-Lovelock models of gravity – which have a second-order equation of motion, and hence they do not suffer from the issues of having ghost modes or the Ostrogradsky instability. These theories have a topological origin, related to the Euler characteristic of the manifold. Subsequently, in 1976, Horndeski [8] put forward a model in four dimensions with the electromagnetic field strength non-minimally coupled to gravity which also have a second-order equation of motion. Recently, these theories have served as an interesting playground to understand the role of higher-derivative terms in black hole thermodynamics, cosmology and astrophysical phenomena. More recent works aim at finding new theories by requiring ghost-free behaviour in certain specific backgrounds or by imposing that they admit specific spherically symmetric solutions. This include the Einsteinian cubic gravity which is the unique theory in four dimensions which is ghost-free in maximally symmetric backgrounds. Subsequently, a new class of theories known as Generalized Quasi-Topological theories [9–12] was proposed – encapsulating the Lanczos-Lovelock theories as a special subset of this broader class.

The outline of the thesis is as follows: In Chapter 1, we study in detail the presence of ghost modes in higher-derivative theories and demonstrate it with some examples. Subsequently, we shall discuss the Ostrogradsky instability which plagues such theories. Further, we will look into the freedom of field redefinitions in an effective field theory of gravity.

In Chapter 2, we will discuss the structure of generic Lagrangians composed of contractions of curvature tensor with the metric. With this insight, we shall construct the Lanczos-Lovelock theories and provide a proof for the uniqueness of these models along with some key observations which are omitted in the literature.

In Chapter 3, we aim to investigate if theories with an Abelian gauge field coupled to gravity could be Lovelockian. We put forward a novel class of Lovelockian theories – at arbitrary order in derivatives – for any form degree of the Abelian gauge field and recover the well-known 'Horndeski gravity' as a special case. Subsequently, we turn our attention to theories with covariant derivatives of the curvature tensor and highlight the problems associated in the effort to see whether such theories can have a second-order equation of motion. We provide strong support to a previous work by analyzing the leading contribution of the flat-space expansion of these theories.

In Chapter 4, we provide an overview of the recently proposed Generalized Quasi-Topological theories of gravity and look into some of their interesting properties which calls for further investigation of these models.

Chapter 1

Stability of higher-derivative theories

In this chapter, we will formally introduce higher-derivative theories of gravity and underline the motivation behind studying such theories. Next, we will perform a detailed review of the instabilities and pathologies associated with generic higher-derivative theories as commonly mentioned in the literature. This chapter forms the foundation for all of the subsequent discussion in the thesis.

1.1 Ostrogradsky Instability

Ostrogradsky instability ([4]) is the statement that there is a linear instability in the Hamiltonian associated with a Lagrangian which depend upon more than one time derivative such that the higher derivatives cannot be eliminated by partial integration.

First, let us analyze a more familiar class of theories which just depend on a single time derivative of a dynamical variable, i.e, it has the form $L \equiv L(q, \dot{q})$. Hence, the variation of the action is given by,

$$\begin{aligned} \delta S &= \int dt \, \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \\ &= \int dt \, \frac{\partial L}{\partial q} \delta q - \partial_t \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q + \partial_t \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \end{aligned} \tag{1.1}$$

Since at the end-points, $\delta q = 0$, the total derivative term vanishes. Now let us look at the coefficient of δq in the second term,

$$\partial_t \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial^2 L}{\partial \dot{q}\, \partial \dot{q}} \ddot{q} + \frac{\partial^2 L}{\partial q\, \partial \dot{q}} \dot{q} \tag{1.2}$$

Note that, only if the first term is non-zero, we would have a second-order equation of motion. This is known as the 'non-degeneracy condition',

$$\frac{\partial^2 L}{\partial \dot{q}\, \partial \dot{q}} \neq 0 \tag{1.3}$$

Hence, if this condition is satisfied, we would have an equation of motion of the form,

$$\ddot{q} = f(q, \dot{q}) \tag{1.4}$$

which we can solve by providing two initial conditions, q_0, $\dot{q}_0$.

Now, consider a Lagrangian which also depends on the second derivatives of the metric and hence has the form $L \equiv L(q, \dot{q}, \ddot{q})$. Using this Lagrangian, the variation of the action is given as,

$$\delta S = \int dt\, \frac{\partial L}{\partial q} \delta q - \partial_t \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q + \partial_t^2 \left(\frac{\partial L}{\partial \ddot{q}} \right) \delta q + \text{total derivatives} \tag{1.5}$$

Now let us look at the coefficient of δq in the last term in the above expression,

$$\partial_t^2 \left(\frac{\partial L}{\partial \ddot{q}} \right) = \frac{\partial^2 L}{\partial \ddot{q}\, \partial \ddot{q}} \ddot{\ddot{q}} + \text{terms with } \dddot{q}, \ddot{q} \text{ and } \dot{q} \tag{1.6}$$

Clearly, the equation of motion for such a Lagrangian will have four time derivatives when this Lagrangian is 'non-degenerate' in the sense,

$$\frac{\partial^2 L}{\partial \ddot{q}\, \partial \ddot{q}} \neq 0 \tag{1.7}$$

In this case, we can solve the equation of motion, to get q by providing 4 initial conditions,

$$q(t) \equiv q(t, q_0, \dot{q}_0, \ddot{q}_0, \dddot{q}_0) \tag{1.8}$$

Now, we want to construct the Hamiltonian corresponding to such a Lagrangian. Call the canonical coordinates as X_1, X_2 and the canonical momenta as P_1, P_2. Then the Legendre transformation which takes us to the Hamiltonian is given as,

$$H = \sum_i P_i \dot{X}_i - L(q, \dot{q}, \ddot{q}) \tag{1.9}$$

The most intuitive choice for the first canonical coordinate X_1 is q. Since the other canonical coordinate X_2 couldn't be a function of X_1, the obvious choice is to take $X_2 = \dot{q}$. With these choices, we must next decide what should be the corresponding canonical momenta such that the Hamilton's equations are satisfied, i.e,

$$\begin{aligned} \frac{\partial H}{\partial X_i} &= -\dot{P}_i \\ \frac{\partial H}{\partial P_i} &= \dot{X}_i \end{aligned} \tag{1.10}$$

From 1.9, we have,

$$\begin{aligned} dH &= \sum_i (P_i \, d\dot{X}_i + \dot{X}_i \, dP_i) - dL \\ &= P_1 \, d\dot{q} + P_2 \, d\ddot{q} + \dot{X}_1 \, dP_1 + \dot{X}_2 \, dP_2 - dL \\ &= P_1 \, d\dot{q} + P_2 \, d\ddot{q} + \dot{X}_1 \, dP_1 + \dot{X}_2 \, dP_2 - \left(\frac{\partial L}{\partial q}\right) dq - \left(\frac{\partial L}{\partial \dot{q}}\right) d\dot{q} - \left(\frac{\partial L}{\partial \ddot{q}}\right) d\ddot{q} \\ &= \dot{X}_1 \, dP_1 + \dot{X}_2 \, dP_2 - \left(\frac{\partial L}{\partial q}\right) dX_1 + \left[P_1 - \left(\frac{\partial L}{\partial \dot{q}}\right)\right] dX_2 + \left[P_2 - \left(\frac{\partial L}{\partial \ddot{q}}\right)\right] d\ddot{q} \end{aligned} \tag{1.11}$$

In the second line, we have expressed dX_i in terms of the dynamical variable. In the fourth line, we have re-expressed q and $\dot{q}$ as the canonical coordinates. Imposing that the Hamilton's equations of motion given by 1.10 are satisfied, we have the following:

$$\begin{aligned} P_2 &= \left(\frac{\partial L}{\partial \ddot{q}}\right) \\ -\dot{P}_2 &= P_1 - \left(\frac{\partial L}{\partial \dot{q}}\right) \end{aligned} \tag{1.12}$$

which implies that,

$$P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)$$
$$P_2 = \frac{\partial L}{\partial \ddot{q}} \tag{1.13}$$

To summarize the above discussion,

$$X_1 = q \qquad P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) \tag{1.14}$$

$$X_2 = \dot{q} \qquad P_2 = \frac{\partial L}{\partial \ddot{q}} \tag{1.15}$$

Now, we see that the non-degeneracy condition 1.7 implies that the equation

$$\frac{\partial L}{\partial \ddot{q}} = P_2 \tag{1.16}$$

can be inverted to find $\ddot{q}$ as function of $q = X_1, \dot{q} = X_2$ and P_2, so we can write $\ddot{q} \equiv \ddot{q}\,(X_1, X_2, P_2)$.

Expressing the Hamiltonian of the theory by explicitly showing the dependence on the canonical variables as,

$$H = P_1\,\dot{q}(X_2) + P_2\,\ddot{q}(X_1, X_2, P_2) - L(q(X_1), \dot{q}(X_2), \ddot{q}(X_1, X_2, P_2)) \tag{1.17}$$

From the above, we see that the Hamiltonian is linear in P_1 and hence it is unbounded below, which leads to the 'instability'. Note that our only assumption in the above discussion is the non-degeneracy of the Lagrangian. Hence, any non-degenerate Lagrangian which contains second-order time derivatives and above is prone to the Ostrogradsky Instability.

(Note: The Legendre transformation in the familiar form, is used to eliminate $\dot{q}$ and write the Hamiltonian in terms of the canonical momenta p. In such a case, the relation $p = \partial L/\partial \dot{q}$ should be satisfied. However, the transformation derived above does not satisfy this relation as we still retain $q = Q_1$ and $\dot{q} = Q_2$ in the Hamiltonian.)

Hence, we conclude that any Lagrangian which leads to higher order equation of motion and is non-degenerate leads to the 'Ostrogradsky' instability.

1.2 Counting derivatives

In the subsequent discussions, we will need a particular scheme of counting the number of derivatives in an action or in the corresponding equation of motion, based on its relevance:

- In the equation of motion, the number of derivatives refers to the order of the differential equation. E.g. $\mathcal{E}_{\mu\nu} = R_{\mu\alpha}R^{\alpha}{}_{\nu}$ is still second-order in derivatives.

- In the action, it refers to the total number of derivatives appearing on any term. For example, both $R^{\mu\nu}R_{\mu\nu}$ and $\Box R$ terms are considered at the four derivative order in an action. This approach is justified because terms with different numbers of derivatives, in this particular sense, are associated with distinct energy scales.

1.3 Ghost modes

When the equation of motion of a theory is higher-derivative, there arises some additional modes which are massive. Further, one can show that some of these modes have the wrong sign of the propagator, which leads to the Hamiltonian being unbounded below [1]. As a simple example, consider a scalar field theory with the Lagrangian,

$$\begin{aligned} L &= -\partial_\mu\phi\,\partial^\mu\phi + c\,\phi\,\Box^2\phi \\ &= \phi\left(\Box + c\,\Box^2\right)\phi \end{aligned} \tag{1.18}$$

where c is an arbitrary constant. From the above Lagrangian, we can read off the momentum space propagator as,

$$\frac{1}{-p^2 + c\,p^4} = -\frac{1}{p^2} + \frac{1}{p^2 - \frac{1}{c}} \tag{1.19}$$

[1]However, this is not an issue for first-order theories, like Dirac's

The contribution from the second pole comes with the wrong sign irrespective of the sign of c – this is known as a 'ghost' mode in the theory. The mass of this ghost mode is given by $m = \sqrt{1/c}$. This can be equivalently interpreted in the following way: Consider the same scalar field theory example but now by expressing c in terms of the mass m of the ghost mode:

$$L = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2m^2}\phi \Box^2 \phi \tag{1.20}$$

Now, take another scalar field χ and consider a new Lagrangian,

$$L = -\frac{1}{2}(\partial_\mu \phi)^2 + (\partial_\mu \chi)(\partial^\mu \phi) + \frac{1}{2}m^2\chi^2 \tag{1.21}$$

This is equivalent to our original Lagrangian when χ is integrated out using its equation of motion,

$$\chi = \frac{1}{m^2}\Box\phi \tag{1.22}$$

Next, we diagonalize the new Lagrangian by defining a new scalar field $\psi = \phi - \chi$,

$$L = -\frac{1}{2}(\partial_\mu \psi)^2 + \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}m^2\chi^2 \tag{1.23}$$

In the above, we see that kinetic term in the of the field χ has the wrong sign. This is the characteristic property of ghost modes. It is also clear that χ is the same massive mode from our analysis of the momentum-space propagator of the theory. The only way to remove the this ghost mode would be to take $c \to 0$, which makes the mass of the ghost mode (which goes as $\sqrt{1/c}$) infinite. Observe that the mass of a ghost is a function of the couplings in the theory, which will help us constrain them later, by choosing these couplings to be such that the masses of the ghost modes are infinite. In summary, higher derivative terms introduce massive modes in our theory, some of which are ghost-like.

Having understood this problem of ghost modes in simple scalar field theories, we turn to demonstrating it in gravity with the help of some simple higher-derivative terms. Before adding any higher-derivative corrections, let us look at the structure of the propagator of the Einstein-Hilbert action which gives rise to the massless spin-2 graviton.

$$S = \int d^4x \sqrt{-g}\, R \tag{1.24}$$

To find the free propagator and analyze the kinetic term of the graviton, we expand the Lagrangian at quadratic order in perturbation around flat space with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We denote the trace of the peturbation as $h = g^{\mu\nu} h_{\mu\nu}$. After explicitly calculating the above Lagrangian at $\mathcal{O}(h^2)$, one obtains (excluding boundary terms produced by partial integration),

$$S = \int d^4x \left(\frac{1}{2} h^{\alpha\gamma}{}_{,\sigma\gamma} h^{\sigma}{}_{\alpha} - \frac{1}{2} h\, h^{\alpha\gamma}{}_{,\alpha\gamma} - \frac{1}{4} h^{\alpha\sigma} \Box h_{\alpha\sigma} + \frac{1}{4} h \Box h \right) \tag{1.25}$$

Note that we still haven't exploited the gauge freedom in our theory, which might help us simplify terms in the above expression. Besides that, fixing the gauge is essential in path integrals because they contain an integration over field configurations. Without gauge fixing, we might integrate over multiple field configurations which corresponds to the same physical configuration.

The linearized gauge transformation generated by an infinitesimal diffeomorphism parametrized by a vector field ξ can be expressed as,

$$h_{\mu'\nu'} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \tag{1.26}$$

There exists a gauge choice known as the 'de-Donder' gauge which will be useful to simplify 1.25. This gauge condition is given by,

$$h^{\mu\nu}{}_{,\nu} = \frac{1}{2} h^{,\mu} \tag{1.27}$$

It resembles the 'Lorenz' gauge in electromagnetism and is also referred to by the same name in the literature.

Imposing the de-Donder gauge, the action 1.25 becomes quite simple,

$$S = -\frac{1}{4}\int d^4x \left(h^{\alpha\sigma}\Box h_{\alpha\sigma} - \frac{1}{2}h\Box h \right) \tag{1.28}$$

Now we can use the action in this form to find the free propagator for the theory. In field theory, if the action for the free theory is given as,

$$S = \int d^Dx \, \eta(x) \, \mathcal{O}(x, x') \, \eta(x') \tag{1.29}$$

Then we know that the propagator for the theory is given by the inverse of $\mathcal{O}$. In our case, if we can write the action 1.28 as,

$$S = \int d^4x \int d^4x' \, h_{\alpha\beta}(x) \, \mathcal{O}^{\alpha\beta\mu\nu}(x - x') \, h_{\mu\nu}(x') \tag{1.30}$$

Following [13], we can now show that $\mathcal{O}$ can be written as,

$$\mathcal{O}^{\mu\nu\alpha\beta} = -\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta} \right) \Box \, \delta^4(x - x') \tag{1.31}$$

where,

$$I^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} \right) \tag{1.32}$$

is known as the 'Identity tensor'.

To verify the above statement,

$$\begin{aligned} & h_{\mu\nu} \left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta} \right) \Box h_{\alpha\beta} \\ &= \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta} \right) h_{\mu\nu}\Box h_{\alpha\beta} \\ &= \frac{1}{2}(h^{\alpha\beta} + h^{\beta\alpha})\Box h_{\alpha\beta} - \frac{1}{2}h\Box h \\ &= h^{\alpha\beta}\Box h_{\alpha\beta} - \frac{1}{2}h\Box h \end{aligned} \tag{1.33}$$

as expected.

We know that in field theory, the inverse of $\mathcal{O}$ should satisfy the following relation by definition,

$$\int d^4x''\, \mathcal{O}(x-x'')\, \mathcal{O}^{-1}(x''-x') = \delta^D(x-x') \tag{1.34}$$

So for the $\mathcal{O}$ given by 1.31, we can write 1.34 as,

$$-\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)\Box\mathcal{O}^{-1}_{\alpha\beta\rho\sigma}(x-x') = I^{\mu\nu}{}_{\rho\sigma}\,\delta^4(x-x') \tag{1.35}$$

If we assume an ansatz for $\mathcal{O}^{-1}$ as,

$$\mathcal{O}^{-1}_{\alpha\beta\rho\sigma}(x-x') = Q_{\alpha\beta\rho\sigma}\,\delta^4(x-x') \tag{1.36}$$

where $Q_{\alpha\beta\rho\sigma}$ is a tensor independent of the space-time coordinates (hence constructed from the flat space metric). We can write 1.35 in Fourier space as,

$$-(-k^2)\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right) Q_{\alpha\beta\rho\sigma} = I^{\mu\nu}{}_{\rho\sigma} \tag{1.37}$$

We can verify that,

$$Q_{\alpha\beta\rho\sigma} = \frac{1}{k^2}\left(I_{\alpha\beta\rho\sigma} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\rho\sigma}\right) \tag{1.38}$$

satisfies 1.37, and hence $\mathcal{O}^{-1}_{\alpha\beta\rho\sigma}$, i.e, the propagator for the Lagrangian 1.28 when $x^0 > x'^0$ is given as,

$$\mathcal{O}^{-1}_{\alpha\beta\rho\sigma}(x-x') = \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{2}\left(\eta_{\alpha\rho}\eta_{\beta\sigma} + \eta_{\alpha\sigma}\eta_{\beta\rho} - \eta_{\alpha\beta}\eta_{\rho\sigma}\right)\frac{e^{-ik\cdot(x-x')}}{k^2} \tag{1.39}$$

where we have expanded $Q_{\alpha\beta\rho\sigma}$. From the above, we see that the propagator has a pole at $k=0$. This demonstrates that the Einstein-Hilbert action propagates just the massless spin-2 graviton mode in flat space. Later in our notes, we will argue that any theory (whose Lagrangian is a scalar density comprising solely of the metric and its derivatives) propagates just the graviton mode in any arbitrary background if it has a second-order equation of motion.

Now, consider the addition of the following higher-derivative term to the Einstein-Hilbert action.

$$S = \int d^D x \sqrt{-g}\,(R + \,a\,R^2) \tag{1.40}$$

where a is an arbitrary constant. This is the simplest case of a class of theories known as '$f(R)$' theories, which comprises of polynomial in Ricci scalar corrections to the gravitational action. First, we expand this Lagrangian at quadratic order in perturbation around a flat background and then impose the de-Donder gauge,

$$\begin{aligned} S &= -\frac{1}{4}\int d^D x \left(h^{\alpha\sigma}\Box h_{\alpha\sigma} - \frac{1}{2} h\Box h + a\,(h^{\alpha\gamma}_{,\alpha\gamma})^2 \right) \\ &\equiv -\frac{1}{4}\int d^D x \left(h^{\alpha\sigma}\Box h_{\alpha\sigma} - \frac{1}{2} h\Box h + a\,h\Box^2 h \right) \\ &= -\frac{1}{4}\int d^4 x \left(h^{<\alpha\sigma>}\Box h_{<\alpha\sigma>} + \left(\frac{1}{D} - \frac{1}{2}\right) h\Box h + a\,h\Box^2 h \right) \end{aligned} \tag{1.41}$$

where $h_{<\alpha\beta>}$ denotes the traceless part of the metric perturbation, and we have decomposed it as,

$$h_{\alpha\beta} = h_{<\alpha\beta>} + \frac{1}{D}\eta_{\alpha\beta} h \tag{1.42}$$

Focusing just on the trace part of the action, we realize that it is identical to the higher derivative scalar field theory example which we studied before, and hence one might hastily conclude that this theory leads to massive ghost modes. However, we still have residual gauge freedom of the metric which we can utilize. Here, one can choose the transverse-traceless (TT) gauge (See Appendix A), where $h = 0$ and $h_{0i} = 0$. Having completely fixed the gauge, which removes the trace degree of freedom, we see that – at order $\mathcal{O}(h^2)$ – the contribution of the higher-derivative correction vanishes, and the only contribution is from the Einstein-Hilbert term. Hence, the theory is devoid of any ghost modes around flat space, even though it has a higher-derivative equation of motion which come at $\mathcal{O}(h^3)$. Note that this argument easily extends to any theory with higher-derivative corrections as scalar polynomials of the Ricci scalar – which are formally known as $f(R)$ theories.

To emphasize, $f(R)$ theories have the remarkable property that they do not propagate any ghost modes in flat space even though they seem to have a higher-derivative equation of motion. All the higher-derivatives are carried by the trace degree of freedom, which can be removed utilizing the gauge freedom of the theory.

But what if we had a $R + R_{\mu\nu}R^{\mu\nu}$ theory instead? First, we expand this Lagrangian at quadratic order in perturbation around a flat background and then impose the de-Donder gauge:

$$\begin{aligned} S &= -\frac{1}{4}\int d^D x \left(h^{\alpha\sigma}\Box h_{\alpha\sigma} - \frac{1}{2}h\Box h + a\, h^{\mu\nu}\Box^2 h_{\mu\nu} \right) \\ &= -\frac{1}{4}\int d^D x \Big(h^{<\alpha\sigma>}\Box h_{<\alpha\sigma>} + a\, h^{<\alpha\sigma>}\Box^2 h_{<\alpha\sigma>} \\ &\qquad\qquad -\frac{1}{2}h\Box h + \frac{a}{D}\, h\Box^2 h \Big) \end{aligned} \tag{1.43}$$

Here, even if we remove the trace mode by residual gauge freedom, the traceless components still carry higher derivatives and we do not have sufficient gauge freedom in our theory to eliminate them. Hence, such a theory has ghosts on a flat background.

In the discussions above, we have always taken a flat background and analysed the linearized spectrum around that. A crucial observation is that the presence of absence of ghost modes relies on the background around which we are linearising the theory. As an example, if the higher-derivative corrections are of order Riem^3 (without any covariant derivatives of the curvature tensor) and higher, around a flat background, they never contribute to the linearized equations of motion. This is because, when we expand the Lagrangian to quadratic order around a flat background, any of the curvature tensors (besides the two which are linearized) evaluated on the background is zero. We can investigate the presence of ghost modes only when a background is specified. Furthermore, the above examples also demonstrate that even if the the full theory has a higher-order equation of motion, the linearized equation of motion might be second-order, but the converse doesn't hold.

One might ask if there's a relation between the presence of ghost modes and Ostrogradsky instability. As we have emphasized before, the presence of ghosts depends on the linearized action (at $\mathcal{O}(h^2)$), while the Ostrogradsky instability pertains to the entire theory. However, one must first ask the question if it is possible to uniquely reconstruct the full higher-derivative theory from the linearized action of that theory. A recent paper [14] argues that this is not possible in flat space and the full reconstruction can only be made for just the Einstein-Hilbert action. This suggests that one cannot comment on the Ostrogradsky stability of the theory from the linearized action in some background.

A related point is that while a second-order linearized action does not guarantee that the full equations of motion remain second-order on a generic background, the converse is true. The discussions in this section provides us a physically motivated criterion based on which we can classify theories. In the next section, we will discuss the subtleties associated with such a classification scheme.

1.4 Equation of motion vs. Linearized equation of motion

The previous section elucidated the crucial point that the presence or absence of ghosts in a theory depends on the background on which the modes propagate. Now, we will proceed by proposing two criterion based on which we can classify models of gravity.

1.4.1 Order of equation of motion

If the equation of motion of a given theory is second order in derivatives, the linearized equation of motion will be second order too around any arbitrary background. As we shall prove soon in the next section, if the linearized equation of motion is second-order around some background, then the theory only propagates the massless graviton (spin-2) around that background. The unique class of theories with this property is the 'Lanczos-Lovelock' models, which we will study in the next chapter.

On the other hand, any Lagrangian which leads to a higher-order equation of motion, may or may not propagate ghost modes, depending on the background around which it is linearized.

1.4.2 Order of linearized equation of motion

For a generic Lagrangian comprised of the metric and its derivatives, if the linearized equation of motion is second-order in derivatives in some specific background, it implies necessarily that our theory propagates a single massless spin-2 mode – the graviton – on the background, i.e, the theory is 'Einsteinian' on that specific background. To see this, consider the following argument:

The linearized equation of motion, by definition, contains only a single metric perturbation $h_{\mu\nu}$ in each term, which is contributed by a single linearized Riemann tensor, two of whose indices are contracted by either (two) covariant derivatives or using (once) the background metric. If we use covariant derivatives, the equation of motion will no longer be second order. So going by the only option left – to contract indices with the background metric – the only two covariant tensors we are left with are: $R^{(1)}_{\mu\nu}$ and $\bar{g}_{\mu\nu}R^{(1)}$, where $\bar{g}$ refers to the background metric and the superscript (1) denotes linearization in the perturbation $h_{\mu\nu}$. Hence, the most general expression of a second-order linearized equation of motion – which is equated to zero – is given by,

$$\mathcal{E}_{\mu\nu} = c_1\, R^{(1)}_{\mu\nu} + c_2\, \bar{g}_{\mu\nu} R^{(1)} \tag{1.44}$$

where c_1 and c_2 are constants. Since the Lagrangian is a diffeomorphsim invariant scalar, this quantity must be divergence-free, i.e, $\nabla^\mu \mathcal{E}_{\mu\nu} = 0$ which relates the two coefficients. Using the contracted Bianchi identity, we get $c_1 = -2c_2$ and hence the linearized equation of motion reads,

$$\mathcal{E}_{\mu\nu} = c_1\, G^{(1)}_{\mu\nu} \tag{1.45}$$

which propagates only a single massless spin-2 mode - as can be seen after fixing the gauge – which we have already discussed before. Hence, there is an equivalence between having second order linearized equation of motion and the spectrum being Einsteinian.

