Study of Weighted Pseudo Compact Almost Automorphic Solutions and its Stability Analysis

A Thesis

submitted to

Indian Institute of Science Education and Research Pune in partial fulfilment of the requirements for the BS-MS Dual Degree Programme

by

Priyanshu Chourasiya

IISER PUNE

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2025

Supervisor - Prof. Syed Abbas

Certificate

This is to certify that this dissertation entitled **Study of Weighted Pseudo Compact Almost Automorphic Function and its Stability Analysis** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by **Priyanshu Chourasiya** at Indian Institute of Technology, Mandi under the supervision of **Prof. Syed Abbas**, Professor, Department of Mathematics, during the academic year 2024-25.

Prof. Syed Abbas

Committee :

Prof. Syed Abbas

Dr. Anisa Chorwadwala

This thesis is dedicated to

My Parents

Declaration

I hereby declare that the matter embodied in the report entitled **Study of Weighted Pseudo Compact Almost Automorphic Function and its Stability Analysis** are the results of the work carried out by me at the Department Mathematics , Indian Institute of Science Education and Research, Pune, under the supervision of **Syed Abbas** and the same has not been submitted elsewhere for any other degree

Priyanshu Chourasiya

Abstract

In this thesis, we are looking a differential equation with time-varying delays and prove the uniqueness and existence of weighted pseudo compact almost automorphic solution of the differential equation with certain assumptions and then we have done it's stability analysis. In proving the result, we use fixed point theorem and some of the properties of a weighted pseudo-compact almost automorphic function. We use some results and obtain the result for the existence and uniqueness of weighted pseudo-compact almost automorphic solutions for the Nicholson model. In this thesis, we have also mentioned weighted pseudo-compact almost automorphic functions's properties. We have also done it's stability analysis.

Acknowledgement

I would like to express my profound gratitude to **Prof. Syed Abbas**, of IIT Mandi - Mandi School of Mathematical and Statistical Sciences, and **Dr. Anisa Chorwadwala**, of IISER Pune, Department of Mathematics, for their contributions and guidance to the completion of my project **'Study of Weighted Pseudo Compact Almost Automorphic Function and its Stability Analysis'.**

Thanks to both of them for their time and effort that they provided me throughout the year. Their useful advice and suggestions were really helpful to me during the completion of the project. In this regard, I am forever grateful to you.

Priyanshu Chourasiya

Table of Contents

1 Introduction

In the past fourteen years, questions related to existence, uniqueness, and properties of the solutions for time-delay differential equations have been looked by a lot of mathematicians. We can see time delays in several population models because of several reasons such as migration, maturity, etc. Large number of time delay population models are based on delay differential equations. The qualitative and quantitative analysis of time-delay population models have been studied by many mathematicians.

In the theory of the dynamics of several biological and ecological phenomena we can see that dynamics may be periodic with certain errors. This consideration arrives at the theory of almost periodic (AP), almost automorphic (AA), compact almost automorphic, weighted pseudo almost automorphic functions.

Weighted pseudo-compact almost automorphic functions are more generalized than periodic, almost automorphic , pseudo almost automorphic, compact almost automorphic functions. It is much wider than all other functions and it covers a larger range of dynamics. In this thesis, our focus will be on these functions.

The purpose of the work is to validate the uniqueness and existence of weighted positive pseudo compact almost automorphic solution for delay differential equations and do it's stability analysis.

The stability analysis of the solution is done by using Halanay's inequality and the Lyapnov stability.

The explanation of the thesis are as follows: In the second section, the literature of compact almost automorphic, weighted pseudo compact almost automorphic functions and delay differential equation is given. In the third section, we have given the application of pseudo almost automotphic functions to solve abstract differential equation. Then, we explain some model that is based on delay differential equation . And we present the existence and uniqueness of of pseudo compact almost automorphic solution of the Nicholson Model and it's stability analysis. The main result of the thesis is presented in the sixth section,i.e. the uniqueness and existence of the weighted pseudo-compact almost automorphic solution of Nicholson Model under some suitable assumptions. We also provide conditions for the stability of the weighted pseudo compact almost automorphic solution using Lyapnov function. The seventh section contains examples.

2 Preliminaries

2.1 Delay Differential Equation

Delay differential equation (DDEs) also known as time delay differential equation, is differential equation in which the derivative of the unknown function at a certain time is obtained in terms of the values of the function at previous times.

Following are the notations for delay differential equations.

For $\zeta \geq 0$, we have $C = C([-\zeta, 0], \mathbb{R})$, that is the Banach space with the norm

$$\|\psi\|_\zeta = \sup_{-\zeta \leq \phi \leq 0} |\psi(\phi)|,$$

where $|\cdot|$ is the absolute value in $\mathbb{R}$. $\forall\ c \in \mathbb{R}$ is identified in C with the constant function $\psi(\phi) = c\ \forall\ \phi \in [-\zeta, 0]$. A general functional differential equation is

$$\dot{x}(t) = g(t, x_t)$$

with the functional $g : \mathbb{R} \times C \supset B \to \mathbb{R}$ and x_t corresponds to the translation of a solution $x(t)$ on the interval $[t - \zeta, t]$ to the interval $[-\zeta, 0]$; i.e., $x_t \in C$ is given by $x_t(\phi) = x(t+\phi)$ for $\phi \in [-\zeta, 0]$.

A function x is said to be a solution of the above equation on $[-\zeta, P)$ if $\exists\ P > 0$ such that $x \in C([-\zeta, P), \mathbb{R})$, $(t, x_t) \in B$, and $x(t)$ satisfies above equation for $t \in [0, P)$.

$\forall\ \psi \in C$, $x(t; 0, \psi)$ is a solution of above equation with initial value ψ at 0 if $\exists$ an $P > 0$ such that $x(t; 0, \psi)$ is a solution of above equation on $[-\zeta, P)$ and $x_0(t; 0, \psi) = \psi$.

We consider that the functional g is regular and continuous to make sure the uniqueness and existence of the differential equation with initial value ψ at 0.

2.2 Definitions

2.2.1 Uniformly Persistent[1]: A delay differential equation $\dot{x} = g(t, x_t)$ is known as *uniformly persistent* in C_0 if $\forall$ solutions $x(t, 0, \psi)$ with $\psi \in C_0$ are defined on $[0, \infty)$ and $\exists\ \delta_1 > 0$ such that

$$\liminf_{t \to \infty} x(t, 0, \psi) \geq \delta_1 \quad \text{for} \quad \psi \in C_0.$$

2.2.2 Permanent [1]: The delay differential equation $\dot{x} = g(t, x_t)$ is said to be *permanent* in C_0 if $\forall$ solutions $x(t, 0, \psi)$ with $\psi \in C_0$ are defined on $[0, \infty)$ and $\exists$ positive constants δ_1, δ_2 such that, $\forall$ $\psi \in C_0$, $\exists$ $t_0 = t_0(\psi)$ for which

$$\delta_1 \leq x(t, 0, \psi) \leq \delta_2 \quad \text{for} \quad t \geq t_0.$$

2.2.3 Relatively Dense: A set $P \subset \mathbb{R}$ is called relatively dense in $\mathbb{R}$ $\exists$ $r > 0$ such that $\forall a \in \mathbb{R}$, $[a, a + r] \cap P \neq \emptyset$.

2.2.4 Almost Periodic: A continuous function $g : \mathbb{R} \to Y$ is called *almost periodic* if $\forall$ $\delta > 0$ $\exists$ a relatively dense set $Q(\delta, g)$ such that

$$\sup_{t \in \mathbb{R}} |g(t + \zeta) - g(t)| < \delta$$

$\forall$ $\zeta \in Q(\delta, g)$.

We can denote it by $AP(Y)$.

2.2.5 Almost Automorphic [1]: Suppose Y is a Banach space. A function $g(t) \in C(\mathbb{R}, Y)$ is known as *almost automorphic* if $\forall$ sequence of $\mathbb{R}$ (a_n), $\exists$ a subsequence (a_{n_k}) such that

$$j(t) := \lim_{k \to \infty} g(t + a_{n_k})$$

is well defined $\forall$ $t \in \mathbb{R}$ and

$$\lim_{k \to \infty} j(t - a_{n_k}) = g(t)$$

$\forall$ $t \in \mathbb{R}$.

We can denote it by AA.

2.2.6 Compact Almost Automorphic Function [1] Suppose Y is a Banach space. A function $g(t) \in C(\mathbb{R}, Y)$ is known to be almost automorphic if $\forall$ sequence of $\forall$ (a_n), $\exists$ a subsequence (a_{n_k}) such that

$$j(t) := \lim_{k \to \infty} g(t + a_{n_k})$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{k \to \infty} j(t - a_{n_k}) = g(t)$$

for each $t \in \mathbb{R}$.