- 2nd order linearized equation of motion $\Longleftrightarrow$ Einsteinian spectrum

In the presence of higher derivative terms in the equation of motion, the linearized equation of motion will (after suitable gauge choices and redefinitions of $h_{\mu\nu}$) take the form,

$$\kappa\,(\Box - m_1)\,(\Box - m_2)\,(\Box - m_3)\,...\,(\Box - m_n)\,t_{<\mu\nu>} = 0 \tag{1.46}$$

for massive or massless spin-2 modes – where $t_{<\mu\nu>}$ represents the traceless degrees of freedom and,

$$\kappa\,(\Box - m_1')\,(\Box - m_2')\,(\Box - m_3')\,...\,(\Box - m_n')\,h = 0 \tag{1.47}$$

for the scalar modes which represent the trace degree of freedom. κ represents the effective gravitational constant of the theory, m_i and m_i' are the masses of the modes such that atleast one of the m_i's is zero if the theory has an Einstein gravity limit when the higher derivative couplings vanish. The momentum space propagator for such a theory will necessarily have multiple poles with the wrong sign, which represents ghost degrees of freedom. Hence, we see that their exists another correspondence as,

. Higher order linearized equation of motion $\Longleftrightarrow$ Ghost modes

1.5 Further discussion on $f(R)$ theories

In the previous section, we have seen that we can utilize the gauge freedom in GR to remove the scalar degree of freedom which carries higher-derivatives in the equation of motion of $f(R)$ theories, and hence these theories do not propagate any ghost modes. An alternative way to see this is by performing a conformal transformation which maps the theory to the Einstein-Hilbert action with a minimally coupled scalar field. Let us proceed by first deriving the equation of motion of these theories:

1.5.1 Equation of motion

The action for a general $f(R)$ theory is given as,

$$S = \int d^4x \sqrt{-g}\, f(R) \tag{1.48}$$

Taking an arbitrary infinitesimal variation of the action with respect to the metric:

$$\delta S = \int d^4x \; (\delta\sqrt{-g}) f(R) + \sqrt{-g}\, \delta f(R) \tag{1.49}$$

We focus our attention on the second term in the above expression. Using the chain rule:

$$\begin{aligned} \delta f(R) &= \left(\frac{\partial f}{\partial R}\right) \delta R \\ &= f'(R)\, \delta R \end{aligned} \tag{1.50}$$

where we have defined $\partial f/\partial R = f'(R)$. Taking an infinitesimal variation of the Ricci scalar,

$$\begin{aligned} \delta R &= \delta(g^{\beta\nu} R_{\beta\nu}) \\ &= (\delta g^{\beta\nu}) R_{\beta\nu} + g^{\beta\nu} \delta R_{\beta\nu} \\ &= (\delta g^{\beta\nu}) R_{\beta\nu} + g^{\beta\nu} \left(\delta \Gamma^{\alpha}{}_{\beta\nu;\alpha} - \delta \Gamma^{\alpha}{}_{\beta\alpha;\nu}\right) \end{aligned} \tag{1.51}$$

An infinitesimal variation of the Christoffel symbol is given as:

$$\delta \Gamma^{\alpha}{}_{\beta\nu} = \frac{1}{2} g^{\alpha\sigma} (\delta g_{\sigma\beta;\nu} + \delta g_{\sigma\nu;\beta} - \delta g_{\beta\nu;\sigma}) \tag{1.52}$$

Substituting the above expression in 1.51 after contracting the indices as required, we have,

$$g^{\beta\nu}\left(\delta\Gamma^{\alpha}{}_{\beta\nu;\alpha} - \delta\Gamma^{\alpha}{}_{\beta\alpha;\nu}\right) = -\delta g^{\alpha\beta}{}_{;\alpha\beta} + g_{\alpha\sigma}\Box(\delta g^{\alpha\sigma}) \tag{1.53}$$

Finally, using the expressions 1.53, 1.51 and 1.50, we can write the variation of the action 1.49 as,

$$\delta S = \int d^4x\,(\delta\sqrt{-g})f(R) + \sqrt{-g}\,f'(R)R_{\beta\nu}\delta g^{\beta\nu} - f'(R)\nabla_\alpha\nabla_\beta\delta g^{\alpha\beta} + f'(R)g_{\alpha\sigma}\nabla_\mu\nabla^\mu(\delta g^{\alpha\sigma}) \tag{1.54}$$

On using the relation $\delta\sqrt{-g} = -(1/2)\sqrt{-g}g_{\beta\nu}\delta g^{\beta\nu}$, performing integration by parts twice on the last two terms and ignoring the boundary terms:

$$\delta S = \int d^4x\,\sqrt{-g}\left(-\frac{1}{2}g_{\beta\nu}f(R) + f'(R)R_{\beta\nu} - \nabla_\beta\nabla_\nu f'(R) + g_{\beta\nu}\nabla^\mu\nabla_\mu f'(R)\right)\delta g^{\beta\nu} \tag{1.55}$$

From the above, we can finally write the equation of motion of a general $f(R)$ theory as,

$$\{f'(R)\,R_{\beta\nu} - \frac{1}{2}g_{\beta\nu}f(R) - (\nabla_\beta\nabla_\nu - g_{\beta\nu}\Box)f'(R) = 0 \tag{1.56}$$

The last two terms have two covariant derivatives of $f'(R)$ and hence contain terms with four derivatives of the metric.

In the simple case of the Einstein-Hilbert action, where $f(R)$ is linear in R, $f'(R)$ is a constant, so the last two terms vanishes and we get back the Einstein's equations.

Having derived the equation of motion of generic $f(R)$ theories, we proceed to investigate whether they propagate any additional modes by making a conformal transformation to map the action to that of Einstein-Hilbert with a minimally coupled scalar.

1.5.2 Conformal transformation to Einstein frame

The goal of the section will be to show that $f(R)$ theories do not suffer from the Ostrogradsky instability by making a conformal transformation which demonstrates that the higher derivatives are carried by a single scalar degree of freedom. We shall show that on performing the field redefinition, we can write the action as the Einstein-Hilbert action with a minimally coupled scalar field. Let us perform this conformal transformation step-by-step [15] [16] [17].

We write the action for a general $f(R)$ theory as,

$$S = \int d^4x\sqrt{-g}\,f(R) \tag{1.57}$$

Let us introduce a scalar field χ and write a new action as,

$$S = \int d^4x\sqrt{-g}\,(f(\chi) + f'(\chi)(R - \chi)) \tag{1.58}$$

This action is 'dynamically equivalent' to the action 1.57 for general $f(R)$ theories in the sense that it leads to the same equation of motion. To see this, consider the variation of the action with respect to the scalar field χ:

$$\delta_\chi S = \int d^4x\,\sqrt{-g}\,(f'(\chi) - f''(\chi)(R - \chi) - f'(\chi))\,\delta\chi = 0 \tag{1.59}$$

which leads to the equation of motion by varying χ,

$$f''(\chi)(R - \chi) = 0 \tag{1.60}$$

Let us now assume that $f''(\chi) \neq 0$ for any χ. This is an important condition which we shall discuss below. This assumption leads to $\chi = R$. Now, if we vary the action 1.58 with respect to the metric $g^{\mu\nu}$, we get back the equation of motion 1.56 but in terms of $f(\chi)$ and $f'(\chi)$. Using the constraint $\chi = R$ as we got above, it reproduces the equation of motion 1.56.

(Note: It might be tempting to substitute $\chi = R$ back into the action in any of the proceeding calculation, but it is an on-shell relation, and we will take care to not do so.)

Now let us define a scalar ϕ which depends on χ as,

$$\phi = f'(\chi) \tag{1.61}$$

The condition $f'' \neq 0$ implies that the above relation can be inverted to give χ as a function of ϕ. To see this:

$$\frac{d\phi}{d\chi} = f''(\chi) \neq 0 \tag{1.62}$$

The above equation implies that f' is injective if f' is a smooth function, and in such a case, we can invert 1.61 to obtain $\chi \equiv \chi(\phi)$.

Let us use this scalar ϕ to rewrite the action 1.58 as,

$$S = \int d^4x\sqrt{-g}\,(\phi R - V(\phi)) \tag{1.63}$$

where we have defined,

$$V(\phi) = \chi(\phi)\,\phi - f(\chi(\phi)) \tag{1.64}$$

(The above action is known as the 'Jordan' frame representation of $f(R)$ theories). Next, we perform a 'conformal transformation' ($g_{\mu\nu} \to \tilde{g}_{\mu\nu}$) on our spacetime, which is given by,

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{1.65}$$

where Ω is known as the conformal factor.

Under such a conformal transformation, the Ricci scalar (in 4D) transforms as,

$$R = \Omega^{-2}(\tilde{R} + 6\Box(\ln\Omega) - 6\tilde{g}^{\mu\nu}\nabla_\mu(\ln\Omega)(\nabla_\nu \ln\Omega)) \tag{1.66}$$

and the metric determinant transforms as,

$$\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}} \tag{1.67}$$

Substituting the above two relation in 1.63, we have,

$$S = \int d^4x\sqrt{-\tilde{g}}\left(\phi\,\Omega^{-2}\left(\tilde{R} + 6\Box(\ln\Omega) - 6\tilde{g}^{\mu\nu}\nabla_\mu(\ln\Omega)\nabla_\nu(\ln\Omega)\right) - \Omega^{-4}V(\phi)\right) \tag{1.68}$$

We make the choice of the conformal factor as,

$$\phi = \Omega^2 \tag{1.69}$$

and drop the second term in the above action as it is a total derivative term, which leads to the action of the $f(R)$ theory in the 'Einstein frame' as,

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\left(\tilde{R} - \frac{6}{4} \tilde{g}^{\mu\nu} \nabla_\mu (\ln\phi) \nabla_\nu (\ln\phi) \right) - \frac{V(\phi)}{\phi^2} \right) \tag{1.70}$$

Let us re-define the scalar ϕ as $\tilde{\phi}$ in the following way:

$$\tilde{\phi} = \sqrt{3}\, \ln\phi \tag{1.71}$$

Substituting the above definition in 1.70, we can write the action as:

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{2}\, \tilde{g}^{\alpha\beta} \nabla_\alpha \tilde{\phi} \nabla_\beta \tilde{\phi} - U(\tilde{\phi}) \right) \tag{1.72}$$

where we defined U to be a function of $\tilde{\phi}$ as given below, by using 1.71 to express ϕ in terms of $\tilde{\phi}$,

$$U(\tilde{\phi}) = \frac{V(e^{\frac{\tilde{\phi}}{\sqrt{3}}})}{e^{\frac{2\tilde{\phi}}{\sqrt{3}}}} \tag{1.73}$$

Note that the above is the Einstein-Hilbert action with a minimally coupled scalar $\tilde{\phi}$. Hence, the field equations corresponding to the above action is given as,

$$\tilde{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \tilde{R} = \frac{1}{2} \nabla_\alpha \tilde{\phi}\, \nabla_\beta \tilde{\phi} - \frac{1}{4} \tilde{g}_{\alpha\beta} \nabla^\mu \tilde{\phi}\, \nabla_\mu \tilde{\phi} - \frac{1}{2} g_{\alpha\beta}\, U(\tilde{\phi}) \tag{1.74}$$

Taking the trace above to find R and then substituting the result back, we have,

$$\tilde{R}_{\mu\nu} = \frac{1}{2} \nabla_\mu \tilde{\phi}\, \nabla_\nu \tilde{\phi} + \frac{1}{2} \tilde{g}_{\mu\nu} U(\tilde{\phi}) \tag{1.75}$$

We can verify that the right hand side of 1.74 is indeed divergence-free with the help of the equation of motion obtained by varying ϕ, given as,

$$\nabla_{\mu}\nabla^{\mu}\tilde{\phi} = \frac{\partial U(\tilde{\phi})}{\partial\tilde{\phi}} \tag{1.76}$$

Hence, we see that the redefined field-equation 1.75 contains atmost two derivatives of the metric $\tilde{g}_{\mu\nu}$. The scalar $\tilde{\phi}$ provides the source terms, and its evolution is governed by 1.76.

1.6 Field Redefinitions

In an effective field theory (EFT), one always has the freedom to redefine the metric to achieve a desired change at a certain order in the perturbative expansion, and ignore the effect of such a redefinition at all higher orders by truncation. We want to understand what sort of terms one can add or remove from the theory by such field redefinitions. As we shall see below, field redefinitions might introduce or eliminate additional poles in the theory. We will investigate these ambiguities in this section and discuss whether this freedom bears any physical significance for an EFT.

As a start, consider a metric redefinition of the form,

$$g_{\mu\nu} \to g_{\mu\nu} + \eta\,(a\,R_{\mu\nu} + b\,R\,g_{\mu\nu}) \tag{1.77}$$

where we have defined $\eta = \kappa^{\frac{2}{D-2}} \sim [L]^2$ and $\kappa = 8\pi G/c^4$. We will consider the above redefinition as a perturbation of the metric and hence only keep terms which are linear in a and b. The factors of η in the perturbative terms are introduced to maintain the correct dimensionality of the action.

Consider the Einstein-Hilbert action as,

$$S = \frac{1}{2\kappa}\int d^D x\,\sqrt{-g}\,R \tag{1.78}$$

and hence the Lagrangian is given as,

$$L_{EH} = \frac{1}{2\kappa}(\sqrt{-g}\,R) \tag{1.79}$$

We want to find the change at linear order in perturbation of this Lagrangian due to the redefinition 1.77. In 1.77, we have only considered perturbative terms with 2-derivatives of the metric. Hence, the first-order changes in the curvature tensor will be at 4-derivative order.

For a small perturbation of the metric ($\delta g^{\mu\nu}$), the Lagrangian changes upto linear order in perturbation as,

$$\delta(\sqrt{-g}\,R) = (\delta\sqrt{-g})\,R + \sqrt{-g}\,\delta g^{\mu\nu}\,R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu} \tag{1.80}$$

Now, we notice that the last term can be written as a total derivative, as shown below, and hence we can ignore it.

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \left(g^{\alpha\beta}\delta\Gamma^{\mu}{}_{\alpha\beta} - g^{\alpha\mu}\;\delta\Gamma^{\beta}{}_{\alpha\beta}\right)_{;\mu} \tag{1.81}$$

Next, on using the relation $\delta\sqrt{-g} = (1/2)\,\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}$, we have:

$$\int d^D x\;\delta(\sqrt{-g}R) = \int d^D x\,\sqrt{-g}\left(\frac{1}{2}R\,g^{\mu\nu} + R^{\mu\nu}\right)\delta g_{\mu\nu} \tag{1.82}$$

On replacing 1.77 for $\delta g^{\mu\nu}$, we can write the integrand in the RHS of 1.82 as:

$$\left(\frac{1}{2}R\,g^{\mu\nu} + R^{\mu\nu}\right)\delta g_{\mu\nu} = \left(\eta\,\frac{a + (D+2)b}{2}\right)R^2 + (\eta\,a)\,R^{\mu\nu}R_{\mu\nu} \tag{1.83}$$

where D is the dimension of the spacetime. Let us assume that we start with an initial EFT action given by,

$$S = \frac{1}{2\kappa}\int d^D x\,\sqrt{-g}\left(R + \eta\,d_1\,R^2 + \eta\,d_2\,R^{\mu\nu}R_{\mu\nu} + \eta\,d_3\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) + \mathcal{O}(\eta^2 a^2, \eta^2 b^2) \tag{1.84}$$

where d_1, d_2 and d_3 are constants. After performing the above-discussed field redefinition, the EFT action will become:

$$S = \frac{1}{2\kappa}\int d^D x\,\sqrt{-g}\left(R + \eta\left(\frac{a + b(D+2)}{2} + d_1\right)R^2 + \eta\,(a + d_2)\,R^{\mu\nu}R_{\mu\nu} + d_3\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) + \mathcal{O}(\eta^2 a^2, \eta^2 b^2) \tag{1.85}$$

If we want to map this theory to GR, assuming $d_3 = 0$ initially, we choose the coefficients a and b such that the $R^{\mu\nu}R_{\mu\nu}$ and R^2 terms vanish from the above action. Notice that if $d_3 \neq 0$ in the initial theory, we cannot use field redefinitions to map the theory to the Einstein-Hilbert action.

In 1.77, we have only considered terms in the perturbation which contain second-order in derivatives of the metric. Now, if we would like to see what terms we can remove with field redefinitions at order Riem^3, we should consider the most general redefinition with four derivatives:

$$\begin{aligned} g_{\mu\nu} \to\ g_{\mu\nu} + \eta^2 \big(& a_1\, R^2\, g_{\mu\nu} + a_2\, R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} + a_3\, R_{\mu\rho\sigma\eta} R_\nu{}^{\rho\sigma\eta} + a_4\, \Box R_{\mu\nu} \\ & + a_5\, R_\mu{}^\beta R_{\beta\nu} + a_6\, R R_{\mu\nu} + a_7\, g_{\mu\nu} \Box R + a_8\, g_{\mu\nu} R_{\alpha\beta\gamma\sigma} R^{\alpha\beta\gamma\sigma} \big) \end{aligned} \tag{1.86}$$

With the above metric redefinition, we have,

$$\begin{aligned} & \left(\frac{1}{2} R\, g^{\mu\nu} + R^{\mu\nu} \right) \delta g_{\mu\nu} = \\ & + \eta^2 \left(\left(\frac{a_1(D+2) + a_6}{2} \right) R^3 + \left(\frac{a_2(D+2) + a_5 + 2a_6}{2} \right) R R^{\mu\nu}\, R_{\mu\nu} \right) \\ & + \left(\frac{a_3 + a_8(D+2)}{2} \right) R R_{\alpha\beta\sigma\gamma} R^{\alpha\beta\sigma\gamma} + \left(\frac{a_7(D+2) + a_4}{2} \right) R \Box R \\ & + a_3 R^{\mu\nu} R_{\mu\rho\sigma\eta} R_\nu{}^{\rho\sigma\eta} + a_4 R^{\mu\nu} \Box R_{\mu\nu} + a_5\, R^{\mu\nu} R_\mu{}^\beta R_{\beta\nu} \big) \end{aligned} \tag{1.87}$$

As an example, starting from $R + d_1\, R^2 + d_2\, R^3$, theory, if we want to map it to the Einstein-Hilbert action + corrections at order Riem^4 and higher, we perform field redefintions order-by-order. First, we remove the R^2 term using the redefinition 1.77, However, this produces corrections at order Riem^3 and higher. Then, we use the redefinition 1.86, to set all the terms at order Riem^3 (this includes new contributions from the previous redefinition besides the R^3 term originally present) to zero. This can always be done as each scalar invariant in the Lagrangian (containing a Ricci tensor or scalar), is accompanied by one parameter from the redefinition, which is free, i.e, it doesn't accompany any other scalar invariant, as can be inferred from 1.82. In the above example, a_1 for R^3, a_2 for $RR^{\mu\nu}R_{\mu\nu}$ and so on, are the free parameters referred to in the previous line.

From the above discussion, we make the following observations:

1. No terms where the indices of the curvature tensors are contracted such that it does not involve a Ricci tensor or scalar – can ever be introduced by field redefinitions. For example, scalars such as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\gamma\delta}R^{\rho\sigma}{}_{\gamma\delta}$ are unaffected by field redefinitions.

2. One can always remove any term which includes a Ricci tensor or a Ricci scalar. For example, $R^{\mu\nu}R_{\mu\nu}$, $RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ can be removed by field redefinitions

One particular reason why the field redefinitions we studied above are useful is as follows: if a certain theory does not admit particular black hole solutions (say spherically symmetric solutions like the Schwarzschild), one can try to use field redefinitions to arrive at a Lagrangian which does admit the particular solution.

However, consider the following thought. Suppose an EFT truncated at some energy scale has a higher-derivative equation of motion and hence it has ghost-modes. Now one performs appropriate field redefinitions to introduce or remove terms such that the theory now has a second-order equation of motion. Does this mean that simply by making field redefinitions one is able to fix the issue of ghosts in an EFT Lagrangian?

As we saw in the previous sections, the mass of a ghost mode depends inversely on the coefficients of the higher-derivative terms present in the theory. Hence, in an EFT, this mass lies beyond the regime of validity of the EFT and hence the presence or absence of ghosts is not an issue one should be concerned with. An early argument in favour of string theory, as first put forward in [18], suggested that the fact that the gravitational effective action of bosonic string theory at fourth order in derivatives can be completed to the Gauss-Bonnet Lagrangian, is a hint towards the theories' consistency. However, later works pointed out that the theory only predicts that the gravitational effective action must have the $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ term, while keeping the coefficients of the $R_{\mu\nu}R^{\mu\nu}$ and R^2 terms ambiguous, which one can manipulate through field redefinitions. However, as we discussed above, ghost modes do not have any physical significance in an EFT and this is hence not an insightful analysis.

Chapter 2

Lanczos-Lovelock models

In the previous chapter, we saw how higher-derivative extensions to GR generically lead to the presence of ghost-like massive modes in the theory. This encourages us to investigate if there exists theories with higher-derivative corrections which still have a second-order equation of motion. In this chapter, we will show that this is indeed possible and this unique class of theories is known as the 'Lanczos-Lovelock models' [19], [7] (henceforth referred to as – the Lovelock models – as is common in the literature). But before that, we shall explore some properties of Lagrangians composed of contractions of the curvature tensor with the metric, as has been developed in [20], [21] and [22]. This insight into their structure will enable us to construct the Lovelock models of gravity.

2.1 Relations on general Lagrangians

Consider a Lagrangian which is composed of the curvature tensor and its contractions with the metric:

$$L \equiv L(R^{\alpha\beta}{}_{\sigma\gamma}, g^{\alpha\beta}) \tag{2.1}$$

where we have taken $(R^{\alpha\beta}{}_{\sigma\gamma}, g^{\alpha\beta})$ (with the given index placement) as the independent quantities. Let us define two quantities which we will use extensively below,

$$
\begin{aligned}
P_{\alpha\beta}{}^{\gamma\sigma} &= \left(\frac{\partial L}{\partial R^{\alpha\beta}{}_{\gamma\sigma}}\right)_{g_{\mu\nu}} \\
P_{\alpha\beta} &= \left(\frac{\partial L}{\partial g^{\alpha\beta}}\right)_{R^{\mu\nu}{}_{\sigma\gamma}}
\end{aligned}
\tag{2.2}
$$

From the above definitions, we notice that $P_{\alpha\beta}^{\gamma\sigma}$ has all the symmetries of the curvature tensor and $P_{\alpha\beta}$ is symmetric just like the metric.

Now, we proceed to prove an identity which will be quite significant in all of our subsequent discussion, especially in the construction of Lovelock theories. The identity is given as,

$$P_{\alpha\beta} = 0 \tag{2.3}$$

Note that the above relation does not represent the equation of motion with respect to the metric, as from 2.2, one can see that the curvature tensor is held fixed while evaluating $P_{\alpha\beta}$.

For the proof, consider an infinitesimal change in the Lagrangian with $(R^{\alpha\beta}{}_{\gamma\sigma}, g^{\mu\nu})$ as the independent variables which, by the chain rule, involves both the quantities in 2.2,

$$
\begin{aligned}
dL &= \left(\frac{\partial L}{\partial R^{\alpha\beta}{}_{\sigma\gamma}}\right)_{g^{\mu\nu}} dR^{\alpha\beta}{}_{\sigma\gamma} + \left(\frac{\partial L}{\partial g^{\alpha\beta}}\right)_{R^{\mu\nu}{}_{\rho\sigma}} dg^{\alpha\beta} \\
&= P_{\alpha\beta}{}^{\sigma\gamma}\, dR^{\alpha\beta}{}_{\sigma\gamma} + P_{\alpha\beta}\, dg^{\alpha\beta}
\end{aligned}
\tag{2.4}
$$

We would like to evaluate both sides of the above equation for 1-parameter class of diffeomorphisms which is given by: $x^\beta \to x^\beta + \xi^\beta(x)$, where ξ^β is an arbitrary vector field. We evaluate the left hand side of 2.4 as,

$$
\begin{aligned}
\mathcal{L}_\xi L &= \xi^\beta \nabla_\beta L \\
&= \xi^\mu (P_{\alpha\beta}{}^{\sigma\gamma} \nabla_\mu R^{\alpha\beta}{}_{\sigma\gamma} + P_{\alpha\beta} \nabla_\mu g^{\alpha\beta}) \\
&= \xi^\mu P_{\alpha\beta}{}^{\sigma\gamma} \nabla_\mu R^{\alpha\beta}{}_{\sigma\gamma}
\end{aligned}
\tag{2.5}
$$

In the first line, we have used the fact that L is a scalar. In the second line, we have used the chain rule. Finally, in the third line, we have used the covariant constancy of the metric.

Next we express the right hand side of 2.4 in terms of the Lie derivative as,

$$\mathcal{L}_\xi L = P_{\alpha\beta}{}^{\sigma\gamma}\mathcal{L}_\xi R^{\alpha\beta}{}_{\sigma\gamma} + P_{\alpha\beta}\mathcal{L}_\xi g^{\alpha\beta} \tag{2.6}$$

The Lie derivative on the metric is given as,

$$\mathcal{L}_\xi g^{\alpha\beta} = -\nabla^\alpha \xi^\beta - \nabla^\beta \xi^\alpha \tag{2.7}$$

Evaluating the Lie derivative on the curvature tensor,

$$\mathcal{L}_\xi R^{\alpha\beta}{}_{\sigma\gamma} = \xi^\mu \nabla_\mu R^{\alpha\beta}{}_{\sigma\gamma} - (\nabla^\alpha \xi_\mu) R^{\mu\beta}{}_{\sigma\gamma} - (\nabla^\beta \xi_\mu) R^{\alpha\mu}{}_{\sigma\gamma} + (\nabla_\sigma \xi^\mu) R^{\alpha\beta}{}_{\mu\gamma} + (\nabla_\gamma \xi^\mu) R^{\alpha\beta}{}_{\sigma\mu} \tag{2.8}$$

When contracted with $P^{\alpha\beta\sigma\gamma}$ (and using its symmetries), the last four terms cancel each other. So, we are left with,

$$P_{\alpha\beta}{}^{\sigma\gamma}\mathcal{L}_\xi R^{\alpha\beta}{}_{\sigma\gamma} = P_{\alpha\beta}{}^{\sigma\gamma}\xi^\mu \nabla_\mu R^{\alpha\beta}{}_{\sigma\gamma} \tag{2.9}$$

Combining the results 2.7 and 2.9,

$$\mathcal{L}_\xi L = 2P^{\alpha\beta}\nabla_\alpha \xi_\beta + P_{\alpha\beta}{}^{\sigma\gamma}\xi^\mu \nabla_\mu R^{\alpha\beta}{}_{\sigma\gamma} \tag{2.10}$$

Comparing the two equations 2.5 and 2.10, we finally get the relation:

$$P_{\alpha\beta}\nabla^\alpha \xi^\beta = 0 \tag{2.11}$$

Since ξ^β is an arbitrary vector field and the above relation holds at every point on the manifold, we reach to the conclusion that,

$$P_{\alpha\beta} = 0 \tag{2.12}$$

2.2 Equation of motion and Lovelock condition

Now, we finally proceed to study the equation of motion for our theory, utilizing all the machinery we have developed above. With the Lagrangian 2.1, we have the following action,

$$S = \int d^D x \sqrt{-g}\, L(g^{\alpha\beta}, R^{\alpha\beta}{}_{\sigma\gamma}) \tag{2.13}$$

To find the equation of motion, let us take a variation of this action as,

$$\begin{aligned}
\delta S &= \int d^D x\, \delta(\sqrt{-g})\, L + \sqrt{-g}\, \delta L \\
&= \int d^D x \sqrt{-g} \left(\frac{\partial L}{\partial g^{\alpha\beta}}\right) \delta g^{\alpha\beta} + \sqrt{-g} \left(\frac{\partial L}{\partial R^{\alpha\beta}{}_{\gamma\delta}}\right) (\delta R^{\alpha\beta}{}_{\gamma\delta}) - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta} \\
&= \int d^D x \sqrt{-g}\, P_{\alpha\beta}\, \delta g^{\alpha\beta} + \sqrt{-g}\, P^{\gamma\delta}{}_{\alpha\beta}\, \delta(g^{\beta\theta} R^{\alpha}{}_{\theta\gamma\delta}) - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta} \\
&= \int d^D x \sqrt{-g}\, P_{\alpha\beta}\, \delta g^{\alpha\beta} + \sqrt{-g}\, P^{\gamma\delta}{}_{\alpha\beta}\, R^{\alpha}{}_{\theta\gamma\delta} (\delta g^{\beta\theta}) + P^{\gamma\delta}{}_{\alpha\beta}\, g^{\beta\theta} (\delta R^{\alpha}{}_{\theta\gamma\delta}) - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}
\end{aligned} \tag{2.14}$$

We will need the following relation:

$$\begin{aligned}
\delta R^{\alpha}{}_{\beta\sigma\gamma} &= \nabla_\sigma(\delta\Gamma^{\alpha}{}_{\beta\gamma}) - \nabla_\gamma(\delta\Gamma^{\alpha}{}_{\beta\sigma}) \\
&= \frac{1}{2}\nabla_\sigma(g^{\alpha\mu}(\nabla_\gamma \delta g_{\beta\mu} + \nabla_\beta \delta g_{\gamma\mu} - \nabla_\mu \delta g_{\gamma\beta})) - (\sigma \leftrightarrow \gamma)
\end{aligned} \tag{2.15}$$

Now,

$$\begin{aligned}
P_{\alpha}{}^{\beta\sigma\gamma}\delta R^{\alpha}{}_{\beta\sigma\gamma} &= P_{\alpha}{}^{\beta\sigma\gamma} \left(\frac{1}{2}\nabla_\sigma(g^{\alpha\mu}(\nabla_\gamma \delta g_{\beta\mu} + \nabla_\beta \delta g_{\gamma\mu} - \nabla_\mu \delta g_{\gamma\beta}))\right) - (\sigma \leftrightarrow \gamma) \\
&= \frac{1}{2} P^{\mu\beta\sigma\gamma} \left(\nabla_\sigma(\nabla_\gamma \delta g_{\beta\mu} + \nabla_\beta \delta g_{\gamma\mu} - \nabla_\mu \delta g_{\gamma\beta})\right) - (\sigma \leftrightarrow \gamma)
\end{aligned} \tag{2.16}$$

The first term drops out since $P^{\mu\beta\sigma\gamma}$ is antisymmetric in β and μ, leaving,

$$P_{\alpha}{}^{\beta\sigma\gamma}\delta R^{\alpha}{}_{\beta\sigma\gamma} = \frac{1}{2} P^{\mu\beta\sigma\gamma} \left(\nabla_\sigma(\nabla_\beta \delta g_{\gamma\mu} - \nabla_\mu \delta g_{\gamma\beta})\right) - (\sigma \leftrightarrow \gamma) \tag{2.17}$$

The two terms above are equal along with the corresponding antisymmetrised $\sigma \leftrightarrow \gamma$, so we find:

$$P_{\alpha}^{\ \beta\sigma\gamma}\delta R^{\alpha}_{\ \beta\sigma\gamma} = 2P^{\mu\beta\sigma\gamma}\nabla_\sigma\nabla_\beta(\delta g_{\gamma\mu}) \tag{2.18}$$

We use the chain rule to free the $\delta g_{\gamma\mu}$ term from the derivatives,

$$\begin{aligned} 2P^{\mu\beta\sigma\gamma}\nabla_\sigma\nabla_\beta(\delta g_{\gamma\mu}) &= 2\nabla_\sigma(P^{\mu\beta\sigma\gamma}\nabla_\beta\delta g_{\gamma\mu}) - 2(\nabla_\sigma P^{\mu\beta\sigma\gamma})(\nabla_\beta\delta g_{\gamma\mu}) \\ &= 2\nabla_\sigma(P^{\mu\beta\sigma\gamma}\nabla_\beta\delta g_{\gamma\mu}) - 2\nabla_\beta(\nabla_\sigma P^{\mu\beta\sigma\gamma}\delta g_{\gamma\mu}) + 2(\nabla_\beta\nabla_\sigma P^{\mu\beta\sigma\gamma})\,\delta g_{\gamma\mu} \end{aligned} \tag{2.19}$$

We notice that the first two terms are total derivatives while the last term is a double derivative of $P^{\alpha\beta\sigma\gamma}$. Using the relation $\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}$ to raise the indices on $\delta g_{\alpha\beta}$ and substituting the result 2.19 in 2.14, we finally obtain the equation of motion as:

$$P_{\alpha}^{\ \mu\nu\rho}R_{\beta\mu\nu\rho} - \frac{1}{2}\,g_{\alpha\beta}\,L - 2\nabla^\sigma\nabla^\gamma P_{\alpha\sigma\gamma\beta} = \frac{1}{2}T_{\alpha\beta} \tag{2.20}$$

In the above equation of motion, only the last term contributes higher-derivatives of the metric. Hence, if we want a theory with a second-order equation of motion, the last term should either be second-order or zero. Choosing the latter constraint, we have the 'Lovelock condition':

$$\nabla_\alpha P^{\alpha\beta\sigma\gamma} = 0 \tag{2.21}$$

Note that the above condition implies that $P^{\alpha\beta\sigma\gamma}$ is divergence-free in all of its indices. The Lanczos-Lovelock theories constitute the unique solutions which satisfies the above relation – which we shall show in the next section. Another question which remains to be addressed is whether there exists any theories which satisfies the more general condition:

$$\nabla^\sigma\nabla^\gamma P_{\alpha\sigma\gamma\beta} = \text{all second order terms} \tag{2.22}$$

However, as we shall see in the subsequent sections, the above condition can never be satified by any theory composed solely of the metric and its derivatives.