If all of the above limits are uniformly maintained in compact subsets of $\mathbb{R}$, then g is known as *compact almost automorphic*.

We can denote it by KAA.

2.2.7 Bi-Almost Automorphic Function [1]: A function $Q(t, s)$ is known to be *bi-almost automorphic*, if $\forall$ sequence q_n there is a subsequence q_{n_k} and $Q^*(t,q)$ such that

$$Q(t+q_{n_k}, q+q_{n_k}) - Q^*(t,q) \to 0 \quad \text{as} \quad n \to \infty,$$

and

$$Q^*(t-q_{n_k}, q-q_{n_k}) - Q(t,q) \to 0 \quad \text{as} \quad n \to \infty.$$

We denote

$$PAA_0(\mathbb{R}, Y) = \left\{ g(t) \in BC(\mathbb{R}, Y) : \lim_{s\to+\infty} \frac{1}{2s} \int_{-s}^{s} \|g(t)\|\, dt = 0 \right\}.$$

Define:

$$PAA_0(\mathbb{R}\times Y, Y) = \left\{ \psi(t,y) \in BC(\mathbb{R}\times Y, Y) : \lim_{s\to\infty} \frac{1}{2s} \int_{-s}^{s} \|\psi(t,y)\| dt = 0 \right\}$$

uniformly for y in any bounded subset of Y.

Now, we define, for $\eta \in V_\infty$, define

$$PAA_0(\mathbb{R}\times Y, \eta) := \left\{ g \in BC(\mathbb{R}\times Y, Y) : \lim_{t\to\infty} \mu(t,\eta)^{-1} \int_{-t}^{t} \|g(s)\| \eta(s)\, ds = 0 \right\}.$$

2.2.8 Pseudo Almost Automorphic [**1**]: A continuous function $g : \mathbb{R} \to Y$ is known as *pseudo almost automorphic* if we can decomposed it in terms of

$$g = g_1 + g_2$$

where $g_1 \in AA(\mathbb{R}, Y))$ and $g_2 \in PAA_0(\mathbb{R}, Y)$.

We can denote it by $PAA(\mathbb{R}, Y)$.

2.2.9 Pseudo Compact Almost Automorphic [**1**]: A continuous function $g : \mathbb{R} \to Y$ is known as *pseudo compact almost automorphic*, if we can decompose it in terms of

$$g = g_1 + g_2$$

where $g_1 \in KAA(\mathbb{R}, Y)$ and $g_2 \in PAA_0(\mathbb{R}, Y)$.

We can denote it by $PKAA(\mathbb{R}, Y)$.

Suppose V is the set of all functions $\eta : \mathbb{R} \to (0, \infty)$ which are locally integrable and positive over $\mathbb{R}$.
$\forall\ t > 0$, define

$$\mu(t,\eta) := \int_{-t}^{t} \eta(x)\, dx$$

$\forall\ \eta \in V$.

Define

$$V_\infty := \left\{ \eta \in V : \lim_{t\to\infty} \mu(t,\eta) = \infty \right\},$$

$$V_b := \left\{ \eta \in V_\infty : \eta \text{ is bounded and } \inf_{x\in\mathbb{R}} \eta(x) > 0 \right\}.$$

We have, $V_b \subset V_\infty \subset V$.

Now, we define, for $\eta \in V_\infty$, define

$$PAA_0(\mathbb{R},\eta) := \left\{ g \in BC(\mathbb{R},Y) : \lim_{t\to\infty} \mu(t,\eta)^{-1} \int_{-t}^{t} \|g(s)\|\eta(s)\,ds = 0 \right\}.$$

2.2.10 Weighted ergodic zero: Let $\eta \in V_\infty$. A set $P \subset \mathbb{R}$ is known as *weighted ergodic zero set* if

$$\lim_{s\to+\infty} \frac{\int_{[-s,s]\cap P} \eta(t)\,dt}{\mu(s,\eta)} = 0.$$

2.2.11 Weighted Pseudo Almost Automorphic[1]: A continuous function $g : \mathbb{R} \to Y$ is called *weighted pseudo almost automorphic* if we can decompose it in terms of

$$g = g_1 + g_2$$

where $g_1 \in AA(\mathbb{R},Y)$ and $g_2 \in PAA_0(\mathbb{R},\eta)$.