As a sanity check, we need to show that left hand side of 2.20, in the case of Lovelock theories, i.e, in the absence of the last term, is divergence-free. We need,

$$\begin{aligned} &\nabla_\alpha \left(P^{\alpha\mu\nu\rho} R^\beta{}_{\mu\nu\rho} - \frac{1}{2} g^{\alpha\beta} L \right) = 0 \\ \implies &\nabla_\alpha (P^{\alpha\mu\nu\rho} R^\beta{}_{\mu\nu\rho}) - \frac{1}{2} g^{\alpha\beta} \nabla_\alpha L = 0 \end{aligned} \tag{2.23}$$

Evaluating the left hand side of this expression, we have,

$$\begin{aligned} \nabla_\alpha (P^{\alpha\mu\nu\rho} R^\beta{}_{\mu\nu\rho}) - \frac{1}{2} g^{\alpha\beta} \nabla_\alpha L &= P^{\alpha\sigma\gamma\mu} \nabla_\alpha R^\beta{}_{\sigma\gamma\mu} - \frac{1}{2} \left(\frac{\partial L}{\partial R_{\alpha\sigma\gamma\mu}} \right) \nabla^\beta R_{\alpha\sigma\gamma\mu} \\ &= P^{\alpha\sigma\gamma\mu} \nabla_\alpha R^\beta{}_{\sigma\gamma\mu} - \frac{1}{2} P^{\alpha\sigma\gamma\mu} \nabla^\beta R_{\alpha\sigma\gamma\mu} \end{aligned} \tag{2.24}$$

In the first line, we have used the fact that $P^{\alpha\gamma\sigma\mu}$ and $g^{\alpha\beta}$ are divergence-free.

Lowering the β index, and taking $P^{\alpha\sigma\gamma\mu}$ common from both terms: (We also switch to semi-colons for covariant derivatives for a short while to clearly express a few steps)

$$\begin{aligned} P^{\alpha\sigma\gamma\mu} \left(R_{\beta\sigma\gamma\mu;\alpha} - \frac{1}{2} R_{\alpha\sigma\gamma\mu;\beta} \right) &= P^{\alpha\sigma\gamma\mu} \left(R_{\beta\sigma\gamma\mu;\alpha} + \frac{1}{2} R_{\alpha\sigma\beta\gamma;\mu} + \frac{1}{2} R_{\alpha\sigma\mu\beta;\gamma} \right) \\ &= P^{\alpha\sigma\gamma\mu} \left(R_{\beta\sigma\gamma\mu;\alpha} - R_{\beta\mu\alpha\sigma;\gamma} \right) \\ &= 0 \end{aligned} \tag{2.25}$$

In the first line we have used the cyclic identity and in the second line, we used the symmetries of $P^{\alpha\sigma\gamma\mu}$. Hence, we have explicitly verified that the left-hand side of 2.20, is indeed divergence-free, in the case of Lovelock Lagrangians. This is expected due to the diffeomorphism invariance of the action 2.13.

2.3 Construction of Lovelock theories

In this section, we will see how Lovelock theories can be constructed based on intuition gained from the previous discussions. First, let us start with the simple case, i.e, for a Lagrangian linear in the curvature tensor, where we trivially know the Einstein-Hilbert action to be a second-order theory. If we construct it using the index placement $R^{\alpha\beta}{}_{\gamma\delta}$, the result 2.3

implies that all the indices must be contracted with Kronecker deltas only. The unique way to do this is given by,

$$L_1 = \frac{1}{2}\delta^{\mu\nu}_{\rho\sigma}\, R^{\rho\sigma}{}_{\mu\nu} \tag{2.26}$$

where,

$$\delta^{\mu\nu}_{\rho\sigma} = \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma} \tag{2.27}$$

Now, we come to case when the Lagrangian is quadratic in the curvature tensor. Let us start with the most general such Lagrangian:

$$L = a\, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + b\, R_{\mu\nu}R^{\mu\nu} + c\, R^2 \tag{2.28}$$

where a, b and c are arbitrary coefficients. Starting from the above Lagrangian and differentiating with respect to the curvature tensor, we have:

$$\begin{aligned} P^{\alpha\beta\gamma\delta} =& 2a\, R_{\alpha\beta\gamma\delta} + \frac{1}{2}b\left(g^{\alpha\gamma}R^{\beta\delta} - g^{\beta\gamma}R^{\alpha\delta} - g^{\alpha\delta}R^{\beta\gamma} + g^{\beta\delta}R^{\alpha\gamma}\right) \\ &+ c\,\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \end{aligned} \tag{2.29}$$

On rearranging the above terms, we have,

$$\begin{aligned} P^{\alpha\beta\gamma\delta} =& - g^{\beta\gamma}(\frac{1}{2}b\, R^{\alpha\delta} + c\, g^{\alpha\delta}R) + g^{\beta\delta}(\frac{1}{2}b\, R^{\alpha\gamma} + c\, g^{\alpha\gamma}R) \\ &+ \frac{1}{2}b\, g^{\alpha\gamma}R^{\beta\delta} - \frac{1}{2}b\, g^{\alpha\delta}R^{\beta\gamma} + 2a\, R^{\alpha\beta\gamma\delta} \end{aligned} \tag{2.30}$$

Now, implementing the Lovelock condition 2.21:

$$\begin{aligned} \nabla P^{\alpha\beta\gamma\delta} =& - g^{\beta\gamma}\nabla_\alpha(\frac{1}{2}b\, R^{\alpha\delta} + a\, g^{\alpha\delta}R) + g^{\beta\delta}\nabla_\alpha(\frac{1}{2}b\, R^{\alpha\gamma} + a\, g^{\alpha\gamma}R) + \\ &\frac{1}{2}b\, g^{\alpha\gamma}\nabla_\alpha R^{\beta\delta} - \frac{1}{2}b\, g^{\alpha\delta}\nabla\alpha R^{\beta\gamma} + 2c\,\nabla_\alpha R^{\alpha\beta\gamma\delta} \end{aligned} \tag{2.31}$$

From the Bianchi identity, we have the following relation,

$$\nabla_\alpha R^{\alpha\beta\gamma\delta} = \nabla^\gamma R^{\beta\delta} - \nabla^\delta R^{\beta\gamma} \tag{2.32}$$

Using this to simplify 2.30,

$$\begin{aligned}\nabla_\alpha P^{\alpha\beta\gamma\delta} = & - g^{\beta\gamma}\nabla_\alpha(\frac{1}{2}b\,R^{\alpha\delta} + a\,g^{\alpha\delta}R) + g^{\beta\delta}\nabla_\alpha(\frac{1}{2}b\,R^{\alpha\gamma} + a\,g^{\alpha\gamma}R) \\ & + \left(\frac{1}{2}b + 2c\right)\,\nabla^\gamma R^{\beta\delta} - \left(\frac{1}{2}b + 2c\right)\,\nabla^\delta R^{\beta\gamma}\end{aligned} \tag{2.33}$$

Notice that when $b/2 = -2a$, the first two terms vanish as $\nabla_\alpha G^{\alpha\delta} = 0$, where $G^{\alpha\delta}$ is the Einstein tensor. The last two terms vanish when $-b/2 = 2c$. This represents the unique solution which satisfies the Lovelock condition.

The above leads to the famous 'Gauss-Bonnet' Lagrangian (upto an overall constant factor):

$$L_{GB} = R^{\alpha\beta\sigma\gamma}R_{\alpha\beta\sigma\gamma} - 4\,R^{\alpha\sigma}R_{\alpha\sigma} + R^2 \tag{2.34}$$

The above discussion is valid in arbitrary dimensions. However, in 4D, the Gauss-Bonnet term does not contribute to the equations of motion as it is a total-derivative in 4D. This is given by the Chern-Gauss-Bonnet theorem [23], which shows that upon integration, the Gauss-Bonnnet term leads to the Euler characteristic of the manifold.

Notice that the Gauss-Bonnet Lagrangian above can we written in the form,

$$L_2 = \frac{1}{2^2}\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\,R^{\mu\nu}_{\alpha\beta}\,R^{\rho\sigma}_{\alpha\beta} \tag{2.35}$$

The 'determinant tensor' $\delta^{\alpha_1\beta_1\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_n}$ is defined as follows,

$$\delta^{\alpha_1\beta_1\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_n} = det\begin{bmatrix}\delta^{\alpha_1}_{\beta_1} & \delta^{\alpha_1}_{\beta_2} & \delta^{\alpha_1}_{\beta_3} & \ldots \\ \delta^{\alpha_2}_{\beta_1} & \delta^{\alpha_2}_{\beta_2} & \delta^{\alpha_2}_{\beta_3} & \ldots \\ \ldots & \ldots & \ldots & \ldots\end{bmatrix} \tag{2.36}$$

It can be shown that this tensor is completely antisymmetric in the upper or lower row of indices.

The two cases discussed above suggests the following generalization to Lovelock Lagrangians which are higher-order in curvature:

$$L_n = \frac{1}{2^n} \delta^{\alpha_1 \beta_1 \alpha_2 \beta_2 \dots \alpha_n \beta_n}_{\sigma_1 \delta_1 \sigma_2 \delta_2 \dots \sigma_n \delta_n} R^{\sigma_1 \delta_1}{}_{\alpha_1 \beta_1} R^{\sigma_2 \delta_2}{}_{\alpha_2 \beta_2} \dots R^{\sigma_n \delta_n}{}_{\alpha_n \beta_n} \tag{2.37}$$

such that a general Lovelock Lagrangian upto nth order in the curvature tensor is given by,

$$L = \sum_{i=1}^{n} c_i \, L_i \tag{2.38}$$

To verify that this is a correct generalization, we must verify that any Lagrangian given by 2.37 satisfies the Lovelock condition 2.21. We have:

$$P^{\alpha\beta}{}_{\sigma\delta} = n \, \delta^{\alpha\beta\alpha_2\beta_2 \dots \alpha_n \beta_n}_{\sigma\delta\sigma_2\delta_2 \dots \sigma_n \delta_n} R^{\sigma_2 \delta_2}{}_{\alpha_2 \beta_2} R^{\sigma_3 \delta_3}{}_{\alpha_3 \beta_3} \dots R^{\sigma_n \delta_n}{}_{\alpha_n \beta_n} \tag{2.39}$$

where we have defined, The Lovelock condition 2.21 implies that $\nabla_a P^{\alpha\beta}{}_{\sigma\delta} = 0$. To verify this statement, proceed as:

$$\begin{aligned} \nabla_\alpha P^{\alpha\beta}{}_{\sigma\delta} &= \nabla_\alpha \delta^{\alpha\beta \dots \alpha_i \beta_i}_{\sigma\delta \dots \sigma_i \delta_i} \, R^{\sigma_2 \delta_2}{}_{\sigma_2 \delta_2} R^{\sigma_3 \delta_3}{}_{\alpha_3 \beta_3} \dots R^{\sigma_n \delta_n}{}_{\alpha_n \beta_n} \\ &= n \, \delta^{\alpha\beta \dots \alpha_i \beta_i}_{\sigma\delta \dots \sigma_i \delta_i} \, \nabla_{[\alpha} R^{\sigma_2 \delta_2}{}_{\alpha_2 \beta_2]} \dots R^{\sigma_i \delta_i}{}_{\alpha_i \beta_i} \\ &= 0 \end{aligned} \tag{2.40}$$

In the second line, we used the fact that the determinant tensor is completely antisymmetric in both rows of indices. In the third line, we used the Bianchi identity $\nabla_{[\mu} R^{\alpha\beta}{}_{\gamma\delta]} = 0$. This completes the above result.

At a given order in curvature, the Lanczos-Lovelock Lagrangian given by 2.37 is trivial below a certain dimension. If the dimension D is lower than the curvature order n, a certain index value (out of $2n$ in either of the upper or lower row of indices) has to be repeated on the determinant tensor, which gives zero.

Also, the nth curvature order Lovelock theory is topological (related to the Euler characteristic of the manifold) for $D = 2n$ – we will soon demonstrate this fact by analyzing the equation of motion of the Lovelock theories.

Hence, in terms of affecting the local dynamics of a given theory, the Lovelock theories are only active for $D > 2n$. For example, the Gauss Bonnet Lagrangian is zero for $D < 4$. In $4D$, it is a total-derivative, hence it does not contribute to the equation of motion. However, it can still evaluate to non-zero values and affect the global dynamics in non-trivial manifolds.

2.3.1 Equation of motion of Lovelock theories

The equation of motion of the Lovelock theories as given by 2.37 can be easily found from the generic equation of motion 2.20 since the last term becomes zero. First, we calculate $P^{\alpha\beta}{}_{\gamma\delta}$ from 2.20:

$$P^{\alpha\beta}{}_{\sigma\delta} = \frac{n}{2^n}\delta^{\alpha\beta\alpha_2\beta_2\,...\,\alpha_n\beta_n}_{\sigma\delta\sigma_2\delta_2\,...\,\sigma_n\delta_n} R^{\sigma_2\delta_2}{}_{\alpha_2\beta_2} \,...\, R^{\sigma_n\delta_n}{}_{\alpha_n\beta_n} \tag{2.41}$$

Using this in the equation of motion 2.37:

$$\frac{n}{2^n}\delta^{\alpha\beta\alpha_2\beta_2\,...\,\alpha_n\beta_n}_{\sigma\delta\sigma_2\delta_2\,...\,\sigma_n\delta_n} R^{\sigma\delta}{}_{\gamma\beta} R^{\sigma_2\delta_2}{}_{\alpha_2\beta_2} \,...\, R^{\sigma_n\delta_n}{}_{\alpha_n\beta_n} - \frac{1}{2}\delta^{\alpha}_{\gamma} L_n = 0 \tag{2.42}$$

On expanding the determinant tensor in the first term, it can be shown after a tedious calculation that the above expression is equivalent to:

$$-\frac{1}{2^{n+1}}\delta^{\alpha\alpha_1\beta_1\,...\,\alpha_n\beta_n}_{\gamma\sigma_1\delta_1\,...\,\sigma_n\delta_n} R^{\sigma_1\delta_1}{}_{\alpha_2\beta_1} \,...\, R^{\sigma_n\delta_n}{}_{\alpha_n\beta_n} = 0 \tag{2.43}$$

Notice that now each row of the determinant tensor has $2n+1$ indices. Hence, the expression vanishes for $D \leq 2n$. Since, the Lovelock Lagrangian given by 2.20 are zero for $D < 2n$, we conclude that L_n is a total derivative at $D = 2n$.

2.4 Uniqueness of Lanczos-Lovelock Lagrangians

In the previous sections, we have shown that the Lovelock theories indeed have a second-order equation of motion and satisfy the Lovelock condition. But we had raised two question during our discussion of the Lovelock condition 2.21: Are the Lovelock theories unique in having a second-order equation of motion? This question was first tackled by Lovelock [7] in his seminal paper. In this section, we propose an alternative argument for the uniqueness of Lovelock theories, which is more transparent and intuitive.

Consider a Lagrangian L composed of polynomial contractions of the Riemann tensor and the metric, i.e, $L \equiv L(g^{\alpha\beta}, R^{\alpha\beta}{}_{\gamma\delta})$,

$$L = A^{\gamma_1\delta_1\gamma_2\delta_2\cdots\gamma_n\delta_n}_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n} R^{\alpha_1\beta_1}{}_{\gamma_1\delta_1} R^{\alpha_2\beta_2}{}_{\gamma_2\delta_2} \ldots R^{\alpha_{n-1}\beta_{n-1}}{}_{\gamma_{n-1}\delta_{n-1}} R^{\alpha_n\beta_n}{}_{\gamma_n\delta_n} \tag{2.44}$$

where the indices of A inherits all the symmetries of the curvature tensors. In addition, it is symmetric under the collective exchange of indices $(\alpha_i, \beta_i, \gamma_i, \delta_i) \leftrightarrow (\alpha_j, \beta_j, \gamma_j, \delta_j)$ for any i, j. As we derived earlier, the equation of motion of such a theory is given by,

$$P_{\alpha\mu\nu\rho} R_{\beta}{}^{\mu\nu\rho} - \frac{1}{2} g_{\alpha\beta} L - 2\nabla_\gamma \nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} = 0 \tag{2.45}$$

The condition that the equation of motion should be second-order is equivalent to requiring that the last term is either second-order or zero. This leads to two important questions which we raised:

1. Can $\nabla_\gamma \nabla^\beta P_{\alpha\beta}{}^{\gamma\delta}$ ever be non-zero and second-order?
2. What is the complete set of solutions of $\nabla_\gamma \nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} = 0$? Are they all exhausted by the stronger condition $\nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} = 0$?

On computing $P_{\alpha\beta}{}^{\gamma\delta}$ for the Lagrangian 2.44, we obtain

$$P_{\alpha\beta}{}^{\gamma\delta} = n\, A^{\gamma\delta\gamma_2\delta_2\cdots\gamma_n\delta_n}_{\alpha\beta\alpha_2\beta_2\cdots\alpha_n\beta_n} R^{\alpha_2\beta_2}{}_{\gamma_2\delta_2} \ldots R^{\alpha_{n-1}\beta_{n-1}}{}_{\gamma_{n-1}\delta_{n-1}} R^{\alpha_n\beta_n}{}_{\gamma_n\delta_n} \tag{2.46}$$

For the Lovelock condition, taking the covariant divergence of the above expression, we arrive at,

$$\nabla^{\beta} P_{\alpha\beta}{}^{\gamma\delta} = n(n-1)\, A^{\gamma\delta\gamma_2\delta_2\cdots\gamma_n\delta_n}_{\alpha\beta\alpha_2\beta_2\cdots\alpha_n\beta_n} \nabla^{\beta} R^{\alpha_2\beta_2}{}_{\gamma_2\delta_2} \dots R^{\alpha_{n-1}\beta_{n-1}}{}_{\gamma_{n-1}\delta_{n-1}} R^{\alpha_n\beta_n}{}_{\gamma_n\delta_n} \tag{2.47}$$

Consider the leading order non-trivial term in the flat-space expansion ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) of the above expression, which occurs at order h^{n-1}:

$$\partial^{\beta} P^{(n-1)}{}_{\alpha\beta}{}^{\gamma\delta} = n(n-1)\, A^{\gamma\delta\gamma_2\delta_2\cdots\gamma_n\delta_n}_{\alpha\beta\alpha_2\beta_2\cdots\alpha_n\beta_n} \partial^{\beta} \hat{R}^{\alpha_2\beta_2}{}_{\gamma_2\delta_2} \dots \hat{R}^{\alpha_{n-1}\beta_{n-1}}{}_{\gamma_{n-1}\delta_{n-1}} \hat{R}^{\alpha_n\beta_n}{}_{\gamma_n\delta_n} \tag{2.48}$$

where ˆ(hat) refers to the corresponding linearized quantity. Notice that $\partial^{\beta} P^{(n-1)}{}_{\alpha\beta}{}^{\gamma\delta}$ is either third order in derivatives or zero. If 2.46 is non-trivial or not a total derivative, it implies that 2.48 does not vanish unless some identity satisfied by the Riemann tensor is utilized. Let us clarify this statement. Any choice of the tensor A possesses symmetries, beyond the symmetries imposed by contracting with the curvature tensors (it must have the additional symmetries since such a choice picks out one of out of multiple distinct way to contract n Riemann tensors to form a scalar). These additional symmetries must also be obeyed by the part consisting of the curvature tensors in 2.48 as all of the indices of the curvature tensors are contracted with A. We want to choose the additional symmetries of A such that they lead to a vanishing identity on the curvature tensors in 2.48, without making the Lagrangian 2.44 trivial.

Since 2.48 has a derivative of a curvature tensor, we must utilize it in the identity we seek to use, as otherwise any identities involving only indices on the curvature tensor will make even 2.44 trivial. Immediately, one can see that by making β antisymmetric with the indices α_2 and β_2, the Bianchi identity can be utilized, and the expression vanishes. However, since β could be any one of the β_i positions of the A tensor in 2.44 and the choice of having ∇_{β} on $R_{\alpha_2\beta_2\gamma_2\delta_2}$ was also arbitrary, antisymmetrizing β with α_2 and β_2 (and that α_i, β_i are antisymmetric for any i) implies that $\beta_i, \alpha_k, \beta_k$ is completely antisymmetrized for any i, k. Hence, the entire lower row of indices of A is antisymmetrised by imposing β, α_2 and β_2 to be antisymmetrized. Since the condition $\partial^{\beta} P^{(n-1)}{}_{\alpha\beta}{}^{\gamma\delta} = 0$ is equivalent to $\partial_{\gamma} P^{(n-1)}{}_{\alpha\beta}{}^{\gamma\delta} = 0$ due to the symmetries of P, the above arguments similarly requires the entire upper row of indices being antisymmetrised. In any dimension, the unique tensor which has the above properties is the determinant tensor, which leads us to the Lanczos-Lovelock Lagrangians given as,

$$L = \delta^{\gamma_1\delta_1\gamma_2\delta_2\cdots\gamma_n\delta_n}_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\, R^{\alpha_1\beta_1}{}_{\gamma_1\delta_1}\, R^{\alpha_2\beta_2}{}_{\gamma_2\delta_2}\ldots R^{\alpha_n\beta_n}{}_{\gamma_n\delta_n} \tag{2.49}$$

Whenever the cyclic identity is utilized on a curvature tensor to make 2.48 vanish, the Lagrangian 2.44 also becomes trivial. This is expected as the cyclic identity does not involve any derivatives of the curvature tensors. Hence the symmetries which act on the curvature tensors in 2.48 to make the expression vanish using the cyclic identity on one or multiple curvature tensors, will similarly act in 2.44, making it vanish too.

Since the cyclic and the Bianchi identity are the only two properties of the Riemann tensor which were left to be considered (remember that we had already considered the symmetry/antisymmetries of the indices of the curvature tensor while defining the A tensor), from the above arguments, we conclude that 2.49 is the unique solution to $\nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} = 0$.

Now, we are interested in whether the more general condition $\nabla^\gamma\nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} = 0$ leads to solutions distinct from the above class. We claim that the answer is no, and demonstrate it by showing that $\nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} \neq 0$ implies that $\nabla_\gamma\nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} \neq 0$. To see this, let us explicitly compute the latter quantity,

$$\begin{aligned}\nabla_\gamma\nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} = n(n-1)A^{\gamma\delta\gamma_2\delta_2\cdots\gamma_n\delta_n}_{\alpha\beta\alpha_2\beta_2\cdots\alpha_n\beta_n}\Big(&(\nabla_\gamma\nabla^\beta R^{\alpha_2\beta_2}{}_{\gamma_2\delta_2})\ldots R^{\alpha_{n-1}\beta_{n-1}}{}_{\gamma_{n-1}\delta_{n-1}} R^{\alpha_n\beta_n}{}_{\gamma_n\delta_n}\\ &+(n-2)(\nabla^\beta \hat{R}^{\alpha_2\beta_2}{}_{\gamma_2\delta_2})(\nabla_\gamma R^{\alpha_3\beta_3}{}_{\gamma_3\delta_3})\ldots R^{\alpha_{n-1}\beta_{n-1}}{}_{\gamma_{n-1}\delta_{n-1}} R^{\alpha_n\beta_n}{}_{\gamma_n\delta_n}\Big)\end{aligned} \tag{2.50}$$

Again, consider the leading order non-trivial term in the flat-space expansion of the above expression, which occurs at order h^{n-1}

$$\begin{aligned}\partial_\gamma\partial^\beta P^{(n-1)}{}_{\alpha\beta}{}^{\gamma\delta} = n(n-1)A^{\gamma\delta\gamma_2\delta_2\cdots\gamma_n\delta_n}_{\alpha\beta\alpha_2\beta_2\cdots\alpha_n\beta_n}\Big(&(\partial_\gamma\partial^\beta \hat{R}^{\alpha_2\beta_2}{}_{\gamma_2\delta_2})\ldots \hat{R}^{\alpha_{n-1}\beta_{n-1}}{}_{\gamma_{n-1}\delta_{n-1}} \hat{R}^{\alpha_n\beta_n}{}_{\gamma_n\delta_n}\\ &+(n-2)\,(\partial^\beta \hat{R}^{\alpha_2\beta_2}{}_{\gamma_2\delta_2})(\partial_\gamma \hat{R}^{\alpha_3\beta_3}{}_{\gamma_3\delta_3})\ldots \hat{R}^{\alpha_{n-1}\beta_{n-1}}{}_{\gamma_{n-1}\delta_{n-1}} \hat{R}^{\alpha_n\beta_n}{}_{\gamma_n\delta_n}\Big)\end{aligned} \tag{2.51}$$

The first line contains four-derivative terms in the metric while the second line has three. The assumption $\nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} \neq 0$ (which leads to $\partial^\beta P^{(n-1)}{}_{\alpha\beta}{}^{\gamma\delta} \neq 0$ in flat space) implies that the above expression cannot vanish. This is because, as before, the only way both the lines above can vanish is through the Bianchi identity. Based closely on our previous arguments, this leads to the Lovelock solution 2.49, which satisfies $\nabla^\beta P_{\alpha\beta}{}^{\gamma\delta} = 0$, hence violating the assumption.

From the above analysis and 2.51, we readily see that $\partial_\gamma \partial^\beta P^{(n-1)}{}_{\alpha\beta}{}^{\gamma\delta}$ can never be second-order, as unless its trivial, it necessarily contains third and fourth derivative terms. In general spacetimes, this translates to the fact that $\nabla_\gamma \nabla^\beta P_{\alpha\beta}{}^{\gamma\delta}$ is always either a higher-derivative term or zero, which answers the first question we raised.

Chapter 3

Beyond Lovelock theories

Till now, we have studied theories which are purely composed of the metric and its derivatives – or equivalently those formed by contractions of the curvature tensor and the metric. In nature, gravity is coupled to various matter fields, including Abelian and non-Abelian gauge fields which may have arbitrary form degree and spins. Such couplings are not just theoretical possibilities but are known to arise in complete frameworks like string theory. These theories generically contain higher-derivative corrections to gravity coupled to gauge fields, often originating from integrating out additional modes from the underlying quantum theory.

However, the questions remains, analogous to the pure gravity case, as to which of these theories are natural – one of the criterion being the absence of ghost modes. As we saw in the previous chapters, this is ensured if the theory has second-order equation of motions. In this chapter, we will explore how Abelian gauge fields of arbitrary form degree can non-minimally couple to gravity to have second-order equations of motion in arbitrary backgrounds. Further, we will investigate if there exists Lovelockian theories which involve covariant derivatives of the curvature tensor or Abelian gauge fields.

We will start by investigating the case with an Abelian gauge field strength of form degree 2 – which we shall take as the electromagnetic (EM) field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (with A_μ as the vector potential) – at fourth order in derivatives.

3.1 1-form gauge field

3.1.1 At fourth order in derivatives

At fourth order in derivatives, the most general Lagrangian (barring terms with covariant derivatives of the field strengths, which we shall study in a later section) with the EM field strength non-minimally coupled to gravity can be written as,

$$S = \int d^D x \sqrt{-g} \Big(a_1 \, R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + a_2 \, R_{\alpha\beta} R^{\alpha\beta} + a_3 \, R^2 \\ + a_4 \, F^{\alpha\beta} F^{\gamma\delta} R_{\alpha\beta\gamma\delta} + a_5 \, F_\alpha{}^\mu F^{\alpha\nu} R_{\mu\nu} + a_6 \, F^{\alpha\beta} F_{\alpha\beta} R + L(F, \nabla F) \Big) \tag{3.1}$$

where $L(F, \nabla F)$ represents terms purely composed of contractions of the field strength and its covariant derivatives with the metric. We will set aside such terms for now and analyze their contribution of such terms in a later section.

The electromagnetic field tensor $F_{\mu\nu}$ satisfies a Bianchi identity,

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0 \tag{3.2}$$

The equation of motion with respect to the vector potential A_μ can be written as,

$$\nabla^\mu K_{\mu\nu} = 0 \tag{3.3}$$

where

$$K^{\alpha\mu} = \left(\frac{\partial L}{\partial F_{\alpha\mu}} \right) \tag{3.4}$$

Let us calculate this quantity for each of the three coupling terms in the above Lagrangian and look specifically at the higher derivative contributions.

$$K_{\mu\sigma} = 2 \, \nabla^\mu \left(a_1 \, \delta^{\alpha\beta}_{\mu\sigma} \, F^{\gamma\delta} \, R_{\alpha\beta\gamma\delta} + a_3 \, \delta^{\alpha\beta}_{\mu\sigma} F_{\alpha\beta} R + a_2 \, \delta^{\alpha\nu}_{\mu\sigma} F^\rho{}_\alpha \, R_{\nu\rho} \right) \tag{3.5}$$

Therefore, the higher-derivative terms in the above occurs when ∇ acts on the curvature tensors. We procced by evaluating just the higher-derivative contributions:

$$\begin{aligned}
\nabla^\mu K_{\mu\sigma} &\to 4a_4\, F^{\gamma\delta}\nabla^\mu R_{\mu\sigma\gamma\delta} + 4a_6\, F_{\mu\sigma}\nabla^\mu R + 2a_5\, \delta^{\alpha\nu}_{\mu\sigma} F^\rho{}_\alpha\, \nabla^\mu R_{\nu\rho} \\
&\to 4a_4\, F^{\gamma\delta}(\nabla_\gamma R_{\delta\sigma} - \nabla_\delta R_{\gamma\delta}) + 4a_6\, F_{\mu\sigma}\nabla^\mu R + 2a_5 \Big(F_\mu{}^\rho (\nabla^\mu R_{\sigma\rho}) - F_\sigma{}^\rho (\nabla^\mu R_{\mu\rho}) \Big) \\
&\to 4a_4\, F^{\gamma\delta}(\nabla_\gamma R_{\delta\sigma} - \nabla_\delta R_{\gamma\delta}) + 4a_6\, F_{\mu\sigma}\nabla^\mu R + 2a_5\, F_{[\mu}{}^{\rho]} (\nabla^{[\mu|} R_{\sigma|\rho]}) - a_5\, F_\sigma{}^\rho (\nabla_\rho R) \\
&\to 4a_4\, F^{\gamma\delta}(\nabla_\gamma R_{\delta\sigma} - \nabla_\delta R_{\gamma\delta}) + 4a_6\, F_{\mu\sigma}\nabla^\mu R + a_5 \Big\{ F_{\mu\rho}\, (\nabla^\mu R_{\rho\sigma} - \nabla_\rho R^\mu{}_\sigma) \Big\} + F^\rho{}_\sigma\, (\nabla_\rho R)
\end{aligned} \tag{3.6}$$

In the second line and third lines, we have used the Bianchi identity and the contracted Bianchi identity. Comparing the first and third term, and the second and fourth term, we get the relations

$$\begin{aligned} 4a_4 + a_5 &= 0 \\ 4a_6 + a_5 &= 0 \end{aligned} \tag{3.7}$$

or equivalently.

$$a_4 = a_6 = -\frac{1}{4} a_5 \tag{3.8}$$

Finally, with the constraint 3.8, we conclude that the following Lagrangian has a second-order equation of motion with respect to the EM vector potential,

$$L = F^{\alpha\beta} F^{\gamma\delta} R_{\alpha\beta\gamma\delta} - 4\, F_\alpha{}^\mu F^{\alpha\nu} R_{\mu\nu} + \; F^{\alpha\beta} F_{\alpha\beta} R \tag{3.9}$$

Now, let us look into the equation of motion with respect to the metric and analyze the higher-derivative contribution, so that we can verify that 3.8 is indeed the correct constraint. To begin with, remember the identity from the previous section,

$$\delta R_{\alpha\beta\gamma\delta} = R^\mu{}_{\beta\gamma\delta}\, \delta g_{\mu\alpha} + \frac{1}{2}(\delta g_{\alpha\delta;\beta\gamma} - \delta g_{\beta\delta;\alpha\gamma} - \delta g_{\alpha\gamma;\beta\delta} + \delta g_{\beta\gamma;\alpha\delta} - R^\eta{}_{\alpha\gamma\delta} \delta g_{\eta\beta} - R^\eta{}_{\beta\gamma\delta} \delta g_{\sigma\eta}) \tag{3.10}$$

The first four terms in brackets with two derivatives of $\delta g_{\mu\nu}$ are the only ones which may lead to the higher derivative terms in the equation of motion (from here onwards we shall use $\rightarrow$ to denote exclusively the higher derivative contribution):

$$F^{\beta\delta}F^{\alpha\gamma}\delta R_{\alpha\beta\gamma\delta} \rightarrow \frac{1}{2}F^{\beta\delta}F^{\alpha\gamma}(\delta g_{\alpha\delta;\beta\gamma} - \delta g_{\beta\delta;\alpha\gamma} - \delta g_{\alpha\gamma;\beta\delta} + \delta g_{\beta\gamma;\alpha\delta}) \tag{3.11}$$

The covariant derivatives in the second and third terms are antisymmetrised by contraction with the F indices, and hence do not lead to any higher derivative term. Using integration by parts on the first and fourth terms, we have,

$$F^{\beta\delta}F^{\alpha\gamma}\delta R_{\alpha\beta\gamma\delta} \rightarrow 2(-F^{\gamma\mu}\nabla_\beta\nabla_\gamma F^{\beta\sigma} - F^{\beta\sigma}\nabla_\beta\nabla_\gamma F^{\gamma\mu})\,\delta g_{\mu\sigma} \tag{3.12}$$

Notice that the term in the brackets is symmetric in μ and σ as expected.

Now let us look at the $F^{\alpha\gamma}F^{\beta}{}_{\gamma}R_{\alpha\beta}$ term. We will use the following identity which we derived in the previous section,

$$\delta R_{\alpha\gamma} = \delta g^{\beta\delta}\,R_{\alpha\beta\gamma\delta} + \frac{1}{2}(\delta g_{\alpha\delta;}{}^{\delta}{}_{\gamma} - g^{\beta\delta}\delta g_{\beta\delta;\alpha\gamma} - \Box(\delta g_{\alpha\gamma}) + \delta g_{\beta\gamma;\alpha}{}^{\beta} - R^{\eta}{}_{\alpha\gamma}{}^{\beta}\,\delta g_{\eta\beta} - R^{\eta}{}_{\gamma}\,\delta g_{\alpha\eta}) \tag{3.13}$$

In the above expression, only the first four terms, after integration by parts, lead to higher derivative contribution. On grouping them based on similar contractions, we have,

$$\begin{aligned} &F^{\alpha\theta}F^{\gamma}{}_{\theta}\,\delta R_{\alpha\gamma} \\ \rightarrow&\frac{1}{2}\Big[\Big\{(\nabla^\sigma\nabla_\gamma F^{\mu\theta})F^{\gamma}{}_{\theta} + F^{\alpha\theta}(\nabla_\alpha\nabla^\mu F^{\sigma}{}_{\theta})\Big\} + \Big\{(\nabla^\sigma\nabla_\gamma F^{\gamma}{}_{\theta})F^{\mu\theta} + (\nabla_\alpha\nabla^\mu F^{\alpha\theta})F^{\sigma}{}_{\theta}\Big\} \\ &- 2g^{\mu\sigma}(\nabla_\alpha\nabla_\gamma F^{\alpha\theta})F^{\gamma}{}_{\theta} + \Big\{-(\Box F^{\mu\theta})F^{\sigma}{}_{\theta} - F^{\sigma\theta}\Box F^{\mu}{}_{\theta}\Big\}\Big]\delta g_{\mu\sigma} \end{aligned} \tag{3.14}$$

Note that in the above expression, one is free to exchange the order of covariant derivatives in any of the terms, as such a change only requires an extra term with a Riemann tensor and no covariant derivatives. The grouped terms in parenthesis are similar upto such an exchange. Also, each group of terms is symmetric in μ and σ as expected.

Let us use the Bianchi identity in the first term in the second curly brackets,

$$
\begin{aligned}
&(\nabla_\gamma \nabla^\sigma F^\gamma{}_\theta) F^{\mu\theta} \\
=&\Big(-\Box F_\theta{}^\sigma - \nabla_\gamma \nabla_\theta F^{\sigma\gamma}\Big) F^{\mu\theta} \\
=&F^{\mu\theta} \Box F^{\sigma\theta} - (\nabla_\gamma \nabla_\theta F^{\sigma\gamma}) F^{\mu\theta}
\end{aligned}
\tag{3.15}
$$

Similarly, by exchanging indices μ and σ, we can write the other term in the second curly brackets as,

$$
(\nabla_\gamma \nabla^\mu F^\gamma{}_\theta) F^{\sigma\theta} = F^{\sigma\theta} \Box F^\mu{}_\theta - (\nabla_\gamma \nabla_\theta F^{\mu\gamma}) F^{\sigma\theta} \tag{3.16}
$$

The $\Box$ terms in 3.15 and 3.16 are the same as those in the third curly brackets in 3.14 but with opposite sign, hence they cancel out. Hence we arrive at,

$$
\begin{aligned}
&F^{\alpha\theta} F^\gamma{}_\theta \delta R_{\alpha\gamma} \\
\rightarrow \frac{1}{2} \Big[&\Big\{ (\nabla^\sigma \nabla_\gamma F^{\mu\theta}) F^\gamma{}_\theta + F^{\alpha\theta} (\nabla_\alpha \nabla^\mu F^\sigma{}_\theta) \Big\} - \Big\{ (\nabla_\gamma \nabla_\theta F^{\sigma\gamma}) F^{\mu\theta} + (\nabla_\gamma \nabla_\theta F^{\mu\gamma}) F^{\sigma\theta} \Big\} \\
&- 2 g^{\mu\sigma} (\nabla_\alpha \nabla_\gamma F^{\alpha\theta}) F^\gamma{}_\theta \Big] \delta g_{\mu\sigma}
\end{aligned}
\tag{3.17}
$$

Finally, we see that the second curly brackets in 3.17 is the same as 3.12 upto some overall factor. This gives us our first constraint,

$$
4a_4 + a_5 = 0 \tag{3.18}
$$

To eliminate the rest of the terms in 3.17, let us turn our attention to $F^{\alpha\beta} F_{\alpha\beta} R$. Repeating the same steps as we have done earlier, the terms from δR which may lead to higher-derivative contributions are,

$$
\delta R \rightarrow \nabla^\delta \nabla^\alpha \delta g_{\alpha\delta} - g^{\alpha\beta} \Box (\delta g_{\alpha\beta}) \tag{3.19}
$$

So,

$$
F^{\alpha\beta} F_{\alpha\beta} \delta R \tag{3.20}
$$

$$
\rightarrow \Big(\nabla^{(\mu} \nabla^{\sigma)} (F^{\alpha\beta} F_{\alpha\beta}) - g^{\mu\sigma} \Box (F^{\alpha\beta} F_{\alpha\beta}) \Big) \delta g_{\mu\sigma} \tag{3.21}
$$

$$
\rightarrow 2 \Big(\nabla^{(\mu} \nabla^{\sigma)} (F^{\alpha\beta}) F_{\alpha\beta} - g^{\mu\sigma} \Box (F^{\alpha\beta}) F_{\alpha\beta} \Big) \delta g_{\mu\sigma} \tag{3.22}
$$

Now, using the Bianchi identity, we try to simplify both the terms to express them similar to the remaining terms from 3.17. We see that,

$$F^{\alpha\beta}(\nabla^{(\mu}\nabla^{\sigma)}F_{\alpha\beta}) = 2(\nabla^{\alpha}\nabla^{(\mu}F^{\sigma)\beta})F_{\alpha\beta} \tag{3.23}$$

and,

$$F^{\alpha\eta}\Box F_{\alpha\eta} = 2\,F^{\alpha\eta}(\nabla^{\theta}\nabla_{\alpha}F_{\eta\theta}) \tag{3.24}$$

Substituting the above relations back in 3.22, we have,

$$F^{\alpha\beta}F_{\alpha\beta}(\delta R) \tag{3.25}$$

$$\to \Big[2\Big\{(\nabla^{\alpha}\nabla^{\mu}F^{\sigma\beta})F_{\alpha\beta} + (\nabla^{\alpha}\nabla^{\sigma}F^{\mu\beta})F_{\alpha\beta}\Big\} - 4\,g^{\mu\sigma}\,F^{\alpha\eta}(\nabla^{\theta}\nabla_{\alpha}F_{\eta\theta})\Big]\delta g_{\mu\sigma} \tag{3.26}$$

Remarkably, the curly bracket above is the same as the first curly bracket of 3.17 and the terms having $g^{\mu\sigma}$ are also identical. We can cancel both of them by having the constraint,

$$a_5 + 4a_6 = 0 \tag{3.27}$$

Hence, combining 3.18 and 3.27, we again arrive at the same conclusion as we had from the equation of motion with respect to A_{μ}. Now let us explicitly write this coupling terms (upto an overall constant) as,

$$L_{RFF} = R_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta} - 4F^{\rho}{}_{\alpha}\,F_{\rho\beta}\,R^{\alpha\beta} + F^{\alpha\beta}F_{\alpha\beta}R \tag{3.28}$$

In d dimensions, the above can be equivalently written as,

$$\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}R^{\mu\nu}_{\alpha\beta}\,F^{\rho\sigma}F_{\gamma\delta} = 4(d-4)!(-4\,F^{\alpha}{}_{\mu}\,F_{\alpha\nu}R^{\mu\nu} + F^{\mu\nu}F_{\mu\nu}\,R + F^{\mu\nu}F^{\alpha\beta}R_{\mu\nu\alpha\beta}) \tag{3.29}$$

The Lagrangian above was first presented in by Horndeski [8] in 1976, along with a proof of its uniqueness in four dimnensions. Hence, it is known as the ‘Horndeski theory’ in the literature.

The structure of the Lagrangian 3.29 suggests the following generalization to arbitrary order in curvature and EM field strength,

$$L = \delta^{\mu_1\nu_1\,\dots\,\mu_m\nu_m\,\eta_1\delta_1\,\dots\,\eta_n\delta_n}_{\alpha_1\beta_1\,\dots\,\alpha_m\beta_m\,\theta_1\sigma_1\,\dots\,\theta_n\sigma_n} F_{\mu_1\nu_1} \dots F_{\mu_m\nu_m}\, F^{\alpha_1\beta_1} \dots F^{\alpha_m\beta_m}\, R^{\theta_1\sigma_1}{}_{\eta_1\delta_1} \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n}$$

We will soon show that indeed the above class of theories has a second-order equation of motion, but before that we need to develop some formalism to demonstrate the same.

3.1.2 Equation of motion

In this section, we will derive the equation of motion of a Lagrangian composed of polynomial contractions of the curvature tensor, the field strength tensor and the metric. Such a Lagrangian can be schematically represented as,

$$L \equiv L(R^{\alpha\beta}{}_{\gamma\delta}, g^{\alpha\beta}, F_{\alpha\beta}) \tag{3.30}$$

Taking a variation of the action with respect to the metric, we have,

$$\begin{aligned}
\delta S &= \int d^Dx\; \delta(\sqrt{-g})\, L + \sqrt{-g}\, \delta L \\
&= \int d^Dx\; \sqrt{-g}\, \frac{\partial L}{\partial g^{\alpha\beta}}\, \delta g^{\alpha\beta} + \sqrt{-g}\, \frac{\partial L}{\partial R^{\alpha\beta}{}_{\gamma\delta}} (\delta R^{\alpha\beta}{}_{\gamma\delta}) - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta} \\
&= \int d^Dx\; \sqrt{-g}\, P_{\alpha\beta}\, \delta g^{\alpha\beta} + \sqrt{-g}\, P^{\gamma\delta}{}_{\alpha\beta}\, \delta(g^{\beta\theta} R^{\alpha}{}_{\theta\gamma\delta}) - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta} \\
&= \int d^Dx\; \sqrt{-g}\, P_{\alpha\beta}\, \delta g^{\alpha\beta} + \sqrt{-g}\, P^{\gamma\delta}{}_{\alpha\beta}\, R^{\alpha}{}_{\theta\gamma\delta} (\delta g^{\beta\theta}) + P^{\gamma\delta}{}_{\alpha\beta}\, g^{\beta\theta} (\delta R^{\alpha}{}_{\theta\gamma\delta}) - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}
\end{aligned} \tag{3.31}$$

Let us now evaluate how $P^{\alpha\beta}$ is related to $P^{\alpha\beta\gamma\delta}$ and $K^{\mu\nu}$. Using the method of evaluating Lie derivatives in two ways for an arbitrary infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ (as we have discussed in the second chapter), we have:

$$\mathcal{L}_\xi L = \xi^\mu \partial_\mu L = P_{\alpha\beta}{}^{\gamma\delta} \xi^\mu \nabla_\mu R^{\alpha\beta}{}_{\gamma\delta} + K^{\mu\nu} \xi^\alpha \nabla_\alpha F_{\mu\nu} \tag{3.32}$$

where we have used the chain rule to evaluate the covariant derivative on the Lagrangian.

Next, we evaluate the Lie derivative in the second way:

$$\begin{aligned}\mathcal{L}_\xi L &= P_{\alpha\beta}\mathcal{L}_\xi g^{\alpha\beta} + P_{\alpha\beta}{}^{\gamma\delta}\mathcal{L}_\xi R^{\alpha\beta}{}_{\gamma\delta} + K^{\mu\nu}\mathcal{L}_\xi F_{\mu\nu} \\ &= P_{\alpha\beta}{}^{\gamma\delta}\xi^\mu\nabla_\mu R^{\alpha\beta}{}_{\gamma\delta} + K^{\mu\nu}\xi^\alpha\nabla_\alpha F_{\mu\nu} - 2P_{\alpha\beta}(\nabla^\alpha\xi^\beta) + 2K^{\mu\nu}(\nabla_\mu\xi^\alpha)F_{\alpha\nu}\end{aligned} \tag{3.33}$$

Note that in the last equation, we have used the fact that $P_{\alpha\beta}{}^{\gamma\delta}\mathcal{L}_\xi R^{\alpha\beta}{}_{\gamma\delta} = P_{\alpha\beta}{}^{\gamma\delta}\xi^\mu\nabla_\mu R^{\alpha\beta}{}_{\gamma\delta}$ 2.8. On equating 3.32 and 3.33, since the equality holds for arbitrary diffeomorphisms and points on the manifold, we have:

$$P_{\alpha\beta} = K_\alpha{}^\nu F_{\nu\beta} \tag{3.34}$$

Contrast the above to the case when $(R^\alpha{}_{\beta\gamma\sigma}, g^{\alpha\beta})$ were the independent variables. In that case, we had an extra $P_\alpha{}^{\nu\rho\sigma}R_{\beta\nu\rho\sigma}$ factor. However, in our case, this factor enters the equation of motion as the second term in 3.31. Hence, assembling all the pieces together in the same way as before, we finally arrive at the equation of motion:

$$P_{\alpha\mu\nu\rho}R_\beta{}^{\mu\nu\rho} + K_{\alpha\mu}F_\beta{}^\mu - \frac{1}{2}g_{\alpha\beta}L - 2\nabla^{(\mu}\nabla^{\nu)}P_{\alpha\mu\nu\beta} = 0 \tag{3.35}$$

where,

$$P_{\gamma\delta}{}^{\alpha\beta} = \left(\frac{\partial L}{\partial R^{\gamma\delta}{}_{\alpha\beta}}\right)_{(g^{\alpha\beta}, F_{\mu\nu})} \tag{3.36}$$

For the equation of motion with respect to A_μ,

$$\begin{aligned}\delta_A S &= \int d^D x \left(\frac{\partial L}{\partial F_{\mu\nu}}\right)\left(\frac{\partial F_{\mu\nu}}{\partial(\nabla_{[\mu}A_{\nu]})}\right)\delta(\nabla_{[\mu}A_{\nu]}) \\ &= \int d^D x\, 2\,(K^{\mu\nu})\,\delta(\nabla_{[\mu}A_{\nu]})\end{aligned} \tag{3.37}$$

On integration by parts, this leads to the equation of motion $\nabla_\mu K^{\mu\nu} = 0$.

3.1.3 At arbitrary order in derivatives

Having realized that it is indeed possible to construct a Lagrangian at fourth order in derivatives, in which the curvature tensor is contracted with the EM field strength, we proceed to generalize this result at arbitrary order in derivatives. It will be soon apparent that the determinant tensor 2.36 continues to play a key role in constructing Lovelockian theories.

Claim: A scalar L composed of $2m$ antisymmetric EM field strength tensors $F_{\mu\nu}$, the metric and n curvature tensor $R_{\alpha\beta\gamma\delta}$, which can be written in the following way,

$$L = \delta^{\mu_1\nu_1\,...\,\mu_m\nu_m\,\eta_1\delta_1\,...\,\eta_n\delta_n}_{\alpha_1\beta_1\,...\,\alpha_m\beta_m\,\theta_1\sigma_1\,...\,\theta_n\sigma_n} F_{\mu_1\nu_1}\,...\,F_{\mu_m\nu_m}\,F^{\alpha_1\beta_1}\,...\,F^{\alpha_m\beta_m}\,R^{\theta_1\sigma_1}{}_{\eta_1\delta_1}\,...\,R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \tag{3.38}$$

is such that the corresponding equation of motions with respect to $g^{\mu\nu}$ and A_μ are second-order in derivatives. In the above, $\delta^{\alpha\beta\gamma\,...}_{\mu\nu\rho\,...}$ is the determinant tensor given as,

$$\delta^{\alpha_1\alpha_2...\alpha_n}_{\beta_1\beta_2...\beta_n} = det \begin{bmatrix} \delta^{\alpha_1}_{\beta_1} & \delta^{\alpha_1}_{\beta_2} & \delta^{\alpha_1}_{\beta_3} & ... \\ \delta^{\alpha_2}_{\beta_1} & \delta^{\alpha_2}_{\beta_2} & \delta^{\alpha_2}_{\beta_3} & ... \\ ... & ... & ... & ... \end{bmatrix} \tag{3.39}$$

We will refer to the above class as 'LA(1)' – (L)ovelockian theories with (A)belian gauge field of form degree (1) non-minimally coupled to gravity.

Proof: First, consider the equation of motion with respect to A_μ which is given by $\nabla_\alpha K^{\alpha\mu} = 0$ 3.4. For the given Lagrangian, this quantity can be written in two parts as,

$$\begin{aligned} K^{\alpha\beta} = \quad & m\,\delta^{\mu_1\nu_1\,...\,\mu_m\nu_m\,\eta_1\delta_1\,...\,\eta_n\delta_n}_{\alpha_1\beta_1\,...\,\alpha_m\beta_m\,\theta_1\sigma_1\,...\,\theta_n\sigma_n}\Big[\\ & (\delta^{\alpha}_{\mu_1}\delta^{\beta}_{\nu_1} - \delta^{\alpha}_{\nu_1}\delta^{\beta}_{\mu_1})F_{\mu_2\nu_2}\,...\,F_{\mu_m\nu_m}\,F^{\alpha_1\beta_1}\,...\,F^{\alpha_m\beta_m}\,R^{\theta_1\sigma_1}{}_{\eta_1\delta_1}\,...\,R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\ & + (g^{\alpha_1\alpha}g^{\beta_1\beta} - g^{\alpha_1\beta}g^{\beta_1\alpha})F_{\mu_1\nu_1}F_{\mu_2\nu_2}\,...\,F_{\mu_m\nu_m}\,F^{\alpha_2\beta_2}\,...\,F^{\alpha_m\beta_m}\,R^{\theta_1\sigma_1}{}_{\eta_1\delta_1}\,...\,R^{\theta_n\sigma_n}{}_{\eta_n\delta_n}\Big] \end{aligned} \tag{3.40}$$

The first line comes from taking out an $F_{\mu\nu}$ for the partial derivative while the second line comes from taking out an $F^{\mu\nu}$. Lets call the first line as $K_1^{\alpha\beta}$ and the second as $K_2^{\alpha\beta}$ so that $K^{\alpha\beta} = K_1^{\alpha\beta} + K_2^{\alpha\beta}$. Now let us focus on the contribution to the equation of motion from $K_1^{\alpha\beta}$.

Schematically, we can write it as,

$$\begin{aligned}\nabla_\alpha K_1^{\alpha\beta} =& m\, \delta^{\mu_1\nu_1\,...\,\mu_m\nu_m\,\eta_1\delta_1\,...\,\eta_n\delta_n}_{\alpha_1\beta_1\,...\,\alpha_m\beta_m\,\theta_1\sigma_1\,...\,\theta_n\sigma_n} \Big[\\ & (m-1)(\delta^\alpha_{\mu_1}\delta^\beta_{\nu_1} - \delta^\alpha_{\nu_1}\delta^\beta_{\mu_1})\nabla_\alpha(F_{\mu_2\nu_2})\, F_{\mu_3\nu_3}\,..\, F_{\mu_n\nu_n}\, F^{\alpha_1\beta_1}\,...\, F^{\alpha_m\beta_m}\, R^{\theta_1\sigma_1}{}_{\eta_1\delta_1}\,...\, R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\ & + m\, F_{\mu_2\nu_2}\, F_{\mu_3\nu_3}\,..\, F_{\mu_n\nu_n}\, (\delta^\alpha_{\mu_1}\delta^\beta_{\nu_1} - \delta^\alpha_{\nu_1}\delta^\beta_{\mu_1})\nabla_\alpha(F^{\alpha_1\beta_1})\,...\, F^{\alpha_m\beta_m}\, R^{\theta_1\sigma_1}{}_{\eta_1\delta_1}\,...\, R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\ & + n\, F_{\mu_2\nu_2}\, F_{\mu_3\nu_3}\,..\, F_{\mu_n\nu_n}\, F^{\alpha_1\beta_1}\,...\, F^{\alpha_m\beta_m}(\delta^\alpha_{\mu_1}\delta^\beta_{\nu_1} - \delta^\alpha_{\nu_1}\delta^\beta_{\mu_1})\nabla_\alpha(R^{\theta_1\sigma_1}{}_{\eta_1\delta_1})\,...\, R^{\theta_n\sigma_n}{}_{\eta_n\delta_n}\Big]\end{aligned} \tag{3.41}$$

Notice that in each line, the covariant derivative is contracted with one of the Kronecker delta functions to become either ∇_{α_1} or ∇_{β_1}. In the first two lines, this acts on F and hence leads to second order terms. In the third line, when it acts on the curvature tensor, the term vanishes due to the Bianchi identity, enabled by the antisymmetrization of the indices $\alpha_1, \eta_1, \delta_1$ or $\beta_1, \eta_1, \delta_1$ by the determinant tensor.