We can denote it by $WPAA(\mathbb{R},Y)$.

2.2.12 Weighted Pseudo Compact Almost Automorphic[1]: A continuous function $g : \mathbb{R} \to Y$ is known as *weighted pseudo compact almost automorphic* if we can write it in terms of

$$g = g_1 + g_2$$

where $g_1 \in KAA(\mathbb{R},Y)$ and $g_2 \in PAA_0(\mathbb{R},\eta)$.

We can denote it by $WPKAA(\mathbb{R},Y)$ or $PKAA(Y,\eta)$.

2.2.13 Weighted Pseudo Almost Periodic [1]: Let $\eta \in V_\infty$. A function $g \in BC(\mathbb{R},Y)$ is known as *weighted pseudo almost periodic* if we can write it in terms of $g = h + l$, where $h \in AP(Y)$ and $l \in PAA_0(Y,\eta)$.

We denote it by $PAP(Y,\eta)$.

2.2.14 Pseudo Almost Automorphic [1]: A continuous function $g : \mathbb{R} \times Y \to Y$ is called pseudo almost automorphic if we can can write it in terms of

$$g = g_1 + g_2$$

where $g_1 \in AA(\mathbb{R} \times Y, Y))$ and $g_2 \in PAA_0(\mathbb{R} \times Y, Y)$.

We can denote it by $PAA(\mathbb{R} \times Y, Y)$.

2.2.15 Pseudo Compact Almost Automorphic[1]: A continuous function $g : \mathbb{R} \times Y \to Y$ is called pseudo compact almost automorphic, if we can decompose it in terms of

$$g = g_1 + g_2$$

where $g_1 \in KAA(\mathbb{R} \times Y, Y)$ and $g_2 \in PAA_0(\mathbb{R} \times Y, X)$.

We can denote it by $PKAA(\mathbb{R} \times Y, Y)$.

2.2.16 Weighted Pseudo Almost Automorphic [1]: A continuous function $g : \mathbb{R} \times Y \to Y$ is called weighted pseudo almost automorphic if we can decompose it in terms of

$$g = g_1 + g_2$$

where $g_1 \in AA(\mathbb{R} \times Y, Y)$ and $g_2 \in PAA_0(\mathbb{R} \times Y, \eta)$.

We can denote it by $WPAA(\mathbb{R} \times Y, Y)$.

2.2.17 Weighted Pseudo Compact Almost Automorphic [1]: A continuous function $g : \mathbb{R} \times Y \to Y$ is called weighted pseudo compact almost automorphic if we can decompose it in terms of

$$g = g_1 + g_2$$

where $g_1 \in KAA(\mathbb{R} \times Y, Y)$ and $g_2 \in PAA_0(\mathbb{R} \times Y, \eta)$.

We can denote it by $WPKAA(\mathbb{R} \times Y, Y)$.

2.2.18 Weighted Pseudo Almost Periodic: Let $\eta \in V_\infty$. A function $g \in BC(\mathbb{R} \times Y, Y)$ is known as *weighted pseudo almost periodic* if it is written as $g = l + f$, where $l \in AP(\mathbb{R} \times Y)$ and $f \in PAA_0(\mathbb{R} \times Y, \eta)$.

2.3 Properties and theorems of Weighted Pseudo Compact Almost Automorphic functions

Lemma 2.3.1 *A function g is almost automorphic and uniformly continuous iff g is compact almost automorphic.[**1**]*

$$AP(\mathbb{R}, Y) \subset KAA(\mathbb{R}, Y) \subset AA(\mathbb{R}, Y).$$
$$PAP(\mathbb{R}, Y) \subset PKAA(\mathbb{R}, Y) \subset WPKAA(\mathbb{R}, Y).$$

Lemma 2.3.2 *Decomposition of pseudo compact almost automorphic function $g = g_1 + g_2$, where $g_1 \in KAA(\mathbb{R}, Y)$ and $g_2 \in PAA_0(\mathbb{R}, Y)$,* *is unique; i.e.,* $g = g_1 \oplus g_2$.

Proof. We know that $g_1(\mathbb{R}) \subset \overline{g(\mathbb{R})}$.
Assume that $g = f_1 + l_1$ and $g = f_2 + l_2$.
Then $0 = (f_1 - f_2) + (l_1 - l_2) \in PKAA(\mathbb{R}, Y)$, where $(f_1 - f_2) \in KAA(\mathbb{R}, Y)$ and $(l_1 - l_2) \in PAA_0(\mathbb{R}, Y)$.
$f_1 - f_2 = 0$. Consequently, $l_1 - l_2 = 0$. □

Lemma 2.3.3 *If* $g(\cdot) \in PKAA(\mathbb{R}, \mathbb{R})$, $\mu(\cdot) \in KAA(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R}^+)$, *and* $\mu(t) \leq \mu^* < 1$, *then* $t \mapsto g(t - \mu(t)) \in PKAA(\mathbb{R}, \mathbb{R})$.

Proof. Since $g(\cdot) \in PKAA(\mathbb{R}, \mathbb{R})$. So, we can write $g(\cdot) = g_1(\cdot) + g_2(\cdot)$, where $g_1(\cdot) \in KAA(\mathbb{R}, \mathbb{R})$ and $g_2(\cdot) \in PAA_0(\mathbb{R}, \mathbb{R})$.

It follows that

$$g(t - \mu(t)) = g_1(t - \mu(t)) + g_2(t - \mu(t)).$$

Claim : $g_1(t - \mu(t)) \in KAA(\mathbb{R}, \mathbb{R})$.

Suppose $(a_m)_m$ is a sequence of $\mathbb{R}$. Then $\exists$ a subsequence $(a'_m)_m \subset (a_m)_m$, a function $g_1 : \mathbb{R} \to \mathbb{R}$, and a function $\mu : \mathbb{R} \to \mathbb{R}^+$ such that

$$g_1(t + a'_m) \to g_1(t), \quad g_1(t - a'_m) \to g_1(t),$$
$$\mu(t + a'_m) \to \mu(t), \quad \mu(t - a'_m) \to \mu(t),$$

in which all the above convergences hold uniformly on compact subsets of $\mathbb{R}$.
Suppose P is a compact subset of $\mathbb{R}$. Then $\exists$ a compact subset $P_N \subset P$ and an integer N such that, $\forall\, t \in P_N$,

$$\sup_{t \in P} |g_1(t + a'_m - \mu(t + a'_m)) - g_1(t - \mu(t))| \to 0 \quad \text{as } n \to \infty.$$

Using the above argument, we can write

$$|g_1(t_m - a_m - \mu(t_m - a'_m)) - g_1(a'_m - \mu(a'_m))| \to 0 \quad \text{as } m \to \infty$$

for each $t \in P$.

We conclude that $t \mapsto g_1(t - \mu(t)) \in AA(\mathbb{R}, \mathbb{R})$.

Claim $t \mapsto g_1(t - \mu(t))$ is uniformly continuous.

If $(b'_m)_m$ and $(a'_m)_m$ are two sequences such that $|b'_m - a'_m| \to 0$, then using the uniform continuity of $\sigma(\cdot)$, we have

$$|b'_m - \mu(t'_m) - a'_m - \mu(a'_m)| \leq |b'_m - a'_m| + |\mu(b'_m) - \mu(a'_m)| \to 0 \quad \text{as } m \to \infty.$$

Now, using uniform continuity of $g_1(\cdot)$, we can write that

$$|g_1(b'_m - \mu(b'_m)) - g_1(a'_m - \mu(a'_m))| \to 0 \quad \text{as } m \to \infty.$$

Therefore, $t \mapsto g_1(t - \mu(t))$ is uniformly continuous.

From previous lemma we can say that
$$t \mapsto g_1(t-\mu(t)) \in KAA(\mathbb{R},\mathbb{R}).$$

Claim : $g_2(t-\mu(t)) \in PAA_0(\mathbb{R},\mathbb{R})$.

Since, we have $\mu(\cdot) \in KAA(\mathbb{R},\mathbb{R}^+) \cap C^1(\mathbb{R},\mathbb{R}^+)$ and $\mu(t) \leq \mu^* < 1$.