It is clear that the above argument also holds in a similar way for $\nabla_\alpha K_2^{\alpha\beta}$ and hence, we conclude that the equation of motion with respect to A_μ is second-order.

Now, we shall look at the equation of motion with respect to to the metric. Consider the Lagrangian composed of the following independent variables,

$$L \equiv L(R^{\alpha\beta}{}_{\gamma\delta}, g^{\alpha\beta}, F_{\alpha\beta}) \tag{3.42}$$

As we have derived before, the equation of motion corresponding to this Lagrangian is given by 3.35 which we restate here for convenience:

$$P_{\alpha\mu\nu\rho} R_\beta{}^{\mu\nu\rho} + K_{\alpha\mu} F_\beta{}^\mu - \frac{1}{2} g_{\alpha\beta} L - 2\nabla^{(\mu}\nabla^{\nu)} P_{\alpha\mu\nu\beta} = 0 \tag{3.43}$$

where,

$$P_{\gamma\delta}{}^{\alpha\beta} = \left(\frac{\partial L}{\partial R^{\gamma\delta}{}_{\alpha\beta}}\right)_{(g^{\alpha\beta}, F_{\mu\nu})} \tag{3.44}$$

Note that $P_{\alpha\beta\gamma\delta} = g_{\alpha\theta} g_{\beta\eta} P^{\theta\eta}_{\gamma\delta}$. Notice that, just like theories with just the Riemann tensor and its contractions, only the last term $\nabla^\mu \nabla^\nu P_{\alpha\mu\nu\beta}$ contributes higher derivative terms. So let's first calculate $P^{\alpha\mu}_{\nu\beta}$ upto an overall constant factor,

$$P_{\nu\beta}{}^{\alpha\mu} = n\, \delta^{\mu_1\nu_1 \dots \mu_m\nu_m\, \alpha\mu\, \eta_2\delta_2 \dots \eta_n\delta_n}_{\alpha_1\beta_1 \dots \alpha_m\beta_m\, \nu\beta\, \theta_2\sigma_2 \dots \theta_n\sigma_n} F_{\mu_1\nu_1} \dots F_{\mu_m\nu_m}\, F^{\alpha_1\beta_1} \dots F^{\alpha_m\beta_m}\, R^{\theta_2\sigma_2}{}_{\eta_2\delta_2} \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \quad (3.45)$$

Taking the covariant derivatives, we can express the result schematically as,

$$\begin{aligned}
&\nabla_{(\mu}\nabla^{\nu)} P_{\nu\beta}{}^{\alpha\mu} = \\
&n\, \delta^{\mu_1\nu_1 \dots \mu_m\nu_m\, \alpha\mu\, \eta_2\delta_2 \dots \eta_n\delta_n}_{\alpha_1\beta_1 \dots \alpha_m\beta_m\, \nu\beta\, \theta_2\sigma_2 \dots \theta_n\sigma_n} \Big[m\, (\nabla_{(\mu}\nabla^{\nu)} F_{\mu_1\nu_1}) \dots F_{\mu_m\nu_m}\, F^{\alpha_1\beta_1} \dots F^{\alpha_m\beta_m}\, R^{\theta_2\sigma_2}{}_{\eta_2\delta_2} \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + m\, F_{\mu_1\nu_1} \dots F_{\mu_m\nu_m}\, (\nabla_{(\mu}\nabla^{\nu)} F^{\alpha_1\beta_1}) \dots F^{\alpha_m\beta_m}\, R^{\theta_2\sigma_2}{}_{\eta_2\delta_2} \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + (n-1)\, F_{\mu_1\nu_1} \dots F_{\mu_m\nu_m}\, F^{\alpha_1\beta_1} \dots F^{\alpha_m\beta_m}\, (\nabla_{(\mu}\nabla^{\nu)} R^{\theta_2\sigma_2}{}_{\eta_2\delta_2}) \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + (n-1)m\, (\nabla^{(\nu} F_{\mu_1\nu_1}) \dots F_{\mu_m\nu_m}\, F^{\alpha_1\beta_1} \dots F^{\alpha_m\beta_m}\, (\nabla_{\mu)} R^{\theta_2\sigma_2}{}_{\eta_2\delta_2}) \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + (n-1)m\, F_{\mu_1\nu_1} \dots F_{\mu_m\nu_m}\, (\nabla^{(\nu} F^{\alpha_1\beta_1}) \dots F^{\alpha_m\beta_m}\, (\nabla_{\mu)} R^{\theta_2\sigma_2}{}_{\eta_2\delta_2}) \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + (n-1)(n-2)\, F_{\mu_1\nu_1} \dots F_{\mu_m\nu_m}\, F^{\alpha_1\beta_1} \dots F^{\alpha_m\beta_m} (\nabla_{(\mu} R^{\theta_2\sigma_2}{}_{\eta_2\delta_2})(\nabla^{\nu)} R^{\theta_3\sigma_3}{}_{\eta_3\delta_3})\, R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + m^2\, (\nabla^{(\nu} F_{\mu_1\nu_1}) \dots F_{\mu_m\nu_m}\, (\nabla_{\mu)} F^{\alpha_1\beta_1}) \dots F^{\alpha_m\beta_m}\, R^{\theta_2\sigma_2}{}_{\eta_2\delta_2} \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + m(m-1)\, F_{\mu_1\nu_1} \dots F_{\mu_m\nu_m} (\nabla_{(\mu} F^{\alpha_1\beta_1})(\nabla^{\nu)} F^{\alpha_2\beta_2}) \dots F^{\alpha_m\beta_m} R^{\theta_2\sigma_2}{}_{\eta_2\delta_2} \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + m(m-1)\, (\nabla_{(\mu} F_{\mu_1\nu_1})(\nabla^{\nu)} F_{\mu_2\nu_2}) \dots F_{\mu_m\nu_m}\, F^{\alpha_1\beta_1} F^{\alpha_2\beta_2} \dots F^{\alpha_m\beta_m} R^{\theta_2\sigma_2}{}_{\eta_2\delta_2} \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \Big]
\end{aligned} \quad (3.46)$$

Let us analyze each of the contributions in the above equation. We see that the last three terms are second-order in derivatives as they have a single covariant derivative of the EM field strength. All the remaining terms contain higher-derivatives of either the metric – through covariant derivatives of the curvature tensor – or the EM vector potential – through two covariant derivatives of the EM field strength. However, due to the antisymmetrization of indices by the determinant tensor, all of the higher-derivative contributions vanish by the use of Bianchi identities: $\nabla_{[\mu} F_{\alpha\beta]} = 0$ and $\nabla_{[\mu} R^{\alpha\beta}{}_{\gamma\delta]} = 0$. Hence, the above equation of motion simplifies to,

$$
\begin{aligned}
&\nabla_{(\mu}\nabla^{\nu)}P_{\nu\beta}{}^{\alpha\mu} = \\
&n\,\delta^{\mu_1\nu_1\,...\,\mu_m\nu_m\,\alpha\mu\,\eta_2\delta_2\,...\,\eta_n\delta_n}_{\alpha_1\beta_1\,...\,\alpha_m\beta_m\,\nu\beta\,\theta_2\sigma_2\,...\,\theta_n\sigma_n}\Big[m^2\,(\nabla^{(\nu}F_{\mu_1\nu_1})\,...\,F_{\mu_m\nu_m}\,(\nabla_{\mu)}F^{\alpha_1\beta_1})\,...\,F^{\alpha_m\beta_m}\,R^{\theta_2\sigma_2}{}_{\eta_2\delta_2}\,...\,R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + m(m-1)\,F_{\mu_1\nu_1}\,...\,F_{\mu_m\nu_m}(\nabla_{(\mu}F^{\alpha_1\beta_1})(\nabla^{\nu)}F^{\alpha_2\beta_2})\,...\,F^{\alpha_m\beta_m}R^{\theta_2\sigma_2}{}_{\eta_2\delta_2}\,...\,R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \\
&\quad + m(m-1)\,(\nabla_{(\mu}F_{\mu_1\nu_1})(\nabla^{\nu)}F_{\mu_2\nu_2})\,...\,F_{\mu_m\nu_m}\,F^{\alpha_1\beta_1}F^{\alpha_2\beta_2}\,...\,F^{\alpha_m\beta_m}R^{\theta_2\sigma_2}{}_{\eta_2\delta_2}\,...\,R^{\theta_n\sigma_n}{}_{\eta_n\delta_n}\Big]
\end{aligned}
\tag{3.47}
$$

Hence, we see that the contribution from the last term in 3.35 is second-order for the Lagrangian given by 3.38. This concludes the proof of our claim.

In the above discussion, we have leveraged the fact that only the last term in the the equation of motion 3.43 – $\nabla^{(\mu}\nabla^{\nu)}P_{\alpha\mu\nu\beta}$ – contributes higher-derivatives. In the case of theories with just contractions of the curvature tensor with the metric, we have seen that this term can never be second-order (see discussion below 2.51) – it is either higher-derivative or zero. The latter condition leads to the Lovelock class of theories. On the contrary, from the above discussion we see that in the presence of EM field strength contracted with curvature tensor, some of the terms become second-order 3.47 rather than being zero.

3.2 2-form gauge field

Now, as an extension of the last section, we will investigate if there exists theories with Abelian gauge field strength of form degree 3 non-minimally coupled to gravity such that the equation of motion is second-order in arbitrary backgrounds. This is an interesting case to consider, since in string theory, the gravitational effective action contains an Abelian field of form degree 2 known as the Kalb-Ramond field, which is coupled to the curvature tensor and to a scalar field known as the dilaton.

Consider a form degree 2 Abelian gauge field $B_{\mu\nu}$, which appears through the field strength $H_{\alpha\mu\nu} = 3\nabla_{[\alpha}B_{\mu\nu]}$ (of form degree 3). The field strength $H_{\alpha\mu\nu}$ satisfies a Bianchi identity,

$$
\begin{aligned}
&\nabla_{[\beta} H_{\alpha\mu\nu]} = 0 \\
\implies &\nabla_\beta H_{\alpha\mu\nu} - \nabla_\alpha H_{\mu\nu\beta} + \nabla_\mu H_{\nu\beta\alpha} - \nabla_\nu H_{\beta\alpha\mu} = 0
\end{aligned} \tag{3.48}
$$

We will try to construct a Lagrangian at fourth order in derivatives with the fields mentioned above, such that the equation of motion is second-order.

The most general action at fourth order in derivatives with a Lagrangian of the form $L \equiv L(R^{\alpha\beta}{}_{\gamma\delta}, g^{\alpha\beta}, H_{\alpha\beta\gamma})$ can be written as,

$$
\begin{aligned}
S = \alpha \int d^D x \sqrt{-g}\, e^{-2\Phi}\, \big[&a_1\, R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + a_2\, R^{\mu\nu} R_{\mu\nu} + a_3\, R^2 + a_4\, H_\eta{}^{\beta\delta} H^{\eta\alpha\gamma} R_{\alpha\beta\gamma\delta} \\
&+ a_5\, H^{\alpha\gamma\delta} H^\beta{}_{\gamma\delta} R_{\alpha\beta} + a_6\, H^{\alpha\beta\gamma} H_{\alpha\beta\gamma} R + \mathcal{L}^{(1)}(H, \nabla H) \,\big]
\end{aligned} \tag{3.49}
$$

In the above action, $\mathcal{L}^{(1)}(H, \nabla H)$ denotes the terms which only contains contractions of the H field, and its covariant derivatives – we will analyze such terms in the subsequent sections. Note that terms of the form H^n aren't of concern, since they always have a second-order contribution to the equation of motion.

Our requirement of having a second-order equation of motion leads to the purely metric terms to assume the Gauss-Bonnet form with $a_1 = -4a_2 = a_3$, as it is the unique theory with four derivatives with the above property. Now, let us turn our attention to the coupling terms between the metric and the B field. First, we will derive the equation of motion by varying $g_{\mu\nu}$ and choose the coefficients a_i such that all higher derivative terms in the equation of motion cancel each other.

Starting with the $H_\eta{}^{\beta\delta} H^{\eta\alpha\gamma} R_{\alpha\beta\gamma\delta}$ term, from the identity 3.10, the first four terms in brackets with two derivatives of $\delta g_{\mu\nu}$ are the only ones which may lead to the higher derivative terms in the equation of motion,

$$
H_\eta{}^{\beta\delta} H^{\eta\alpha\gamma} \delta R_{\alpha\beta\gamma\delta} \to \frac{1}{2} H_\eta{}^{\beta\delta} H^{\eta\alpha\gamma} (\delta g_{\alpha\delta;\beta\gamma} - \delta g_{\beta\delta;\alpha\gamma} - \delta g_{\alpha\gamma;\beta\delta} + \delta g_{\beta\gamma;\alpha\delta}) \tag{3.50}
$$

The covariant derivatives in the second and third terms are antisymmetrised by contraction with the H indices, and hence do not lead to any higher derivative term. Using integration by parts on the first and fourth terms, we have,

$$H_{\eta}{}^{\beta\delta}H^{\eta\alpha\gamma}\delta R_{\alpha\beta\gamma\delta} \to \left(-H^{\gamma\eta\mu}\nabla_{\beta}\nabla_{\gamma}H^{\beta}{}_{\eta}{}^{\sigma} - H^{\beta}{}_{\eta}{}^{\sigma}\nabla_{\beta}\nabla_{\gamma}H^{\gamma\eta\mu}\right)\delta g_{\mu\sigma} \tag{3.51}$$

Notice that the term in the brackets is symmetric in μ and σ as expected.

Now let us look at the $H^{\alpha\gamma\delta}H^{\beta}{}_{\gamma\delta}R_{\alpha\beta}$ term. We will use the identity 3.13, where only the first two terms lead to higher derivatives. Grouping terms with similar contractions to get,

$$\begin{aligned}
&H^{\alpha\eta\theta}H^{\gamma}{}_{\eta\theta}\delta R_{\alpha\gamma} \\
\to&\frac{1}{2}\Big[\Big\{(\nabla^{\sigma}\nabla_{\gamma}H^{\mu\eta\theta})H^{\gamma}{}_{\eta\theta} + H^{\alpha\eta\theta}(\nabla_{\alpha}\nabla^{\mu}H^{\sigma}{}_{\eta\theta})\Big\} + \Big\{(\nabla^{\sigma}\nabla_{\gamma}H^{\gamma}{}_{\eta\theta})H^{\mu\eta\theta} + (\nabla_{\alpha}\nabla^{\mu}H^{\alpha\eta\theta})H^{\sigma}{}_{\eta\theta}\Big\} \\
&- 2g^{\mu\sigma}(\nabla_{\alpha}\nabla_{\gamma}H^{\alpha\eta\theta})H^{\gamma}{}_{\eta\theta} + \Big\{-(\Box H^{\mu\eta\theta})H^{\sigma}{}_{\eta\theta} - H^{\mu\eta\theta}\Box H^{\sigma}{}_{\eta\theta}\Big\}\Big]\delta g_{\mu\sigma}
\end{aligned} \tag{3.52}$$

Note that in the above expression, one is free to exchange the order of covariant derivatives in any of the terms, as such a change only requires an extra term with a Riemann tensor and no covariant derivatives. The grouped terms in parenthesis are equal upto such an exchange. Also, each group of terms is symmetric in μ and σ as expected.

Let us use the identity 3.48 in Eq. 3.52, in the first term in the second curly brackets,

$$\begin{aligned}
&(\nabla_{\gamma}\nabla^{\sigma}H^{\gamma}{}_{\eta\theta})H^{\mu\eta\theta} \\
=&\Big(\Box H_{\eta\theta}{}^{\sigma} - \nabla_{\gamma}\nabla_{\eta}H_{\theta}{}^{\sigma\gamma} + \nabla_{\gamma}\nabla_{\theta}H^{\sigma\gamma}{}_{\eta}\Big)H^{\mu\eta\theta} \\
=&H^{\mu\eta\theta}\Box H_{\eta\theta}{}^{\sigma} - 2(\nabla_{\gamma}\nabla_{\eta}H_{\theta}{}^{\sigma\gamma})H^{\mu\eta\theta}
\end{aligned} \tag{3.53}$$

So, by exchanging indices μ and σ, we can write the other term in the second curly brackets as,

$$(\nabla^{\mu}\nabla_{\gamma}H^{\gamma}{}_{\eta\theta})H^{\sigma\eta\theta} = H^{\sigma\eta\theta}\Box H_{\eta\theta}{}^{\mu} - 2(\nabla_{\gamma}\nabla_{\eta}H_{\theta}{}^{\mu\gamma})H^{\sigma\eta\theta} \tag{3.54}$$

The $\Box$ terms in 3.53 and 3.54 are the same as those in the third curly brackets in 3.52 but with opposite sign, hence they cancel out. Hence we arrive at,

$$\begin{aligned}
&H^{\alpha\eta\theta}H^{\gamma}{}_{\eta\theta}\delta R_{\alpha\gamma} \\
\rightarrow &\frac{1}{2}\Bigg[\Big\{(\nabla^{\sigma}\nabla_{\gamma}H^{\mu\eta\theta})H^{\gamma}{}_{\eta\theta} + H^{\alpha\eta\theta}(\nabla_{\alpha}\nabla^{\mu}H^{\sigma}{}_{\eta\theta})\Big\} - 2\Big\{(\nabla_{\gamma}\nabla_{\eta}H^{\gamma\theta\sigma})H^{\eta}{}_{\theta}{}^{\mu} + (\nabla_{\gamma}\nabla_{\eta}H^{\gamma\theta\mu})H^{\eta}{}_{\theta}{}^{\sigma}\Big\} \\
&- 2g^{\mu\sigma}(\nabla_{\alpha}\nabla_{\gamma}H^{\alpha\eta\theta})H^{\gamma}{}_{\eta\theta}\Bigg]\delta g_{\mu\sigma}
\end{aligned} \tag{3.55}$$

Finally, we see that the second curly brackets in 3.55 is the same as 3.51 (even upto the sign). This gives us our first constraint,

$$a_4 + a_5 = 0 \tag{3.56}$$

To eliminate the rest of the terms in 3.55, let us turn our attention to $H^{\alpha\beta\gamma}H_{\alpha\beta\gamma}R$. Repeating the same steps as we have done earlier, we come to the identity 3.19 So,

$$H^{\alpha\beta\gamma}H_{\alpha\beta\gamma}\delta R \tag{3.57}$$

$$\rightarrow\Big(\nabla^{(\mu}\nabla^{\sigma)}(H^{\alpha\beta\gamma}H_{\alpha\beta\gamma}) - g^{\mu\sigma}\Box(H^{\alpha\beta\sigma}H_{\alpha\beta\sigma})\Big)\delta g_{\mu\sigma} \tag{3.58}$$

Now, using 3.48 we try to simplify both the terms to express them as the terms familiar with us from above. We see that,

$$H^{\alpha\beta\gamma}(\nabla^{(\mu}\nabla^{\sigma)}H_{\alpha\beta\gamma}) = 3(\nabla^{\alpha}\nabla^{(\mu}H^{\sigma)\beta\gamma})H_{\alpha\beta\gamma} \tag{3.59}$$

and,

$$H^{\alpha\eta\gamma}\Box H_{\alpha\eta\gamma} = 3\,H^{\alpha\eta\gamma}(\nabla^{\theta}\nabla_{\alpha}H_{\eta\gamma\theta}) \tag{3.60}$$

Substituting the above relations back in 3.58, we have,

$$H^{\alpha\beta\gamma}H_{\alpha\beta\gamma}(\delta R) \tag{3.61}$$

$$\rightarrow \left[3\Big\{(\nabla^{\alpha}\nabla^{\mu}H^{\sigma\beta\gamma})H_{\alpha\beta\gamma} + (\nabla^{\alpha}\nabla^{\sigma}H^{\mu\beta\gamma})H_{\alpha\beta\gamma}\Big\} - 6\, g^{\mu\sigma}\, H^{\alpha\eta\gamma}(\nabla^{\theta}\nabla_{\alpha}H_{\eta\gamma\theta})\right]\delta g_{\mu\sigma} \tag{3.62}$$

Remarkably, the curly bracket above is the same as the first curly bracket of 3.55 and the terms having $g^{\mu\sigma}$ are also identical. We can cancel both of them by having the constraint,

$$a_5 + 6a_6 = 0 \tag{3.63}$$

Finally, with the constraints 3.56 and 3.63, the action 3.49 takes the form,

$$\begin{aligned} S^{(1)} = \alpha \int d^D x\, \sqrt{-g}\, \Big[a_1\Big\{R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4\, R^{\mu\nu}R_{\mu\nu} + R^2\Big\} + a_4\Big\{\, H_{\eta}{}^{\beta\delta}H^{\eta\alpha\gamma}R_{\alpha\beta\gamma\sigma} \\ -\; H^{\alpha\gamma\delta}H^{\beta}{}_{\gamma\delta}R_{\alpha\beta} + \frac{1}{6}\, H^{\alpha\beta\gamma}H_{\alpha\beta\gamma}R\Big\}\Big] \end{aligned} \tag{3.64}$$

We still haven't looked at the equation of motion when varying with respect to $B_{\mu\sigma}$ and whether it is second order too with such a choice of coefficients. This equation of motion is given by,

$$\begin{aligned} \nabla_{\alpha}P^{\alpha\mu\sigma} = 0 \\ \text{where, } P^{\alpha\mu\sigma} = \left(\frac{\partial L}{\partial H_{\alpha\mu\sigma}}\right)_{(R^{\rho\mu}{}_{\sigma\nu},\, g_{\mu\sigma})} \end{aligned} \tag{3.65}$$

Let us calculate the higher order terms in 3.65 contributed by each of the three terms with coefficients a_4, a_5 and a_6.

For $H_{\eta}{}^{\beta\delta}H^{\eta\theta\gamma}R_{\theta\beta\gamma\delta}$,

$$P^{\alpha\mu\sigma} = 2\, \delta^{\alpha\mu\sigma}_{\eta\beta\delta}\, H^{\eta\theta\gamma}R_{\theta}{}^{\beta}{}_{\gamma}{}^{\delta} \tag{3.66}$$

Note that in 3.65, the covariant derivative acting on the H will only produce a second derivative term, while that acting on R will give a third derivative term. Focusing on only terms of the latter kind, we have,

$$\nabla_{\alpha}P^{\alpha\mu\sigma} \rightarrow 4\Big\{H^{\mu\theta\gamma}\nabla_{\gamma}R_{\theta}{}^{\sigma} - H^{\sigma\theta\gamma}\nabla_{\gamma}R_{\theta}{}^{\mu}\Big\} + 2\, H^{\alpha\theta\gamma}\Big[\nabla_{\alpha}R_{\theta\mu\gamma\sigma} - \nabla_{\alpha}R_{\theta\sigma\gamma\mu}\Big] \tag{3.67}$$

Note that all the terms above are antisymmetric in μ and σ as expected. We will later show that the terms with the square brackets do not lead to any higher derivatives of the metric, so let us ignore them in the meanwhile.

For $H_{\eta\gamma\delta}H_{\rho}{}^{\gamma\delta}R^{\eta\rho}$,

$$\begin{aligned}\nabla_\alpha P^{\alpha\mu\sigma} =& \nabla_\alpha\Big(2\,\delta^{\alpha\mu\sigma}_{\eta\gamma\delta}\,H_{\rho}{}^{\gamma\delta}\,R^{\eta\rho}\Big)\\ &\rightarrow 2H^{\alpha\mu\sigma}(\nabla_\alpha R) + 4\Big\{H_{\rho}{}^{\alpha\mu}(\nabla_\alpha R^{\sigma\rho}) - H_{\rho}{}^{\alpha\sigma}(\nabla_\alpha R^{\mu\rho})\Big\}\end{aligned} \tag{3.68}$$

Again, notice that the terms in the curly brackets in 3.67 and 3.68 are identical. This again leads to the constraint $a_4 + a_5 = 0$.

For $H_{\eta\beta\gamma}H^{\eta\beta\gamma}R$,

$$\nabla_\alpha P^{\alpha\mu\sigma} = \nabla_\alpha\Big(2\,\delta^{\alpha\mu\sigma}_{\eta\beta\gamma}H^{\eta\beta\gamma}R\Big) \tag{3.69}$$

$$\rightarrow 12\,H^{\alpha\mu\sigma}\nabla_\alpha R \tag{3.70}$$

This is identical to the first term in 3.68, and gives us the constraint $a_5 + 6a_6 = 0$, as we have already seen. From the above analysis, we conclude that the Lagrangian 3.64 indeed leads to second order equation of motion under the variation of both $g_{\mu\sigma}$ and $B_{\mu\sigma}$.

Now, we just need to prove the claim that the terms with the square bracket in 3.67 do not contribute any higher derivatives. To see this, consider them linearized on an arbitrary background $\bar{H}_{\alpha\beta\gamma}$ and $\bar{g}_{\mu\nu}$ and let us focus only on the terms which could potentially lead to higher derivatives (below, δ refers to the the linearized quantity),

$$\begin{aligned}&\bar{H}^{\alpha\theta\gamma}\nabla_\alpha(\delta R_{\mu\theta\gamma\sigma}) - (\mu\leftrightarrow\sigma)\\ &\rightarrow \frac{1}{2}\bar{H}^{\alpha\theta\gamma}\nabla_\alpha\Big(\nabla_\gamma\nabla_\mu\delta g_{\sigma\theta} - \nabla_\gamma\nabla_\theta\delta g_{\sigma\mu} - \nabla_\sigma\nabla_\mu\delta g_{\gamma\theta} + \nabla_\sigma\nabla_\theta\delta g_{\gamma\mu}\Big) - (\mu\leftrightarrow\sigma)\end{aligned} \tag{3.71}$$

Due to the antisymmetrisation with μ and σ, the second term drops out while the third term becomes second order in derivatives. Now, just consider the remaining terms,

$$\begin{aligned} & H^{\alpha\theta\gamma}\nabla_\alpha\nabla_\gamma\nabla_\mu\delta g_{\sigma\theta} + H^{\alpha\theta\gamma}\nabla_\alpha\nabla_\theta\nabla_\sigma\delta g_{\gamma\mu} - (\mu \leftrightarrow \sigma) \\ = & - H^{[\alpha\gamma]\theta}\nabla_{[\alpha}\nabla_{\gamma]}\nabla_\mu\delta g_{\sigma\theta} + H^{[\alpha\theta]\gamma}\nabla_{[\alpha}\nabla_{\theta]}\nabla_\sigma\delta g_{\gamma\mu} - (\mu \leftrightarrow \sigma) \end{aligned} \tag{3.72}$$

Hence, this antisymmetrisation of the covariant derivatives removes all higher than two derivative terms. This concludes the proof.

Analogous to the last section, one can check that the interaction terms involving the the B field and curvature tensors in 3.64 can be equivalently expressed (upto overall constant factors) as,

$$L = \delta^{\mu_1\nu_1\rho_1\,\eta_1\delta_1}_{\alpha_1\beta_1\gamma_1\,\theta_1\sigma_1} H_{\mu_1\nu_1\rho_1} H^{\alpha_1\beta_1\gamma_1}\, R^{\theta_1\sigma_1}{}_{\eta_1\delta_1} \tag{3.73}$$

The above result suggest the following generalization to arbitrary order in curvature and the field strength as follows – just like the LA(1) case,

$$L = \delta^{\mu_1\nu_1\rho_1\,\dots\,\mu_m\nu_m\rho_m\,\eta_1\delta_1\,\dots\,\eta_n\delta_n}_{\alpha_1\beta_1\gamma_1\,\dots\,\alpha_m\beta_m\gamma_m\,\theta_1\sigma_1\,\dots\,\theta_n\sigma_n} H_{\mu_1\nu_1\rho_1} \dots H_{\mu_m\nu_m\rho_m}\, H^{\alpha_1\beta_1\gamma_1} \dots H^{\alpha_m\beta_m\gamma_m}\, R^{\theta_1\sigma_1}{}_{\eta_1\delta_1} \dots R^{\theta_n\sigma_n}{}_{\eta_n\delta_n} \tag{3.74}$$

Following closely the analysis in Section 3.1, one can easily check that the above theory represent the LA(2) class.