Then, $\exists$ a positive constant, denoted as
$$\mu^+ := \sup_{a\in\mathbb{R}}\{\mu(s)\}.$$
We consider $a = t-\mu(t)$ and $da = (1-\mu(t))\,dt$, which implies
$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}|g_2(t-\mu(t))|\,dt &= \frac{1}{2T}\int_{-T-\mu(-T)}^{T-\mu(T)}|g_2(a)|\frac{1}{1-\mu(a)}\,da\\ &\leq \frac{1}{1-\mu^*}\frac{T+\mu^+}{2(T+\mu^+)}\int_{-T-\mu^+}^{T+\mu^+}|g_2(a)|\,da.\end{aligned}$$
From above and the fact that $g_2 \in PAA_0(\mathbb{R},\mathbb{R})$, we get
$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|g_2(t-\mu(t))|\,dt = 0.$$
□

Lemma 2.3.4 *Suppose g and l is in $KAA(\mathbb{R},\mathbb{R})$. Then the product $g\cdot l$ is also in $KAA(\mathbb{R},\mathbb{R})$.*

Proof. Suppose $(a_m)_m$ is a sequence such that $\exists$ functions $g:\mathbb{R}\to\mathbb{R}$ and $l:\mathbb{R}\to\mathbb{R}$ satisfying
$$\begin{aligned} g(t+a_m) \to g'(t), \quad g'(t-a_m) \to g(t),\\ l(t+a_m) \to l'(t), \quad l'(t-a_m) \to l(t),\end{aligned}$$
as $m\to\infty$, where all the above convergences maintain uniformly on compact subsets of $\mathbb{R}$.

Suppose P is a compact subset of $\mathbb{R}$. Then, $\forall\ t \in P$, we have
$$\begin{aligned}&|g(t+a_m)l(t+a_m) - g'(t)l'(t)|\\ &\leq |g(t+a_m)l(t+a_m) - g'(t)l(t+a_m)| + |g'(t)l(t+a_m) - g'(t)l'(t)|\\ &\leq \|l\|_\infty|g(t+a_m) - g'(t)| + \|g\|_\infty|l(t+a_m) - l'(t)|.\end{aligned}$$
It follows that $g(t+a_m)l(t+a_m) \to g'(t)l'(t)$ uniformly on P.

Similarly, we will show that
$$g(t-a_m)l(t-a_m) \to g'(t)l'(t)$$
uniformly on any compact interval $[x,y]$.

We conclude that
$$t \mapsto (l\cdot g)(t) = g(t)l(t) \in KAA(\mathbb{R},\mathbb{R}).$$
□

Lemma 2.3.5 *Suppose $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lipschitz continuous function, i.e.,*

$$|g(x) - g(y)| \leq L_g|x - y|, \quad x, y \in [0, +\infty).$$

Then $g(u(t)) \in PKAA(\mathbb{R}, \mathbb{R}^+)$, provided $u(t) = u_1(t) + u_2(t) \in PKAA(\mathbb{R}, \mathbb{R}^+)$, where $u_1(t) \in AA(\mathbb{R}, \mathbb{R}^+)$ and $u_2(t) \in PAA(\mathbb{R}, \mathbb{R}^+)$.

Proof. Since $u = u_1 + u_2 \in PKAA(\mathbb{R}, \mathbb{R}^+)$, the function $g(u(t))$ can be decomposed as

$$g(u(t)) = g(u_1(t)) + [g(u(t)) - g(u_1(t))].$$

Denote

$$Q_1(t) = g(u_1(t)),$$
$$Q_2(t) = g(u(t)) - g(u_1(t)).$$

Claim:$Q_1(t) \in KAA(\mathbb{R}, \mathbb{R}^+)$.

Since $u_1 \in KAA(\mathbb{R}, \mathbb{R}^+)$, $\exists$ a function u_1^* and a subsequence $\{s_{m_i}\}$ $\forall$ sequence $\{s_m\}$ such that

$u_1(t + s_{m_i}) - u_1^*(t) \to 0$ and $u_1^*(t - s_{m_i}) - u_1(t) \to 0$ as $n \to \infty$.

Denote $Q_1^*(t) = g(u_1^*(t))$, then

$Q_1(t+s_{m_i}) - Q_1^*(t) = g(u_1(t+s_{m_i})) - g(u_1^*(t)) \leq L_g|u_1(t+s_{m_i}) - u_1^*(t)|$.