3.3 n-form gauge field

The above results reveal a clear pattern through which we can obtain the LA(n) class of theories for any $n \geq 0$. Consider an Abelian gauge field of form degree n given by $A_{[\mu_1\mu_2\,\dots\,\mu_n]}$. The field strength corresponding to this field is given as $C_{[\mu_1\mu_2\,\dots\,\mu_{n+1}]} = \nabla_{[\mu_1}A_{\mu_2\mu_3\,\dots\,\mu_{n+1}]}$ which has form degree $n+1$. This field strength satisfies a Bianchi identity given by,

$$\nabla_{[\mu_{n+2}}C_{\mu_1\mu_2\,\dots\,\mu_{n+1}]} = 0 \tag{3.75}$$

Then, the class of theories with the Abelian gauge field $A_{[\mu_1\mu_2\,\dots\,\mu_n]}$ non-minimally coupled to gravity such that the equations of motion of the theory is second-order in arbitrary backgrounds – LA(n) – can be given as,

$$L = \delta^{\mu_1^{(1)} \dots \mu_{n+1}^{(1)} \dots \dots \mu_1^{(m)} \dots \mu_{n+1}^{(m)} \, \alpha_1\beta_1 \dots \alpha_s\beta_s}_{\nu_1^{(1)} \dots \nu_{n+1}^{(1)} \dots \dots \nu_1^{(m)} \dots \nu_{n+1}^{(m)} \, \gamma_1\delta_1 \dots \gamma_s\delta_s} C_{\mu_1^{(1)}\mu_2^{(1)} \dots \mu_{n+1}^{(1)}} \dots C_{\mu_1^{(m)}\mu_2^{(m)} \dots \mu_{n+1}^{(m)}} C^{\nu_1^{(1)}\nu_2^{(1)} \dots \nu_{n+1}^{(1)}} \dots C^{\nu_1^{(m)}\nu_2^{(m)} \dots \nu_{n+1}^{(m)}} R^{\gamma_1\delta_1}{}_{\alpha_1\beta_1} \dots R^{\gamma_s\delta_s}{}_{\alpha_s\beta_s} \tag{3.76}$$

We have successfully found the most general way to construct Lovelockian theories with an Abelian gauge field contracted with the curvature tensor and the metric.

Scalar field (LA(0))

Following from the above discussion, we find that the class of Lovelockian theories – LA(0) – constructed from contractions of the curvature tensor with the field strength of a scalar field can be expressed as:

$$L = \delta^{\mu_1\nu_1 \dots \mu_n\nu_n \, \eta_1\eta_2 \dots \eta_m}_{\alpha_1\beta_1 \dots \alpha_n\beta_n \, \theta_1\theta_2 \dots \theta_m} (\nabla_{\eta_1}\phi)\,(\nabla_{\eta_2}\phi) \dots (\nabla_{\eta_m}\phi)\,(\nabla^{\theta_1}\phi)\,(\nabla^{\theta_2}\phi) \dots (\nabla^{\theta_m}\phi)\, R^{\alpha_1\beta_1}{}_{\mu_1\nu_1} \dots R^{\alpha_n\beta_n}{}_{\mu_n\nu_n} \tag{3.77}$$

3.4 Lovelockian theories as differential forms

3.4.1 Notation

We shall denote forms using bold typsetting for clarity. The curvature two form is given as,

$$\mathbf{R}_{ab} = e^\mu_a \, e^\nu_b \, R_{\mu\nu\alpha\beta} \, dx^\alpha \wedge dx^\beta \tag{3.78}$$

The electromagnetic field strength is a two form which is given as the exterior derivative of a one form $\mathbf{A}$,

$$\mathbf{F} = d\mathbf{A} \equiv 2\partial_{[\mu}A_{\nu]} \, dx^\mu \wedge dx^\nu = F_{\mu\nu} \, dx^\mu \wedge dx^\nu \tag{3.79}$$

where $F_{\mu\nu}$ is the antisymmetric field strength tensor given as,

$$F_{\mu\nu} = 2\,\partial_{[\mu}A_{\nu]} \tag{3.80}$$

We can write the field strength tensor with two frame indices as,

$$F_{ab} = e^{\mu}_{a}\,e^{\nu}_{b}\,F_{\mu\nu} \tag{3.81}$$

Alternatively, the field strength can be expressed as a one-form as,

$$\mathbf{F}_a = \mathbf{e}^{\mu}_{a}\,F_{\mu\nu}\,dx^{\nu} \tag{3.82}$$

3.4.2 Lovelock theories in differential forms

Claim: In d dimensions, any top form which can be written as below, is equivalent to a Lanczos-Lovelock Lagrangian 2.37.

$$\mathbf{L} = \mathbf{R}_{a_1b_1} \wedge \mathbf{R}_{a_2b_2} \wedge \,...\, \wedge \mathbf{R}_{a_nb_n} \wedge \mathbf{e}_{g_1} \wedge \mathbf{e}_{g_2} \wedge \,...\, \wedge \mathbf{e}_{g_p}\,\epsilon^{a_1b_1\,...\,a_nb_n\,g_1\,...\,g_p} \tag{3.83}$$

with $d = 2n + p$ and $n, p \geq 0$. ϵ denotes the Levi-Civita tensor.

Evaluating the wedges, we write the scalar Lagrangian corresponding to the above top form as,

$$\begin{aligned} L =& \epsilon^{\mu_1\nu_1\,...\,\mu_n\nu_n\,\gamma_1\,...\,\gamma_n} R_{a_1b_1\mu_1\nu_1}\,R_{a_2b_2\mu_2\nu_2}\,...\,R_{a_nb_n\mu_n\nu_n} e_{g_1\,\gamma_1} e_{g_2\,\gamma_2}\,...\,e_{g_p\,\gamma_p} \\ & \epsilon^{a_1b_1\,...\,a_nb_n\,c_1d_1\,...\,c_md_m\,g_1\,...\,g_p} \end{aligned} \tag{3.84}$$

Note that here any contractions between frame (Latin) indices can be replaced by co-ordinate indices. As an example, for an arbitrary tensor A_{ab} in $2D$, $A_{ab}\epsilon^{ab} = e^{\theta}_{a}e^{\eta}_{b}A_{\theta\eta}\epsilon^{ab} = A_{\theta\eta}\epsilon^{\theta\eta}$. Hence, from 3.84, we find,

$$\begin{aligned} L &= \epsilon^{\mu_1\nu_1\,...\,\mu_n\nu_n\,\gamma_1\,...\,\gamma_p}\;\epsilon^{\alpha_1\beta_1\,...\,\alpha_n\beta_n}{}_{\gamma_1\,...\,\gamma_p} R_{\alpha_1\beta_1\mu_1\nu_1}\,R_{\alpha_2\beta_2\mu_2\nu_2}\,...\,R_{\alpha_n\beta_n\mu_n\nu_n} \\ &= \epsilon^{\mu_1\nu_1\,...\,\mu_n\nu_n\,\gamma_1\,...\,\gamma_p}\;\epsilon_{\alpha_1\beta_1\,...\,\alpha_n\beta_n\,\gamma_1\,...\,\gamma_p} R^{\alpha_1\beta_1}{}_{\mu_1\nu_1}\,R^{\alpha_2\beta_2}{}_{\mu_2\nu_2}\,...\,R^{\alpha_n\beta_n}{}_{\mu_n\nu_n} \end{aligned} \tag{3.85}$$

Now, we use the identity that in d dimensions,

$$\epsilon^{i_1\,\dots\,i_s\;j_{s+1}\,\dots\,j_d}\epsilon_{i_1\,\dots\,i_s\;k_{s+1}\,\dots\,k_d} = s!\,\delta^{j_{s+1}\,\dots\,j_d}_{k_{s+1}\,\dots\,k_d} \tag{3.86}$$

Hence, we can finally write the Lagrangian as,

$$L = p!\,\delta^{\mu_1\nu_1\,\dots\,\mu_n\nu_n}_{\alpha_1\beta_1\,\dots\,\alpha_n\beta_n} R^{\alpha_1\beta_1}{}_{\mu_1\nu_1}\,R^{\alpha_2\beta_2}{}_{\mu_2\nu_2}\dots R^{\alpha_n\beta_n}{}_{\mu_n\nu_n} \tag{3.87}$$

This concludes the proof of our claim.

3.4.3 LA(1) Lagrangians in differential forms

Claim: In d dimensions, any top form which can be written as below, is equivalent to a LA(1) Lagrangian (3.38).

$$\begin{aligned}\mathbf{L} = &\mathbf{R}_{a_1b_1}\wedge\,\mathbf{R}_{a_2b_2}\wedge\,\dots\,\wedge\mathbf{R}_{a_nb_n}\,(F_{c_1d_1}\mathbf{F})\wedge(F_{c_2d_2}\mathbf{F})\wedge\,\dots\,\wedge(F_{c_md_m}\mathbf{F})\\ &\wedge\mathbf{e}_{g_1}\wedge\mathbf{e}_{g_2}\wedge\,\dots\,\wedge\mathbf{e}_{g_p}\epsilon^{a_1b_1\,\dots\,a_nb_n\;c_1d_1\,\dots\,c_md_m\;g_1\,\dots\,g_{2p}}\end{aligned} \tag{3.88}$$

with $d = 2n + 2m + p$ and $n, m, p \geq 0$.

Evaluating the wedges, we write the scalar Lagrangian corresponding to the above top form as,

$$\begin{aligned}L = &\epsilon^{\mu_1\nu_1\,\dots\,\mu_n\nu_n\;\rho_1\sigma_1\,\dots\,\rho_m\sigma_m\;\gamma_1\,\dots\,\gamma_n}\\ &R_{a_1b_1\mu_1\nu_1}\,R_{a_2b_2\mu_2\nu_2}\dots R_{a_nb_n\mu_n\nu_n}\,(F_{c_1d_1}F_{\rho_1\sigma_1})(F_{c_2d_2}F_{\rho_2\sigma_2})\dots(F_{c_md_m}F_{\rho_m\sigma_m})\\ &e_{g_1\,\gamma_1}e_{g_2\,\gamma_2}\dots e_{g_p\,\gamma_p}\epsilon^{a_1b_1\,\dots\,a_nb_n\;c_1d_1\,\dots\,c_md_m\;g_1\,\dots\,g_{2p}}\end{aligned} \tag{3.89}$$

On further simplifying 3.89 by replacing contraction of frame indices with that of space-time indices, we find,

$$\begin{aligned}L &= \epsilon^{\mu_1\nu_1\,...\,\mu_n\nu_n\;\rho_1\sigma_1\,...\,\rho_m\sigma_m\;\gamma_1\,...\,\gamma_p}\;\epsilon^{\alpha_1\beta_1\,...\,\alpha_n\beta_n\;\eta_1\delta_1\,...\,\eta_m\delta_m}{}_{\gamma_1\,...\,\gamma_p} \\ &\qquad R_{\alpha_1\beta_1\mu_1\nu_1}\,...\,R_{\alpha_n\beta_n\mu_n\nu_n}\,(F_{\eta_1\delta_1}F_{\rho_1\sigma_1})\,...\,(F_{\eta_m\delta_m}F_{\rho_m\sigma_m}) \\ &= \epsilon^{\mu_1\nu_1\,...\,\mu_n\nu_n\;\rho_1\sigma_1\,...\,\rho_m\sigma_m\;\gamma_1\,...\,\gamma_p}\;\epsilon_{\alpha_1\beta_1\,...\,\alpha_n\beta_n\;\eta_1\delta_1\,...\,\eta_m\delta_m\gamma_1\,...\,\gamma_p} \\ &\qquad R^{\alpha_1\beta_1}{}_{\mu_1\nu_1}\,...\,R^{\alpha_n\beta_n}{}_{\mu_n\nu_n}\,(F^{\eta_1\delta_1}F_{\rho_1\sigma_1})\,...\,(F^{\eta_m\delta_m}F_{\rho_m\sigma_m})\end{aligned} \tag{3.90}$$

On using the identity 3.86, we can finally write the Lagrangian as,

$$L = p!\,\delta^{\mu_1\nu_1\,...\,\mu_n\nu_n\;\rho_1\sigma_1\,...\,\rho_m\sigma_m}_{\alpha_1\beta_1\,...\,\alpha_n\beta_n\;\eta_1\delta_1\,...\,\eta_n\delta_n}\,R^{\alpha_1\beta_1}{}_{\mu_1\nu_1}\,...\,R^{\alpha_n\beta_n}{}_{\mu_n\nu_n}\,(F^{\eta_1\delta_1}F_{\rho_1\sigma_1})\,...\,(F^{\eta_m\delta_m}F_{\rho_m\sigma_m}) \tag{3.91}$$

This concludes the proof of our claim.

3.4.4 LA(n) Lagrangians in differential forms

From the above discussion, we can infer that a general LA(n) class of theories can be expressed as the following differential form in d dimensions:

$$\begin{aligned}\mathbf{L} = &\mathbf{R}_{a_1b_1}\wedge\,\mathbf{R}_{a_2b_2}\wedge\,...\,\wedge\mathbf{R}_{a_jb_j}\,(C_{k_1^{(1)}k_2^{(1)}\,...\,k_{n+1}^{(1)}}\mathbf{C})\wedge\,...\,\wedge(C_{k_1^{(m)}k_2^{(m)}\,...\,k_{n+1}^{(m)}}\mathbf{C}) \\ &\wedge\mathbf{e}_{g_1}\wedge\mathbf{e}_{g_2}\wedge\,...\,\wedge\mathbf{e}_{g_p}\,\epsilon^{a_1b_1\,...\,a_jb_j\,k_1^{(1)}k_2^{(1)}\,...\,k_{n+1}^{(1)}\,......\,k_1^{(m)}k_2^{(m)}\,...\,k_{n+1}^{(m)}\,g_1\,...\,g_{2p}}\end{aligned} \tag{3.92}$$

where $\mathbf{C} = C_{\mu_1\mu_2\,...\,\mu_{n+1}}\,dx^{\mu_1}\wedge dx^{\mu_2}\wedge\,\ldots\,\wedge dx^{\mu_{n+1}}$ and $d = 2j + m(n+1) + p$.

3.4.5 List of Lovelock and LA(1) top-forms upto 6D

In this section, we will list all **non-zero** Lovelock and LA(1) top-forms upto 6D, which are composed of the wedge product of the following quantities contracted with the ϵ tensor: $\mathbf{R}_{ab}$, $\mathbf{F}$ and F_{ab}. Note that $\mathbf{F}$ and F_{ab} are always added in a pair as the latter requires two indices to contract with the Levi-Civita tensor (hence requiring 2 additional dimensions), while the former increases the form number by two, to maintain being the top form. Note that for the prefactors, we utilize the identity 3.86.

In 2D

1. $\mathbf{e}_a\,\mathbf{e}_b\epsilon^{ab} \equiv \epsilon^{\mu\nu}\epsilon_{\mu\nu} = 2$
2. $\mathbf{R_{ab}}\,\epsilon^{ab} \equiv \delta^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu} = 2R$: Topological and total-derivative in 2D
3. $F_{ab}\,\mathbf{F}\,\epsilon^{ab} \equiv \delta^{\mu\nu}_{\alpha\beta}F^{\alpha\beta}F_{\mu\nu} = 2\,F_{\mu\nu}F^{\mu\nu}$: EM kinetic term

In 3D

1. $\mathbf{e}_a\,\mathbf{e}_b\,\mathbf{e}_c\,\epsilon^{abc} \equiv \epsilon^{\mu\nu\rho}\epsilon_{\mu\nu\rho} = 6$: Constant
2. $\mathbf{R}_{ab}\,\mathbf{e}_c\,\epsilon^{abc} \equiv \delta^{\alpha\beta}_{\mu\nu}R^{\mu\nu}{}_{\alpha\beta} = 2\,R$: Einstein-Hilbert
3. $\mathbf{F}\,F_{ab}\,\mathbf{e}_c\,\epsilon^{abc} \equiv \delta^{\alpha\beta}_{\mu\nu}F_{\alpha\beta}F^{\mu\nu} = 2\,F^{\mu\nu}F_{\mu\nu}$: EM kinetic term

In 4D

1. $\mathbf{e}_a\,\mathbf{e}_b\,\mathbf{e}_c\,\mathbf{e}_d\,\epsilon^{abcd} \equiv \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = 24$: Constant
2. $\mathbf{R}_{ab}\,\mathbf{e}_c\,\mathbf{e}_d\epsilon^{abcd} \equiv 2\,\delta^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu} = 4\,R$: Einstein-Hilbert
3. $\mathbf{R}_{ab}\mathbf{R}_{cd}\epsilon^{abcd} \equiv \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\alpha\beta}\,R^{\rho\sigma}{}_{\gamma\delta} = 4\,(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2)$
   : Gauss-Bonnet, topological and total-derivative
4. $F_{ab}\,\mathbf{F}\,\mathbf{e}_c\,\mathbf{e}_d\,\epsilon^{abcd} \equiv 2\,\delta^{\alpha\beta}_{\mu\nu}F_{\alpha\beta}F^{\mu\nu} = 4F_{\mu\nu}F^{\mu\nu}$: EM kinetic term
5. $F_{ab}\,\mathbf{F}\,F_{cd}\,\mathbf{F}\,\epsilon^{abcd} \equiv \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}F_{\alpha\beta}F^{\mu\nu}F_{\gamma\delta}F^{\rho\sigma} = -16\,F^{\mu\nu}F_{\rho\sigma}F_{\mu}{}^{\rho}F_{\nu}{}^{\sigma} + 8\,F_{\mu\nu}F_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$
   : EM self-interaction term
6. $\mathbf{R}_{ab}\,F_{cd}\,\mathbf{F}\,\epsilon^{abcd} \equiv \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}R^{\mu\nu}_{\alpha\beta}\,F^{\rho\sigma}F_{\gamma\delta} = 4(-4\,F^{\alpha}{}_{\mu}\,F_{\alpha\nu}R^{\mu\nu} + F^{\mu\nu}F_{\mu\nu}\,R + F^{\mu\nu}F^{\alpha\beta}R_{\mu\nu\alpha\beta})$
   : LA(1) term with 4 derivatives

In 5D

1. $\mathbf{e}_a\,\mathbf{e}_b\,\mathbf{e}_c\,\mathbf{e}_d\,\mathbf{e}_f\,\epsilon^{abcdf} \equiv \epsilon^{\mu\nu\rho\sigma\delta}\epsilon_{\mu\nu\rho\sigma\delta} = 120$: Constant

2. $\mathbf{R}_{ab}\,\mathbf{e}_c\,\mathbf{e}_d\,\mathbf{e}_f\,\epsilon^{abcdf} \;\equiv\; 6\,\delta^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu} = 12\,R$: Einstein-Hilbert

3. $\mathbf{R}_{ab}\,\mathbf{R}_{cd}\,\mathbf{e}_f\,\epsilon^{abcdf} \;\equiv\; \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\alpha\beta}\,R^{\rho\sigma}{}_{\gamma\delta} = 4\,(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2)$
   : Gauss-Bonnet

4. $F_{ab}\,\mathbf{F}\,\mathbf{e}_c\,\mathbf{e}_d\,\mathbf{e}_f\,\epsilon^{abcdf} \;\equiv\; 6\,\delta^{\alpha\beta}_{\mu\nu}F_{\alpha\beta}F^{\mu\nu} = 12\,F^{\mu\nu}F_{\mu\nu}$: EM kinetic term

5. $F_{ab}\,\mathbf{F}\,F_{cd}\,\mathbf{F}\,\mathbf{e}_f\,\epsilon^{abcdf} \;\equiv\; \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}F_{\alpha\beta}F^{\mu\nu}F_{\gamma\delta}F^{\rho\sigma} = -16\,F^{\mu\nu}F_{\rho\sigma}F_{\mu}{}^{\rho}F_{\nu}{}^{\sigma} + 8\,F_{\mu\nu}F_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$
   : EM self-interaction term

6. $\mathbf{R}_{ab}\,F_{cd}\,\mathbf{F}\,\mathbf{e}_f\,\epsilon^{abcdf} \;\equiv\; \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}R^{\mu\nu}_{\alpha\beta}\,F^{\rho\sigma}F_{\gamma\delta} = 4(-4\,F^{\alpha}{}_{\mu}\,F_{\alpha\nu}R^{\mu\nu} + F^{\mu\nu}F_{\mu\nu}\,R + F^{\mu\nu}F^{\alpha\beta}R_{\mu\nu\alpha\beta})$
   : LA(1) term with 4 derivatives

In 6D

1. $\mathbf{e}_a\,\mathbf{e}_b\,\mathbf{e}_c\,\mathbf{e}_d\,\mathbf{e}_f\,\mathbf{e}_g\,\epsilon^{abcdfg} \equiv \epsilon^{\mu\nu\rho\sigma\delta\eta}\epsilon_{\mu\nu\rho\sigma\delta\eta} = 720$: Constant

2. $\mathbf{R}_{ab}\,\mathbf{e}_c\,\mathbf{e}_d\,\mathbf{e}_f\,\mathbf{e}_g\,\epsilon^{abcdfg} \;\equiv\; 24\,\delta^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu} = 48\,R$: Einstein-Hilbert

3. $\mathbf{R}_{ab}\,\mathbf{R}_{cd}\,\mathbf{e}_f\,\mathbf{e}_g\,\epsilon^{abcdfg} \;\equiv\; 2\,\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\alpha\beta}\,R^{\rho\sigma}{}_{\gamma\delta} = 8\,(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2)$: Gauss-Bonnet

4. $\mathbf{R}_{ab}\,\mathbf{R}_{cd}\,\mathbf{R}_{fg}\,\epsilon^{abcdfg} \;\equiv\; \delta^{\alpha\beta\gamma\delta\theta\eta}_{\mu\nu\rho\sigma\lambda\tau}\,R^{\mu\nu}{}_{\alpha\beta}\,R^{\rho\sigma}{}_{\gamma\delta}\,R^{\lambda\tau}{}_{\theta\eta}$
   : Cubic-Lovelock, topological and total-derivative

5. $F_{ab}\,\mathbf{F}\,\mathbf{e}_c\,\mathbf{e}_d\,\mathbf{e}_f\,\mathbf{e}_g\,\epsilon^{abcdfg} \;\equiv\; 24\,\delta^{\alpha\beta}_{\mu\nu}F_{\alpha\beta}F^{\mu\nu} = 48F^{\mu\nu}F_{\mu\nu}$: EM kinetic term

6. $F_{ab}\,\mathbf{F}\,F_{cd}\,\mathbf{F}\,\mathbf{e}_f\,\mathbf{e}_g\,\epsilon^{abcdfg} \;\equiv\; 2\,\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}F_{\alpha\beta}F^{\mu\nu}F_{\gamma\delta}F^{\rho\sigma}$
   : EM quartic self-interaction term

7. $F_{ab}\,\mathbf{F}\,F_{cd}\,\mathbf{F}\,F_{fg}\mathbf{F}\,\epsilon^{abcdfg} \;\equiv\; \delta^{\alpha\beta\gamma\delta\theta\eta}_{\mu\nu\rho\sigma\lambda\tau}F_{\alpha\beta}F^{\mu\nu}F_{\gamma\delta}F^{\rho\sigma}F_{\theta\eta}F^{\lambda\tau}$
   : EM hex self-interaction term

8. $\mathbf{R}_{ab}\,F_{cd}\,\mathbf{F}\,\mathbf{e}_f\,\mathbf{e}_g\,\epsilon^{abcdfg} \;\equiv\; 2\,\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}\,F^{\rho\sigma}F_{\gamma\delta} = 8\,(-4\,F^{\alpha}{}_{\mu}\,F_{\alpha\nu}R^{\mu\nu} + F^{\mu\nu}F_{\mu\nu}\,R + F^{\mu\nu}F^{\alpha\beta}R_{\mu\nu\alpha\beta})$
   : LA(1) term with 4 derivatives

9. $\mathbf{R}_{ab}\,\mathbf{R}_{cd}\,F_{fg}\,\mathbf{F}\,\epsilon^{abcdfg} \;\equiv\; \delta^{\alpha\beta\gamma\delta\theta\eta}_{\mu\nu\rho\sigma\lambda\tau}\,R^{\mu\nu}{}_{\alpha\beta}\,R^{\rho\sigma}{}_{\gamma\delta}F^{\lambda\tau}F_{\theta\eta}$
   : LA(1) term with 6 derivatives

10. $\mathbf{R}_{ab}\, F_{cd}\, \mathbf{F}\, F_{fg}\, \mathbf{F}\, \epsilon^{abcdfg} \;\equiv\; \delta^{\alpha\beta\gamma\delta\theta\eta}_{\mu\nu\rho\sigma\lambda\tau}\, R^{\mu\nu}{}_{\alpha\beta}\, F^{\rho\sigma} F_{\gamma\delta}\, F^{\lambda\tau} F_{\theta\eta}$
: LA(1) term with 6 derivatives

From the above list, note two observations: First, we do not have any 'new' term in the odd dimensions apart from the ones already introduced in lower dimensions which are then extended to the odd dimension using $\mathbf{e}$ (However, note that even though Lovelock terms of curvature order n can be first written in dimension $2n$, they start affecting the equation of motion from dimension $2n+1$). Second, we have omitted writing terms with an odd number of F. This is because any such term vanishes due to antisymmetrization by the Levi-Civita symbol. To demonstrate this using an example, consider,

$$\mathbf{R}_{ab}\, \mathbf{F}_c\, \mathbf{F}_d\, \mathbf{F}_f \epsilon^{abcdf} \tag{3.93}$$

On evaluating the wedge products and writing the scalar corresponding to the above Lagrangian, we have,

$$\epsilon^{\mu\nu\eta_1\eta_2\eta_3}\, R_{\lambda\tau\mu\nu}\, F_{\theta_1\eta_1}\, F_{\theta_2\eta_2}\, F_{\theta_3\eta_3}\, \epsilon^{\lambda\tau\theta_1\theta_2\theta_3} \tag{3.94}$$

On exchanging (λ,τ) and (μ,ν) we have,

$$\epsilon^{\lambda\tau\theta_1\theta_2\theta_3}\, R_{\mu\nu\lambda\tau}\, F_{\theta_1\eta_1}\, F_{\theta_2\eta_2}\, F_{\theta_3\eta_3}\, \epsilon^{\mu\nu\eta_1\eta_2\eta_3} \tag{3.95}$$

Note that in the above, we have't yet relabelled any indices, we have just done two things: i) performed the said interchange of indices on the curvature tensor and, ii) exchanged the positions of the two epsilons. We now relabel indices to have,

$$\epsilon^{\mu\nu\theta_1\theta_2\theta_3}\, R_{\lambda\tau\mu\nu}\, F_{\theta_1\eta_1}\, F_{\theta_2\eta_2}\, F_{\theta_3\eta_3}\, \epsilon^{\lambda\tau\eta_1\eta_2\eta_3} \tag{3.96}$$

Next, we perform the interchange of indices $(\theta_i \leftrightarrow \eta_i)$ for $i = (1,2,3)$, which leads to a net negative sign as there is an odd number of F. To summarise,

$$\begin{aligned}&\epsilon^{\mu\nu\eta_1\eta_2\eta_3}\, R_{\lambda\tau\mu\nu}\, F_{\theta_1\eta_1}\, F_{\theta_2\eta_2}\, F_{\theta_3\eta_3}\, \epsilon^{\lambda\tau\theta_1\theta_2\theta_3} \\ &= \epsilon^{\mu\nu\theta_1\theta_2\theta_3}\, R_{\lambda\tau\mu\nu}\, F_{\theta_1\eta_1}\, F_{\theta_2\eta_2}\, F_{\theta_3\eta_3}\, \epsilon^{\lambda\tau\eta_1\eta_2\eta_3} = -\epsilon^{\mu\nu\theta_1\theta_2\theta_3}\, R_{\lambda\tau\mu\nu}\, F_{\eta_1\theta_1}\, F_{\eta_2\theta_2}\, F_{\eta_3\theta_3}\, \epsilon^{\lambda\tau\eta_1\eta_2\eta_3}\end{aligned} \tag{3.97}$$

Now, relabelling $(\theta_i \leftrightarrow \eta_i)$ in the last expression, we find:

$$\epsilon^{\mu\nu\eta_1\eta_2\eta_3}\, R_{\lambda\tau\mu\nu}\, F_{\theta_1\eta_1}\, F_{\theta_2\eta_2}\, F_{\theta_3\eta_3}\, \epsilon^{\lambda\tau\theta_1\theta_2\theta_3} = 0 \tag{3.98}$$

As we saw, this result only relies on the no. of F being odd, and the particular contraction method we are investigating, and hence can be easily extended to any such form.

Also, to introduce additional F's when increasing dimensions, we could have opted to insert $\mathbf{F}_a\,\mathbf{F}_b$ instead of $\mathbf{F}\,F_{ab}$. Hence, we must try to see if both the approaches are equivalent. To start with, consider a very simple example,

$$F_{ab}\mathbf{F}\epsilon^{ab} = \epsilon^{\alpha\beta}\epsilon^{\mu\nu}\,F_{\alpha\beta}\,F_{\mu\nu} \tag{3.99}$$

and

$$\mathbf{F}_a\,\mathbf{F}_b\epsilon^{ab} = \epsilon^{\alpha\beta}\epsilon^{\mu\nu}\,F_{\alpha\mu}\,F_{\beta\nu} = \epsilon^{\alpha\mu}\epsilon^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} \tag{3.100}$$

Hence, we must see if $\epsilon^{\alpha\beta}\epsilon^{\mu\nu} \sim \epsilon^{\alpha\mu}\epsilon^{\beta\nu}$ ($\sim$ denotes equality upto overall constant factors) with α, β and μ, ν antisymmetric.

Evaluating the former, we find:

$$\epsilon^{[\alpha\beta]}\epsilon^{[\mu\nu]} = g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha} \tag{3.101}$$

while the latter gives,

$$\begin{aligned}
\epsilon^{\alpha\mu}\epsilon^{\beta\nu} &= g^{\beta\alpha}g^{\nu\mu} - g^{\mu\beta}g^{\nu\alpha} \;\text{ with } [\alpha,\beta] \text{ and } [\mu,\nu] \text{ antisymmetrized} \\
&= -g^{[\mu|\beta}g^{|\nu]\alpha} \\
&= \frac{1}{2}\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right) \\
&= \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\mu\nu}
\end{aligned} \tag{3.102}$$

Following a similar analysis, it can be shown that the following LA(2) forms are also equivalent (upto overall constants):

$$\epsilon^{\mu_1\nu_1\,...\,\mu_n\nu_n\;\rho_1\,...\,\rho_{2m}\;\gamma_1\,...\,\gamma_p}\;\epsilon^{\alpha_1\beta_1\,...\,\alpha_n\beta_n\;\eta_1\,...\,\eta_{2m}}{}_{\gamma_1\,...\,\gamma_p}$$
$$R_{\alpha_1\beta_1\mu_1\nu_1}\,...\,R_{\alpha_n\beta_n\mu_n\nu_n}\,(F_{\rho_1\eta_1}F_{\rho_2\eta_2})\,...\,(F_{\rho_{2m-1}\eta_{2m-1}}F_{\rho_{2m}\eta_{2m}}) \tag{3.103}$$
$$\equiv \mathbf{R}_{a_1a_2}\,...\,\mathbf{R}_{a_{2n-1}a_{2n}}\mathbf{F}_{b_1}\,...\mathbf{F}_{b_{2m}}\,\mathbf{e}_{c_1}\,...\,\mathbf{e}_{c_p}\;\epsilon^{a_1\,...\,a_{2n}\,b_1\,...\,b_{2m}\,c_1\,...\,c_p}$$

and

$$\epsilon^{\mu_1\nu_1\,...\,\mu_n\nu_n\;\rho_1\,...\,\rho_{2m}\;\gamma_1\,...\,\gamma_p}\;\epsilon^{\alpha_1\beta_1\,...\,\alpha_n\beta_n\;\eta_1\,...\,\eta_{2m}}{}_{\gamma_1\,...\,\gamma_p}$$
$$R_{\alpha_1\beta_1\mu_1\nu_1}\,...\,R_{\alpha_n\beta_n\mu_n\nu_n}\,(F_{\rho_1\rho_2}F_{\eta_1\eta_2})\,...\,(F_{\rho_{2n-1}\rho_{2n}}F_{\eta_{2n-1}\eta_{2n}}) \tag{3.104}$$
$$\equiv \mathbf{R}_{a_1a_2}\,...\,\mathbf{R}_{a_{2n-1}a_{2n}}(\mathbf{F}\,F_{b_1b_2})\,...\,(\mathbf{F}\,F_{b_{2m-1}b_{2m}})\,\mathbf{e}_{c_1}\,...\,\mathbf{e}_{c_p}\;\epsilon^{a_1\,...\,a_{2n}\,b_1\,...\,b_{2m}\,c_1\,...\,c_p}$$

In this sectiom, we have thus shown that there exists a systematic scheme which not only reproduces known invariants (like the Einstein-Hilbert and Gauss-Bonnet terms) but also higher derivative couplings between gravity and gauge fields - such as LA(n) interactions. It also sheds light on the fact that combinatorial constraints dictate the structure of such top-forms at each dimension.

3.5 Theories with covariant derivatives

Till now, we have only studied theories which do not involve any covariant derivatives of the curvature tensor or field strengths in the Lagrangian. This naturally leads to the question whether there exists any such Lovelockian theories, which we shall investigate in this section.

3.5.1 Theories with a single covariant derivative of curvature tensor

Consider a Lagrangian which is composed of the metric, the curvature tensor and its covariant derivative, with the following index placement,

$$L \equiv L(g_{\alpha\beta}, R^{\alpha}{}_{\beta\gamma\delta}, \nabla_{\eta} R^{\alpha}{}_{\beta\gamma\delta}) \tag{3.105}$$

(For now, we work with a single covariant derivative, but later we shall generalize our discussion to theories with an arbitrary no. of derivatives on a curvature tensor). We can decompose this Lagrangian into two parts: the terms which do not have any covariant derivatives of the curvature tensor and those which do,

$$L = L_R + L_{\nabla} \tag{3.106}$$

Since we already have performed the analysis of the equation of motion of the former part, we will now focus only on the latter and take steps motivated by the former analysis. First, we should establish the relation among the dependence of the metric to that on the curvature tensor and its covariant derivative.

Taking $(g^{\alpha\beta}, R^{\alpha}{}_{\beta\gamma\delta}, \nabla_{\eta} R^{\alpha}{}_{\beta\gamma\delta})$ as the independent variables, we evaluate the Lie derivative of the the Lagrangian under an arbitrary infinitesimal diffeomorphism $x^{\mu} \to x^{\mu} + \xi^{\mu}$ in two different ways (just like we did in Section 2.1). First,

$$\begin{aligned}
\mathcal{L}_\xi L =& \xi^\mu \nabla_\mu L \\
=& \xi^\mu \left(\frac{\partial L}{\partial g^{\alpha\beta}}\right)\Big|_{(R^\alpha{}_{\beta\gamma\delta}, \nabla_\eta R^\alpha{}_{\beta\gamma\delta})} \nabla_\mu g^{\alpha\beta} + \xi^\mu \left(\frac{\partial L}{\partial R^\alpha{}_{\beta\gamma\delta}}\right)\Big|_{(g^{\alpha\beta}, \nabla_\eta R^\alpha{}_{\beta\gamma\delta})} \nabla_\mu R^\alpha{}_{\beta\gamma\delta} \\
&+ \xi^\mu \left(\frac{\partial L}{\partial \nabla_\eta R^\alpha{}_{\beta\gamma\delta}}\right)\Big|_{(g^{\alpha\beta}, \nabla_\eta R^\alpha{}_{\beta\gamma\delta})} \nabla_\mu \nabla_\eta R^\alpha{}_{\beta\gamma\delta} \\
=& P_\alpha{}^{\beta\gamma\delta}\, \nabla_\mu R^\alpha{}_{\beta\gamma\delta} + P^\eta{}_\alpha{}^{\beta\gamma\delta}\, \nabla_\mu \nabla_\eta R^\alpha{}_{\beta\gamma\delta}
\end{aligned} \tag{3.107}$$

where we have defined,

$$\begin{aligned}
P_{\alpha\beta} &= \left(\frac{\partial L}{\partial g^{\alpha\beta}}\right)\Big|_{(R^\alpha{}_{\beta\gamma\delta}, \nabla_\eta R^\alpha{}_{\beta\gamma\delta})} \\
P^{\alpha\beta\gamma\delta} &= \left(\frac{\partial L}{\partial R^\alpha{}_{\beta\gamma\delta}}\right)\Big|_{(g^{\alpha\beta}, \nabla_\eta R^\alpha{}_{\beta\gamma\delta})} \\
P^\eta{}_\alpha{}^{\beta\gamma\delta} &= \left(\frac{\partial L}{\partial \nabla_\eta R^\alpha{}_{\beta\gamma\delta}}\right)\Big|_{(g^{\alpha\beta}, R^\alpha{}_{\beta\gamma\delta})}
\end{aligned} \tag{3.108}$$

In the second line, we used the chain rule. From brevity, here onwards we are goning to drop the subscripts denoting which quantities are held constant, unless explicitly required. Next, we can also evaluate the Lie derivative of the Lagrangian as,

$$\mathcal{L}_\xi L = P^{\alpha\beta} \mathcal{L}_\xi g^{\alpha\beta} + P_\alpha{}^{\beta\gamma\delta} \mathcal{L}_\xi R^\alpha{}_{\beta\gamma\delta} + P^\eta{}_\alpha{}^{\beta\gamma\delta} \mathcal{L}_\xi \nabla_\eta R^\alpha{}_{\beta\gamma\delta} \tag{3.109}$$

Evaluating the first two Lie derivatives as before,

$$\begin{aligned}
P_{\alpha\beta}\, \mathcal{L}_\xi g^{\alpha\beta} &= P_{\alpha\beta}\, \xi^\mu\, \nabla_\mu g^{\alpha\beta} - 2\, P_{\alpha\beta}\, \nabla^\alpha \xi^\beta \\
P_\alpha{}^{\beta\gamma\delta}\, \mathcal{L}_\xi R^\alpha{}_{\beta\gamma\delta} &= P_\alpha{}^{\beta\gamma\delta}\, \xi^\mu\, \nabla_\mu R^\alpha{}_{\beta\gamma\delta} + 2\, P_\alpha{}^{\beta\gamma\delta}\, (\nabla_\rho \xi^\alpha)\, R^\rho{}_{\beta\gamma\delta}
\end{aligned} \tag{3.110}$$

Now,

$$\begin{aligned}
P^\eta{}_\alpha{}^{\beta\gamma\delta} \mathcal{L}_\xi \nabla_\eta R^\alpha{}_{\beta\gamma\delta} =& P^\eta{}_\alpha{}^{\beta\gamma\delta}\, \xi^\mu\, \nabla_\mu \nabla_\eta R^\alpha{}_{\beta\gamma\delta} \\
&+ P^\eta{}_\alpha{}^{\beta\gamma\delta}\, (\nabla^\rho \xi_\eta)\, \nabla_\rho R_{\alpha\beta\gamma\delta} + 2\, P^\eta{}_\alpha{}^{\beta\gamma\delta}\, (\nabla_\rho \xi^\alpha)\, \nabla_\eta R^\rho{}_{\beta\gamma\delta}
\end{aligned} \tag{3.111}$$

Substituting 3.111 and 3.110 in 3.109 and equating it to 3.107, we have,

$$\left(-2\,P_{\mu\sigma}+2\,P_{\sigma}{}^{\beta\gamma\delta}\,R_{\mu\beta\gamma\delta}+P_{\sigma\alpha}{}^{\beta\gamma\delta}\,\nabla_{\mu}R_{\alpha\beta\gamma\delta}+2\,P^{\eta}{}_{\sigma}{}^{\beta\gamma\delta}\,\nabla_{\eta}R^{\mu}{}_{\beta\gamma\delta}\right)(\nabla^{\mu}\xi^{\sigma})=0 \tag{3.112}$$

Since the above equation holds for arbitrary ξ and at arbitrary points on the manifold, the term in the brackets vanishes and we finally arrive at the desired relation,

$$P_{\mu\sigma}-\;P_{\sigma}{}^{\beta\gamma\delta}\,R_{\mu\beta\gamma\delta}-\frac{1}{2}P_{\sigma}{}^{\alpha\beta\gamma\delta}\,\nabla_{\mu}R_{\alpha\beta\gamma\delta}-\;P^{\eta}{}_{\sigma}{}^{\beta\gamma\delta}\,\nabla_{\eta}R_{\mu\beta\gamma\delta}=0 \tag{3.113}$$

Let us proceed further to find the equation of motion of the theory. The action can be expressed as (suppressing all dimensional factors),

$$S=\int d^{D}x\,\sqrt{-g}\,\Big(L_{R}(g^{\alpha\beta},R^{\alpha}{}_{\beta\gamma\delta})+L_{\nabla}(g^{\alpha\beta},R^{\alpha}{}_{\beta\gamma\delta},\nabla_{\eta}R^{\alpha}{}_{\beta\gamma\delta})\Big) \tag{3.114}$$

Since we have already analyzed the contribution from the L_R part in the previous sections, we shall only focus on the L_{∇} part. Therefore,

$$\begin{aligned}\delta S_{\nabla}&=\int d^{D}x\,\sqrt{-g}\,\delta L_{\nabla}\\&=\int d^{D}x\,(\delta\sqrt{-g})L_{\nabla}+\sqrt{-g}\,\left(\frac{\partial L_{\nabla}}{\partial g^{\alpha\beta}}\right)\delta g^{\alpha\beta}+\sqrt{-g}\,\left(\frac{\partial L_{\nabla}}{\partial R^{\alpha}{}_{\beta\gamma\delta}}\right)\delta R^{\alpha}{}_{\beta\gamma\delta}\\&\quad+\sqrt{-g}\,\left(\frac{\partial L_{\nabla}}{\partial\nabla_{\eta}R^{\alpha}{}_{\beta\gamma\delta}}\right)\delta\nabla_{\eta}R^{\alpha}{}_{\beta\gamma\delta}\end{aligned} \tag{3.115}$$

On performing integration by parts, we can write the above as,

$$\begin{aligned}\delta S_{\nabla}=\int d^{D}x\,\sqrt{-g}\,\left(-\frac{1}{2}g_{\mu\sigma}L_{\nabla}\right)\delta g^{\mu\sigma}+\sqrt{-g}\,\left(\frac{\partial L_{\nabla}}{\partial g^{\alpha\beta}}\right)\delta g^{\alpha\beta}\\+\sqrt{-g}\,\left(\frac{\partial L_{\nabla}}{\partial R^{\alpha}{}_{\beta\gamma\delta}}-\nabla_{\eta}\frac{\partial L_{\nabla}}{\partial(\nabla_{\eta}R^{\alpha}{}_{\beta\gamma\delta})}\right)\delta R^{\alpha}{}_{\beta\gamma\delta}\end{aligned} \tag{3.116}$$

Next, we can express the above more compactly as,

$$\int d^{D}x\,\sqrt{-g}\,\left(-\frac{1}{2}g_{\mu\sigma}L_{\nabla}\right)\delta g^{\mu\sigma}+\sqrt{-g}\,P_{\alpha\beta}\delta g^{\alpha\beta}+\sqrt{-g}\,\left(P_{\alpha}{}^{\beta\gamma\delta}-\nabla_{\eta}P^{\eta}{}_{\alpha}{}^{\beta\gamma\delta}\right)\delta R^{\alpha}{}_{\beta\gamma\delta} \tag{3.117}$$

On evaluating $\delta R^{\alpha}{}_{\beta\gamma\delta}$ and utilizing the symmetries of the P tensors, we have,

$$\begin{aligned}\left(P_{\alpha}{}^{\beta\gamma\delta} - \nabla_{\eta} P^{\eta}{}_{\alpha}{}^{\beta\gamma\delta}\right) \delta R^{\alpha}{}_{\beta\gamma\delta} &= 2\left(P^{\alpha\beta\gamma\delta} - \nabla_{\eta} P^{\eta\alpha\beta\gamma\delta}\right) \nabla_{\gamma}\nabla_{\beta}\delta g_{\alpha\delta} \\ &= -2\left(P_{\alpha\beta\gamma\delta} - \nabla^{\eta} P_{\eta\alpha\beta\gamma\delta}\right) \nabla^{\gamma}\nabla^{\beta}\delta g^{\alpha\delta} \\ &= -2\, E_{\alpha\beta\gamma\delta}\, \nabla^{\gamma}\nabla^{\beta}\delta g^{\alpha\delta}\end{aligned} \tag{3.118}$$

where we have defined a new quantity $E_{\alpha}{}^{\beta\gamma\delta}$ as,

$$E_{\alpha}{}^{\beta\gamma\delta} = \frac{\partial L_{\nabla}}{\partial R^{\alpha}{}_{\beta\gamma\delta}} - \nabla_{\eta} \frac{\partial L_{\nabla}}{\partial(\nabla_{\eta} R^{\alpha}{}_{\beta\gamma\delta})} \tag{3.119}$$

Finally, on using 3.118 (to replace $P_{\alpha\beta}$) and 3.113 in the action 3.117, we arrive at the equation of motion of the theory,

$$P_{\sigma}{}^{\beta\gamma\delta} R_{\mu\beta\gamma\delta} + \frac{1}{2} P_{\sigma}{}^{\alpha\beta\gamma\delta} \nabla_{\mu} R_{\alpha\beta\gamma\delta} + P^{\eta}{}_{\sigma}{}^{\beta\gamma\delta} \nabla_{\eta} R_{\mu\beta\gamma\delta} - \frac{1}{2} g_{\mu\sigma} L_{\nabla} - 2\nabla^{\alpha}\nabla^{\beta} E_{\mu\alpha\beta\sigma} = 0 \tag{3.120}$$

Notice that if we take $P_{\sigma}{}^{\alpha\beta\gamma\delta} = 0$ the above equation of motion becomes equivalent to 2.20 as expected. Unlike 2.20 – where only the last term contributed higher-derivative terms in the equation of motion – all the terms above have higher derivative contributions in the equation of motion. This is because the Lagrangian itself now includes higher-derivative terms. Hence, we realize that one cannot repeat our previous analysis which had lead us to the Lovelock terms. This calls for more sophisticated approach to answering the question whether there exists theories with covariant derivatives which have second-order equation of motion. The question whether there exists any Lovelockian theories with covariant derivatives of the curvature tensor was tackled in [24] using the framework of BRST cohomology. One of the key assumptions made in this work was inspired from a special property of the flat space expansion of Lovelock theories, which we will refer to as – Zumino's property [25]. In the next section, we will study in detail Zumino's property and later, we will demonstrate this this property should also hold for Lovelockian theories with covariant derivatives – if they exist. The conclusion of [24] was that given this property holds, there exists no Lovelockian theories with covariant derivatives of the curvature tensor.

Before concluding this section, we can easily see that if we started with a Lagrangian with an arbitrary number of covariant derivatives on any curvature tensor, 3.120 will have some additional terms – all of which will again contribute higher-derivative terms in the equation of motion.

3.5.2 Zumino property

As we saw previously, any Lovelock term can be written in terms of the curvature two-form as,

$$L = \mathbf{R}_{a_1\, b_1}\, \mathbf{R}_{a_2\, b_2} \ldots \mathbf{R}_{a_n\, b_n}\, \mathbf{e}_{c_1}\, \mathbf{e}_{c_2} \ldots \mathbf{e}_{c_m}\ \epsilon^{a_1\, a_2\, \ldots a_n\, b_1\, b_2\, \ldots\, b_n\, c_1\, c_2\, \ldots\, c_m} \tag{3.121}$$

The curvature two form in terms of the spin connection $\omega_{ab} \equiv \omega_{\mu\, ab}\, dx^{\mu}$, which is a one-form, can be written as,

$$\mathbf{R}_{ab} = (d\omega)_{ab} + \omega_a^{\ c} \wedge \omega_{c\, b} \tag{3.122}$$

From the above it follows that the infitesimal variation of the curvature two-form can be represented as the covariant derivative of the variation of the spin connection,

$$\delta \mathbf{R}_{ab} = D(\delta \omega_{ab}) \tag{3.123}$$

where we have denoted the covariant derivative as D. From 3.122, it also follows that,

$$D\mathbf{R}_{ab} = 0 \tag{3.124}$$

We want to investigate the perturbative expansion of 3.121 in flat space. Consider the linearized metric perturbation as,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.125}$$

First, let us find the variation of the vierbeins at linear order in h:

$$\begin{aligned} \delta g_{\mu\nu} &= \delta(\mathbf{e}_{a\,\mu}\mathbf{e}_{b\,\nu}\eta^{ab}) \\ \Longrightarrow h_{\mu\nu} &= 2\, \mathbf{e}_{\nu}^{a}\, (\delta \mathbf{e}_{a\,\mu}) \\ \Longrightarrow \frac{1}{2}\mathbf{e}_{a}^{\nu}\, h_{\mu\nu} &= \delta \mathbf{e}_{a\,\mu} \end{aligned} \tag{3.126}$$

Hence, $\delta \mathbf{e}_{a\,\mu}$ at linear order in h is generally a non-zero function of h.

Going forward, we will consider a torsion-free manifold with $D\, \mathbf{e}_a = 0$.

From 3.121, notice that if we consider its perturbative expansion around flat space, the first non-zero contribution will be at order h^n. This is because any curvature two-form evaluated on flat background vanishes, hence at the leading non-zero order, one needs to account for all of them. Hence, expanding around flat space, we find:

$$L = (\hat{\mathbf{R}}_{a_1\,b_1})\,(\hat{\mathbf{R}}_{a_2 b_2})\,...\,(\hat{\mathbf{R}}_{a_n b_n})\,\mathbf{e}_{c_1}\,\mathbf{e}_{c_2}\,...\,\mathbf{e}_{c_m}\;\epsilon^{a_1\,a_2\,...a_n\,b_1\,b_2\,...\,b_n\,c_1\,c_2\,...\,c_m} + \mathcal{O}(h^{n+1}) \tag{3.127}$$

where $\hat{}$ represents the corresponding linearized quantity. Next, using 3.123 on the first curvature two-form we have,

$$L = D(\hat{\omega}_{ab})\,(\hat{\mathbf{R}}_{a_2 b_2})\,...\,(\hat{\mathbf{R}}_{a_n b_n})\,\mathbf{e}_{c_1}\,\mathbf{e}_{c_2}\,...\,\mathbf{e}_{c_m}\;\epsilon^{a_1\,a_2\,...a_n\,b_1\,b_2\,...\,b_n\,c_1\,c_2\,...\,c_m} + \mathcal{O}(h^{n+1}) \tag{3.128}$$

On using the chain rule, the Bianchi identity 3.124, and the torsionless condition, we can write the above as,

$$L = d((\hat{\omega}_{ab})\,(\hat{\mathbf{R}}_{a_2 b_2})\,...\,(\hat{\mathbf{R}}_{a_n b_n})\,\mathbf{e}_{c_1}\,\mathbf{e}_{c_2}\,...\,\mathbf{e}_{c_m}\;\epsilon^{a_1\,a_2\,...a_n\,b_1\,b_2\,...\,b_n\,c_1\,c_2\,...\,c_m}) + \mathcal{O}(h^{n+1}) \tag{3.129}$$

Hence, we see that the leading order term in the expansion about flat space for any Lovelock Lagrangian in arbitrary dimensions is a total derivative. This is the result by Zumino [25], which we have reformulated in a way familiar to us with the previous discussions in this note. This naturally leads to the question whether any theory (which has the form $L \equiv L(R^{\alpha\beta}{}_{\gamma\delta}, g^{\alpha\beta})$, with a second-order equation of motion about flat-space has the above property. Since the question now concerns an arbitray theory with a second-order equation of motion, we will use a slightly different approach to answer this question.

Consider an arbitrary Lagrangian $L \equiv L(R^{\alpha\beta}{}_{\gamma\delta})$ at nth order in curvature (unlike the previous discussion, we are not restricting our analysis to the Lovelock theories now). We can represent its perturbative expansion in flat space, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ as,

$$L = L_n + L_{n+1} + L_{n+2}\,... \tag{3.130}$$

where L_k denotes the term of order h^k. In the above expansion, L_n denotes the leading order non-trivial term in the expansion, which can be expressed in a compact way as,

$$A^{\alpha_1\alpha_2\,...\,\mu_1\mu_2\,...\nu_1\nu_2}\,(\partial_{\alpha_1}\partial_{\alpha_2}\,h_{\mu_1\nu_1})\,(\partial_{\alpha_3}\partial_{\alpha_4}\,h_{\mu_2\nu_2})\,...\,(\partial_{\alpha_{2n-1}}\partial_{\alpha_{2n}}\,h_{\mu_n\nu_n}) \tag{3.131}$$

Here, $A^{\alpha_1\alpha_2\ldots\mu_1\mu_2\ldots\nu_1\nu_2}$ encodes all the information about the symmetries of the curvature tensor and how they were contracted with each other. Let us understand how we got the above expression. First, note that the linearized curvature tensor is given as,

$$\hat{R}_{\mu\nu\rho\sigma} = \partial_\sigma \partial_{[\mu} h_{\nu]\rho} - \partial_\rho \partial_{[\mu} h_{\nu]\sigma} \tag{3.132}$$

Hence, it exclusively contains second derivatives of the linearized metric as expected. Now, for any non-zero contribution to the perturbative expansion around flat space, we should consider atleast the linear order contribution from each curvature tensor, else when evaluated on the background, they vanish. So, the leading non-zero order term L_n is formed by taking just the linear contribution from each curvature tensor (given by 3.132) which leads to 3.131. Now, the contribution to the equation of motion from 3.131 is obtained as,

$$\mathcal{E}^{\alpha\beta}_{n-1} = \left(\frac{\delta L_n}{\delta h_{\alpha\beta}}\right) = \partial_\mu \partial_\nu \left(\frac{\partial L}{\partial_\mu \partial_\nu h_{\alpha\beta}}\right) \tag{3.133}$$

This is of order $n-1$ in h as is denoted by the subscript of $\mathcal{E}$. The above leads to higher derivatives terms of the form

$$\mathcal{E}^{\alpha\beta}_{n-1} = A^{\alpha\beta\cdots}\,(\partial^3 h)(\partial^3 h)\,(\partial^2 h)\ldots(\partial^2 h) + A^{\alpha\beta\cdots}\,(\partial^4 h)(\partial^2 h)\,(\partial^2 h)\ldots(\partial^2 h) \tag{3.134}$$

Since these contributions are of the order $n-1$, they cannot be cancelled by other contributions to the equations coming from L_{n+1}, L_{n+2} etc. as they will be atleast of the order n. Thus if we impose that the equation of motion is second-order, then $\mathcal{E}_{k-1}$ must vanish. This requires that L_k should be a total derivative. Hence, we have shown that the second-order equation of motion implies that the leading order non-zero term in the perturbative expansion around flat space should be non-zero.

3.5.3 Generalized Zumino property

Consider a Lagrangian composed of the curvature tensor, its first covariant derivative and their contractions with the metric, i.e, $L \equiv L(R_{\alpha\beta\sigma\delta}, \nabla_\gamma R_{\alpha\beta\sigma\delta}, g^{\alpha\beta})$. It can be schematically written as follows,

$$L = A^{\gamma_1\gamma_2\cdots\gamma_2 p\,\alpha_1\beta_1\sigma_1\delta_1\alpha_2\beta_2\sigma_2\delta_2\cdots\alpha_n\beta_n\sigma_n\delta_n}\,\nabla_{\gamma_1}R_{\alpha_1\beta_1\sigma_1\delta_1}\nabla_{\gamma_2}R_{\alpha_2\beta_2\sigma_2\delta_2}\dots\nabla_n R_{\alpha_n\beta_n\sigma_n\delta_{2n}} \quad (3.135)$$

where the A tensor inherits the symmetries of the curvature tensor. Additionally it is symmetric under the collective exchange of indices $(\gamma_i, \alpha_i, \beta_i, \sigma_i, \delta_i) \leftrightarrow (\gamma_j, \alpha_j, \beta_j, \sigma_j, \delta_j)$. The equation of motion of this theory is given by,

$$P_\sigma^{\ \beta\gamma\delta}\,R_{\mu\beta\gamma\delta} + \frac{1}{2}P_\sigma^{\ \alpha\beta\gamma\delta}\,\nabla_\mu R_{\alpha\beta\gamma\delta} + P^{\eta\ \beta\gamma\delta}_{\ \sigma}\,\nabla_\eta R_{\mu\beta\gamma\delta} - \frac{1}{2}g_{\mu\sigma}L_\nabla - 2\nabla^\alpha\nabla^\beta E_{\mu\alpha\beta\sigma} = 0 \quad (3.136)$$

where,

$$E_\alpha^{\ \beta\gamma\delta} = \frac{\partial L}{\partial R^\alpha_{\ \beta\gamma\delta}} - \nabla_\eta \frac{\partial L}{\partial(\nabla_\eta R^\alpha_{\ \beta\gamma\delta})} \quad (3.137)$$

First, let us establish several important observation for the equation of motion of such Lagrangians. To start with, we state a trivial fact:

Observation 1. Any contribution to the equation of motion which has the following form,

$$A^{\cdots\cdots}_{\cdots\cdots}\,R\dots\nabla^{(1)}R\dots\nabla^{(2)}R\dots\nabla^{(3)}R\dots\nabla^{(l)}R, \quad l \in \mathbb{Z} \quad (3.138)$$

which has atleast one curvature tensor with a single covariant derivative acting on it, can never be second-order. In $\nabla^{(n)}$, n denotes the number of covariant derivatives on the curvature tensor

Since only a single covariant derivative acts on the curvature tensor, it must necessarily contain three derivatives of the metric unless it vanishes. It can't ever be second-order – there are only two identities on the curvature tensor we can utilize to further simplify the expression: the Bianchi and the cyclic identity. However, both of them will make the term zero.

Now, consider a contribution to the equation of motion of the form,

$$A^{\cdots\cdots}_{\cdots\cdots}\, R \ldots \nabla^{(2)} R \ldots \nabla^{(4)} R \ldots \nabla^{(2p)} R, \quad p \in \mathbb{Z} \tag{3.139}$$

Such contributions are particularly important for us as they have even number of covariant derivatives on each curvature tensor and can potentially lead to a second-order contribution to the equation of motion by antisymmetrizing covariant derivatives.

Observation 2. For a Lovelockian theory, the leading order non-trivial term in the flat space expansion of any contribution of the form 3.139 is either zero or higher derivative.

To see this, first consider that in flat space, the leading order non-trivial term is obtained by replacing ∇ in 3.139 with partial derivatives, and replacing the curvature tensors with the linearized curvature tensor. This can be schematically written as,

$$A^{\cdots\cdots}_{\cdots\cdots}\, \hat{R} \ldots \partial^{(2)} \hat{R} \ldots \partial^{(4)} \hat{R} \ldots \partial^{(2p)} \hat{R}, \quad p \in \mathbb{Z} \tag{3.140}$$

where $\hat{R}$ denotes the linearized curvature tensor. Since partial derivatives commute, any antisymmetrizations of covariant derivatives in 3.139 makes 3.140 zero. This shows that 3.140 can never be made second-order in derivatives.

Now, let us look at the contribution to the equation of motion 3.136 coming from the last term, i.e, $-2\nabla^{\alpha}\nabla^{\beta}E_{\mu\alpha\beta\sigma}$. They are either of the form 3.138 (with atleast one curvature tensor having a single covariant derivative) or 3.139 (with curvature tensors having an even number of derivatives). Consider the flat space-expansion of these contributions with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. With the definition of the E tensor in 3.137, we see that it is the only term which contributes at order h^{n-1} in the flat space expansion (since obtaining E involves differentiating with respect to to the curvature tensor and its derivatives) while all other terms in the equation of motion 3.120 contributes at order h^n.

Assume that there exists a Lagrangian of the form 3.135 which is Lovelockian. In that case, the h^{n-1} contribution in the flat space expansion which have the form 3.138 must be zero, as we have argued in the first observation. Next, the h^{n-1} order contribution arising from terms of the form 3.139 will have the form 3.140. By our second observation, this is never second-order and hence must vanish for a Lovelockian theory. What we have just showed is that the contribution at order h^{n-1} in the equation of motion for such a Lovelockian theory must be zero. From this, we can conclude that the leading order non-trivial contribution in the flat space expansion of such Lagrangians must be a total-derivative.

Now, with the above discussions in mind, we generalize our results as follows,

Observation 3. Consider any contribution to the equation of motion which has the following form,

$$A^{\cdots\cdots}_{\cdots\cdots} R \ldots \nabla^{(1)} R \ldots \nabla^{(2)} R \ldots \nabla^{(3)} R \ldots \nabla^{(l)} R, \quad l \in \mathbb{Z} \tag{3.141}$$

with atleast one curvature tensor having atleast one covariant derivative. Then, the leading order non-trivial term in the flat space expansion of the above contribution is either higher-derivative or zero.

If such a contribution has non-zero number of ∇R's, then by our first claim, we see that it must be zero for a Lovelockian theory. In the other case, by our assumption, it has atleast one curvature tensor with greater than one covariant derivatives acting on it. In general spacetimes, these covariant derivatives must be antisymmetrized to make the contribution second-order. At the leading order non-trivial contribution in flat space, the covariant derivatives are replaced by partial derivatives which commute and hence vanish under antisymmetrization.

The contributions schematically written in 3.141 completely characterize terms in the equation of motion from a general Lagrangian which has arbitrary derivatives of the curvature tensor, i.e,

$$L \equiv L(g^{\mu\nu}, R_{\alpha\beta\gamma\delta}, \nabla R_{\alpha\beta\gamma\delta}, \nabla^{(2)} R_{\alpha\beta\gamma\delta} \ldots \nabla^{(n)} R_{\alpha\beta\gamma\delta}) \tag{3.142}$$

Hence, we conclude that if a general Lagrangian with covariant derivatives of the curvature tensor is Lovelockian, then the leading order non-trivial term in the flat space expansion of the Lagrangian must be a total-derivative, so that its corresponding contribution to the equation of motion is zero, as we have argued above.

Hence, we have successfully demonstrated that Zumino's property indeed holds for Lovelockian theories with covariant derivatives of the curvature tensor – if they exist. However, [24] showed that there exists no theories with covariant derivatives which satisfies the above property – which demonstrates that there does not exist any such Lovelockian theories.

3.5.4 Analysis of $(\nabla F)^n$ theories

In this section, we would like to investigate if there exists any theory of the form $(\nabla F)^n$ which can be Lovelockian. This is a particularly interesting since at first sight it seems analogous to the Riem^n theories – being composed of contraction of tensors with 2 derivatives. As we have studied in detail, the latter admits a class of theories with a second-order equation of motion – the Lovelock theories.

Consider such Lagrangians which are composed of contractions of the EM field strength tensor F and its first covariant derivatives, i.e, $L \equiv L(F_{\alpha\beta}, \nabla_\gamma F_{\alpha\beta})$. They can be expressed as:

$$L = A^{\gamma_1\gamma_2\cdots\gamma_{2p}\,\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\nabla_{\gamma_1}F_{\alpha_1\beta_1}\nabla_{\gamma_2}F_{\alpha_2\beta_2}\cdots\nabla_{\gamma_{2n}}F_{\alpha_{2n}\beta_{2n}} \tag{3.143}$$

where A is a constant tensor, which inherits the $[\alpha_i, \beta_i]$ (for all i) antisymmetry of F. It is also symmetric under the collective exchange of indices $(\gamma_i, \alpha_i, \beta_i) \leftrightarrow (\gamma_j, \alpha_j, \beta_j)$ for $i, j \leq 2p$ and $(\alpha_i, \beta_i) \leftrightarrow (\alpha_j, \beta_j)$ for $i, j > 2p$. Any theory which has a second-order equation of motion in flat space is a possible candidate for being a Lovelockian theory, i.e, having a second-order equation of motion in arbitrary backgrounds. Hence, we shall analyze if the above Lagrangian can ever have a second-order equation of motion around a flat background. In this case, the above Lagrangian simplifies to:

$$L = A^{\gamma_1\gamma_2\cdots\gamma_{2p}\,\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\partial_{\gamma_1}F_{\alpha_1\beta_1}\partial_{\gamma_2}F_{\alpha_2\beta_2}\cdots\partial_{2n}F_{\alpha_{2n}\beta_{2n}} \tag{3.144}$$

The equation of motion of a general theory of this form is given by,

$$\partial_\beta \partial_\gamma \left(\frac{\partial L}{\partial(\partial_\gamma F_{\alpha\beta})} \right) = 0 \tag{3.145}$$

Hence, the equation of motion of the Lagrangian in 3.144 is given by:

$$\begin{aligned} & n(n-1)\, A^{\gamma\gamma_2\cdots\gamma_{2p}\,\alpha\beta\alpha_2\beta_2\cdots\alpha_n\beta_n} (\partial_\beta \partial_\gamma \partial_{\gamma_2} F_{\alpha_2\beta_2}) \cdots \partial_{2n} F_{\alpha_{2n}\beta_{2n}} \\ & + \frac{n(n-1)(n-2)}{2} A^{\gamma\gamma_2\cdots\gamma_{2p}\,\alpha\beta\alpha_2\beta_2\cdots\alpha_n\beta_n} (\partial_\beta \partial_{\gamma_2} F_{\alpha_2\beta_2})(\partial_\gamma \partial_{\gamma_3} F_{\alpha_3\beta_3}) \partial_{\gamma_4} F_{\alpha_4\beta_4} \cdots \partial_{2n} F_{\alpha_{2n}\beta_{2n}} = 0 \end{aligned} \tag{3.146}$$

which follows from:

$$\left(\frac{\partial L}{\partial(\partial_\gamma F_{\alpha\beta})} \right) = n\, A^{\gamma\gamma_2\cdots\gamma_{2p}\,\alpha\beta\alpha_2\beta_2\cdots\alpha_n\beta_n} \partial_{\gamma_2} F_{\alpha_2\beta_2} \cdots \partial_{2n} F_{\alpha_{2n}\beta_{2n}} \tag{3.147}$$

In the equation of motion 3.146, the first line leads to terms with four derivatives of the vector potential while the second line contributes terms with three derivatives of the vector potential. There exists no second-order contribution to the equation of motion. Hence, any such Lagrangian in flat space can never be Lovelockian. Notice that this is completely analogous to our analysis in the Generalized Zumino's property section – this theory behaves like the leading order non-trivial contribution in the flat space expansion of theories which are composed exclusively of the contraction of curvature tensors and their covariant derivatives.

Now, consider the following trivial observation:

Observation. Any contribution to the equation of motion which has the following form,

$$A^{\cdots\cdots}_{\cdots\cdots}\, \partial^{(1)} F \ldots \partial^{(2)} F \ldots \partial^{(3)} F \ldots \partial^{(l)} F, \quad l \in \mathbb{Z} \tag{3.148}$$

with atleast one field strength having atleast one partial derivative – is either higher-derivative or zero.

Next, consider any theory composed of arbitrary contractions of the field strength tensor and its covariant derivatives, i.e,

$$L \equiv L(\nabla_{\gamma_1} F_{\alpha\beta}, \nabla_{\gamma_1} \nabla_{\gamma_2} F_{\alpha\beta}, \ldots) \tag{3.149}$$

In flat space – with all the ∇ replaced by ∂ – all the terms in the equation of motion are of the form 3.148. Hence, we come to conclusion that no generic theory of the form 3.149 can ever be Lovelockian in flat space.

A straightforward corollary of the above discussion would be Lagrangians which are composed of arbitrary contractions of higher than two covariant derivatives of a scalar field:

$$L \equiv L(\nabla_{\alpha_1}\nabla_{\alpha_2}\phi, \nabla_{\alpha_3}\nabla_{\alpha_4}\nabla_{\alpha_5}\phi, \ldots) \tag{3.150}$$

Following the analysis we have done in this section, we can easily check that such theories can also never be Lovelockian around flat space.

Chapter 4

Generalized Quasi-Topological gravities – A Brief Overview

In the previous chapters, we have mainly focused on theories whose equation of motion is second-order, implying that the linearized equation of motion is second-order on any arbitrary background – the theory never propagates any ghost-modes. What if we do not want to be so restrictive, and try to identify theories which are 'Einsteinian' – just having the spin-2 graviton mode – only around specific backgrounds. Assuming that the universe is homogeneous and isotropic at large scales, one might be interested to construct theories which are ghost-free around maximally symmetric backgrounds which includes de Sitter (dS) and anti-de Sitter (AdS) spacetimes, besides flat space. In this chapter, we will study the procedure to construct such theories and look into some of their remarkable properties. We will review most results which are already known in the literature and try to create a map of theories which are constructed with the motivation of being ghost-free in arbitrary or specific backgrounds.

4.1 Einsteinian cubic gravity

First, we will review how one can construct theories at a certain order in derivatives which are Einsteinian around maximally symmetric backgrounds:

- Take the most general Lagrangian at a given order in curvature with arbitrary coefficients of each of the terms in the Lagrangian which we aim to fix.

- Linearize this theory around a maximally symmetric background with metric $\bar{g}_{\mu\nu}$ and curvature tensor $\bar{R}_{\mu\nu\rho\sigma} = 2\Lambda(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})$, where Λ is related to the cosmological constant.

- From the linearized equation of motion, identify the propagating modes of the theory. Some of these modes will be massive and ghost-like. The mass of any mode will be a function of the couplings present inside the Lagrangian and Λ

- Choose the coupling constants such that the mass of each mode – other than the spin-2 graviton – goes to infinity. This eliminates all ghost-like modes and gives us a class of Lagrangians which is Einsteinian.

Going a step further, one can impose that the couplings are dimension-independent – which implies that with the same couplings, the theory would be Einsteinian in any dimensions. With this restriction, at cubic order in curvature, one attains the cubic Lovelock term along with another term known as the 'Einsteinian cubic gravity' (ECG) ([26]) – which is non-trivial in 4D – whose Lagrangian is given by,

$$L_{\text{ECG}} = R_{\mu\nu\rho\sigma}R^{\rho\sigma\gamma\delta}R^{\mu\nu}{}_{\gamma\delta} + 12\, R_{\mu\nu\rho\sigma}R^{\nu\gamma\sigma\delta}R^{\gamma}{}_{\mu}{}^{\delta}{}_{\rho} - 12\, R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + 8\, R_{\mu\nu}R^{\nu\rho}R^{\mu}_{\rho} \tag{4.1}$$

To emphasize, the above theory propagates only the spin-2 graviton in any dimensions (as enforced by its dimension-independent couplings) around maximally symmetric backgrounds. We can extend the procedure described above to higher orders in curvature, but next we will slightly digress from this discussion and look at a class of theories which are closely related to the ECG we just studied.

4.2 Generalized Quasi-Topological gravities

In 2010, Myers et. al [9] first noted that in cubic order in curvature, a theory (without requiring dimension independence of the couplings) which admits static, spherically symmetric solution satisfying $g_{tt}g_{rr} = -1$ and characterized by a single function $f(r)$, i.e,

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2 d\Omega^2 \tag{4.2}$$

always has a second order linearized equation of motion, i.e, is Einsteinian around maximally symmetric backgrounds. These theories came to be known as 'Quasi-topological' (QT) gravity. The Lagrangian of the Quasi-topological theory is given by [9],

$$\begin{aligned} L_{QT} =& R^{\alpha\beta\gamma\delta}R_{\alpha\beta\sigma\rho}R_{\gamma\delta}{}^{\sigma\rho} + \frac{1}{(2D-3)(D-4))}\left(\frac{3(3D-8)}{8}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}R\right. \\ &- 3(D-2)R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma}{}_{\rho}R^{\delta\rho} + 3D\,R_{\alpha\beta\gamma\delta}R^{\alpha\gamma}R^{\beta\delta} \\ &\left.+6(D-2)R_{\alpha}{}^{\beta}R^{\gamma}{}_{\beta}R^{\alpha}{}_{\gamma} - \frac{3(3D-4)}{2}R^{\beta}{}_{\alpha}R^{\alpha}{}_{\beta}R + \frac{3D}{8}R^3\right) \end{aligned} \tag{4.3}$$

However, the above Lagrangian is zero for $D \leq 4$. It turns out that this is not the unique Lagrangian (allowing dimension-dependent couplings) which is Einsteinian around maximally symmetric backgrounds. There exists a broad class of theories known as 'Generalized Quasi-Topological theories' or GQTs, which have the same property. Hence, let us look into the precise definition of these theories [11]:

Consider a higher-derivative Lagrangian density of the form $\mathcal{L} \equiv \mathcal{L}(R_{\alpha\beta\gamma\delta}, g^{\alpha\beta})$ (composed of the curvature tensor and its contractions with the metric), such that taking all the higher-derivative couplings to zero reduces it to the Einstein-Hilbert action. In other words, it must admit an Einstein gravity limit. Evaluate it on the following metric:

$$ds^2 = -N^2(r)f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2 d\Omega^2 \tag{4.4}$$

and define the Lagrangian $L_{N,f}$ as,

$$L_{N,f}\left(r, \{f(r), f'(r), f''(r), ...\}, \{N(r), N'(r), N''(r), ...\}\right) = N(r)\,r^{D-2}\,\mathcal{L}|_{\bar{g}_{\mu\nu}} \tag{4.5}$$

Call the expression obtained by evaluating L on 4.4 with $N = 1$ as L_f. If the variation of L_f with respect to $f(r)$ vanishes 'identically', i.e,

$$\frac{\delta L_f}{\delta f} = \frac{\partial L}{\partial f} - \frac{d}{dr}\left(\frac{\partial L_f}{\partial f'(r)}\right) + \frac{d^2}{dr^2}\left(\frac{\partial L_f}{\partial f''(r)}\right) - .... = 0 \tag{4.6}$$

Then, the following always holds,

- The theory admits solutions of the form 4.4. If L has a maximum of n derivatives of the metric, then $f(r)$ satisfies a differential equation of order $\leq 2n+2$
- Such solutions coincide with the exterior field of a spherically symmetric mass distribution, which is fully characterised by the mass of the body M.
- The theory only propagates the spin-2 graviton in maximally symmetric backgrounds.

Theories for which the solution of $f(r)$ is given by an algebraic equation are known as 'Quasi-topological' (QT), gravities. Those theories which lead to a differential equation for $f(r)$ are known as 'Generalized' Quasi-Topological gravities (GQT).

Let us look into the proof of the first claim since it is quite illuminating for constructing GQTs. First, consider the action with the Lagrangian evaluated on the ansatz 4.4,

$$S[N,f] = \Omega_{(D-2)} \int dt \int dr L_{N,f} \tag{4.7}$$

The equations of motion of this theory are given by the variation of this action with respect to to the metric as,

$$\mathcal{E}_{\alpha\beta} = \frac{1}{\sqrt{-g}} \frac{\delta L}{\delta g^{\alpha\beta}} \tag{4.8}$$

Using the chain rule, one can write the tt and the rr component of the equation of motion as,

$$\begin{aligned} \frac{1}{\Omega_{D-2}\, r^{D-2}} \frac{\delta S}{\delta N} &= \frac{2\,\mathcal{E}_{tt}}{f N^2} \\ \frac{1}{\Omega_{D-2}\, r^{D-2}} \frac{\delta S}{\delta f} &= \frac{\mathcal{E}_{tt}}{N f^2} + N\mathcal{E}_{rr} \end{aligned} \tag{4.9}$$

From the ansatz 4.4, notice that if we scale $N \to \alpha N$, it is equivalent to scaling t the same way. Hence, the action 4.7 also scales as $S \to \alpha S$,

$$S[\alpha N, f] = \Omega_{D-2} \int (\alpha dt) \int dr\, L_{N,f} = \alpha\, S[N,f] \tag{4.10}$$

This implies that we can write the Lagrangian as a homogeneous function in N of degree one. The Lagrangian is composed of the functions f, N and their derivatives. As we defined previously, setting $N = 1$ gives us L_f. Accounting for these properties, we can express the Lagrangian in the following form:

$$L_{N,f} = NL_f + \sum_i N^{(i)} F_i + \sum_i \sum_j \frac{N^{(i)} N^{(j)}}{N^2} F_{ij} + \, ... \tag{4.11}$$

In the above expression, $N^{(i)}$ refers to the ith derivative of N with respect to r. We require this Lagrangian to be such that the corresponding equations of motion admits solution with $N = 1$ or some constant. Since all the terms except the first one contains derivatives of N, their contribution to the equation of motion would also have these derivatives of N and hence would vanish when we put $N = 1$. However, this is not the case for the first term – unless L_f is a total derivative. This requirement is equivalent to the hypothesis 4.6. Say L_f can be expressed as total derivative as $L_f = F'_D$, where F_D is some function of N and f. Then,

$$S[N, f] = \Omega_{D-2} \int dt \int dr \, N \left(F_D + \sum_i N^{i-1} F_i \right)' + \mathcal{O}(N'^2/N^2) \tag{4.12}$$

From the above action notice that the condition $\delta S/\delta f = 0$ is satisfied for $N = 1$ as all terms in the corresponding equation of motion will have derivatives of N. Moving on to the equation of motion with respect to N, we see that the only contribution – on setting $N = 1$ – comes from the bracket in the first term above:

$$\frac{\delta S}{\delta N} = 0 \equiv \left(F_D + \sum_i (-1)^i F_i^{(i-1)} \right)' = 0 \tag{4.13}$$

Integrating this equation once, we have,

$$F_D + \sum_i (-1)^i F_i^{(i-1)} = M \tag{4.14}$$

where M is some constant. Now, let us look into the differential order of the above equation. It must be atmost $2n+4-1 = 2n+3$ if we started with a Lagrangian which has $2n$ covariant derivatives. However, let us inspect the r component of the Bianchi identity $\nabla_\alpha \mathcal{E}^{\alpha\beta} = 0$:

$$\frac{d\mathcal{E}^{rr}}{dr} + \left(\frac{2}{r} - \frac{1}{2}f^{-1}f'\right)\mathcal{E}^{rr} + \frac{1}{2}ff'\mathcal{E}^{tt} - rf\mathcal{E}^{\theta\theta} - sin^2(\theta)rf\mathcal{E}^{\phi\phi} = 0 \tag{4.15}$$

Since each term besides the first would generally have $2n + 4$ derivatives, we infer from the first term that $\mathcal{E}_{rr}$ must contain at most $2n+3$ derivatives, which gives us a $(2n+3)$th order differential equation for f. However, we reached 4.14 by already integrating the entire equation of motion once – so it must have atmost $2n+2$ derivatives of the metric. This concludes the proof of the first claim. The next two claims involve elaborate analytical proofs which do not contribute conceptually, hence we shall not discuss them here.

All the discussion, we have had until now is enough to lay down the procedure of how one can assemble a GQT at any curvature order,

- Take a general Lagrangian (admitting an Einstein gravity limit) with all possible scalar contractions of the curvature tensor at some curvature order.
- Evaluate the Lagrangian on the metric 4.2
- Adjust the couplings of the Lagrangian such that it satisfies $\delta L_f/\delta f = 0$

Then to find $f(r)$ after obtaining the couplings from the above steps:

- Find $L_{N,f}$ using 4.5 and then identify F_D and F_i by grouping terms having N, $N^{(1)}$ and so on.
- Assemble 4.14 and solve the differential equation for $f(r)$

This concludes the discussion on the procedure to construct a GQT at any curvature order and find the corresponding spherically symmetric vacuum solution to the same.

One remarkable property of GQTs [27] is that in the effective field theory regime, any theory (without covariant derivatives of the curvature tensor) can be mapped to a GQT by appropriate field redefinitions. As we saw before, any curvature scalar containing a Ricci tensor or a Ricci scalar can be introduced or removed by a field redefinition. Call such terms as ‘reducible’. Call any curvature invariant which does not contain either the Ricci tensor or scalar as ’irreducible’. Assume there for a given curvature order, there exists atleast one GQT in any dimension. If one can show that all irreducible invariants when evaluated on the metric 4.2 are identical to each other upto the evaluation of reducible terms on the same [27], it implies that any theory can be mapped to a GQT, by our assumption that there exists atleast a single GQT at each curvature order. This relies on the fact is that the evaluation of a Lagrangian on 4.2, which we call L_f in the discussions above, is what solely decides if a theory is GQT or not.

At every curvature order n and $D \geq 5$, there exists $n-2$ GQT densities and 1 unique QT density [12]. In 4D, there exists no QT and a single GQT at each order in curvature, which in cubic order is the Einsteinian cubic gravity.

4.3 Classifying theories

Having discussed a wide range of theories so far, each with their own requirements and properties, let us create a map (of theories composed solely of the curvature tensor and its contractions with the metric) to see how they are related to each other. We shall classify theories in two broad categories, i) those which are Einsteinian on maximally symmetric backgrounds, ii) those which admit vacuum spherically symmetric (VSSS) solutions.

4.3.1 Einsteinian on maximally symmetric backgrounds

- 1. GR
    - Admits Schwarzschild black holes in all dimensions
  2. Lanczos-Lovelock models
    - Lovelock theories always satisfy the condition 2.5 [28] and have dimension independent couplings.
  3. Einsteinian cubic gravity (ECG) [26] and same class of theories at higher order in curvature [29].
    - Have dimension independent couplings

- ECG only admits solutions of the form 2.10 in D=4 [29].
- Higher curvature theories of the same class (dimension independent couplings and Einsteinian in maximally symmetric backgrounds) admits VSSS solutions only in specific dimensions where they coincide with GQTs.

4. GQTs
   - By construction they satisfy 2.5 and have dimension-dependent couplings.

4.3.2 Non-Einsteinian on maximally symmetric backgrounds

- They might admit VSSS solutions of the form 2.10
  - E.g. f(R) gravity: But the solution does not correspond to the exterior gravitational field of a static spherically symmetric (SSS) source [11].

4.3.3 Admits VSSS solutions

- If they satisfy 2.5, they are always Einsteinian [11] [12].
- f(R) gravity violates 2.5, but admits a solution of the form 2.10, which as stated above, is unnatural.

If we neglect the fact that Lovelock theories have a topological origin unlike (G)QTs, Lovelock is actually subset of (G)QTs. Hence, we have the following relations,

$$\begin{aligned}
&\text{Satisfies } (\delta L_f/\delta f = 0) \equiv \text{ GQT}\\
&\text{Lovelock} \subset \text{QT} \subset \text{GQT}\\
&\text{ECG in 4D} \subset \text{GQT;}\\
&\text{GQT} \subset \text{Admits VSSS solutions}
\end{aligned} \tag{4.16}$$

Chapter 5

Appendix

A: Transverse-traceless gauge

In linearized gravity, the transverse-traceless (TT) gauge corresponds to having $h_{0i} = 0$ and $h = 0$. Let us see in details the steps required to arrive at this gauge. The linearized gauge transformations is given as,

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{5.1}$$

First, we impose the 'de-Donder' gauge, which is given as,

$$\partial^\nu h_{\mu\nu} = \frac{1}{2} \partial_\mu h \tag{5.2}$$

This requires one to choose ξ such that,

$$\Box \xi_\mu = \partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h \tag{5.3}$$

where $h_{\mu\nu}$ is the metric perturbation before the gauge transformation. This utilises 4 out of the 8 gauge degrees of freedom available. Now, any further gauge transformations of the form 5.1 must satisfy 5.3, which implies that $\Box \xi_\mu = 0$. We can use this residual freedom to set the trace of the metric to be zero,

$$\begin{aligned} & h' = h - 2\,\partial^\alpha \xi_\alpha = 0 \\ \implies & h + 2\partial_0\xi_0 - 2\,\partial_i \xi_i = 0 \\ \implies & h + 2\partial_0\xi_0 - 2\,\partial_i \int dt(h_{0i} - \partial_i \xi_0) = 0 \\ \implies & h + 2\partial_0\xi_0 - 2 \int dt(\partial_i h_{0i} - \partial_i\partial_i \xi_0) = 0 \end{aligned} \tag{5.4}$$

From $\Box\xi_\mu = 0$, we have,

$$\partial_i\partial_i\xi_0 = \partial_0\partial_0\xi_0 \tag{5.5}$$

Using the de-Donder gauge, we can write,

$$\begin{aligned} \partial^\nu h_{\nu 0} &= \frac{1}{2}\partial_0 h \\ \implies \quad -\partial_0 h_{00} + \partial_i h_{i0} &= \frac{1}{2}\partial_0 h \\ \implies \quad \partial_i h_{i0} &= \partial_0 h_{00} + \frac{1}{2}\partial_0 h \end{aligned} \tag{5.6}$$

Putting the expressions 5.5 and 5.6 in 5.4, we have,

$$\begin{aligned} & h + 2\partial_0\xi_0 - 2\int dt(\partial_0 h_{00} + \frac{1}{2}\partial_0 h - \partial_0\partial_0\xi_0) = 0 \\ \implies & h + 2\partial_0\xi_0 - 2h_{00} - h + 2\partial_0\xi_0 = 0 \\ \implies & \xi_0 = \frac{1}{2}\int dt\, h_{00} \end{aligned} \tag{5.7}$$

Hence, fixing ξ_0 in the above way is sufficient to eliminate the trace h'.

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