Thus

$$Q_1(t + s_{m_i}) - Q_1^*(t) \to 0 \text{ as } i \to \infty.$$

Similarly, we can prove that

$$Q_1^*(t - s_{m_i}) - Q_1(t) \to 0 \text{ as } i \to \infty.$$

Q_1 is uniformly continuous, since $|s_m - t_m| \to 0$ as $m \to \infty$, then

$|Q_1(s_m) - Q_1(t_m)| = |g(u_1(s_m)) - g(u_1(t_m))| \leq L_g|u_1(s_m) - u_1(t_m)| \to 0$ as $n \to \infty$

due to the uniform continuity of u_1.

Claim: $Q_2(t) \in PAA_0(\mathbb{R}, \mathbb{R}^+)$.

We can see that:

$$\frac{1}{2T}\int_{-T}^{T} Q_2(t)\,dt = \frac{1}{2T}\int_{-T}^{T} |[g(u(t)) - g(u_1(t))]|\,dt$$
$$\leq \frac{L_g}{2T}\int_{-T}^{T} |[u(t) - u_1(t)]|\,dt = \frac{L_g}{2T}\int_{-T}^{T} |u_2(t)|\,dt,$$

and since $u_2(t) \in PAA_0(\mathbb{R}, \mathbb{R}^+)$, the claim follows. This completes the proof.□

Lemma 2.3.6 *If the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ belong to*

$PKAA(\mathbb{R})$ and $L^1(\mathbb{R})$, respectively, then $g*f$ belongs to $PKAA(\mathbb{R})$.

Proof. Since $f \in PKAA(\mathbb{R},\mathbb{R})$, we have $(g*f)(t) = (g*f_1)(t) + (g*f_2)(t)$, where $f_1 \in KAA(\mathbb{R},\mathbb{R})$ and $f_2 \in PAA_0(\mathbb{R},\mathbb{R})$.

Claim $g*f_1 \in KAA(\mathbb{R},\mathbb{R})$.

Since $f_1 \in KAA(\mathbb{R},\mathbb{R})$, $\exists$ f_1^* and a subsequence $\{s_{n_i}\}$ for each sequence $\{s_i\}$ such that

$$f_1(t+s_{n_i}) - f_1^*(t) \to 0 \quad \text{and} \quad f_1^*(t-s_{n_i}) - f_1(t) \to 0$$

as $n \to \infty$.

Denote $v_1(t) := (g*f_1)(t) = \int_{-\infty}^{\infty} g(t-s)f_1(s)\,ds$ and $v_1^*(t) = \int_{-\infty}^{\infty} g(t-s)f_1^*(s)\,ds$.

We can take the difference between $v_1(t+s_{n_i})$ and $v_1^*(t)$ and making the substitution $r = t-s$, we obtain

$$v_1(t+t_{s_i}) - v_1^*(t) = \int_{-\infty}^{\infty} g(r)[f_1(t-r-s_{n_i}) - f_1^*(t-r)]\,dr.$$

Since $f_1(t-r-s_{n_i}) - f_1^*(t-r) \to 0$ as $n \to \infty$ and $g \in L^1(\mathbb{R})$, DCT implies $v_1(t+s_n) - v_1^*(t) \to 0$ as $n \to \infty$.

Moreover, v_1 is uniformly continuous. Since $|t_n - s_n| \to 0$ as $n \to \infty$, we get

$$v_1(t_n) - v_1(s_n) = \int_{-\infty}^{\infty} g(r)[f_1(r+t_n) - f_1(r+s_n)]\,dr.$$

Since $f_1(r+t_n) - f_1(r+s_n) \to 0$ as $n \to \infty$, DCT implies $v_1(t_n) - v_1(s_n) \to 0$ as $n \to \infty$.

$g*f_2 \in PAA_0(\mathbb{R},\mathbb{R})$

Now, we show that $g*f_2 \in PAA_0(\mathbb{R},\mathbb{R})$,

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |g*f_2(t)|\,dt = 0.$$

Using Fubini Theorem, we get

$$\frac{1}{2T}\int_{-T}^{T} |g*f_2(t)|\,dt \le \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} |g(t-s)|\,|f_2(s)|\,ds\,dt$$
$$= \int_{-\infty}^{\infty} |g(\xi)| \left(\frac{1}{2T}\int_{-T}^{T} |f_2(t-\xi)|\,dt\right) d\xi.$$

Since $f_2 \in PAA_0(\mathbb{R})$ and using invariance by translation of the mean, we get

$$\frac{1}{2T}\int_{-T}^{T}|f_2(t-\xi)|\,dt \to 0 \quad \text{as} \quad T\to\infty.$$

From $g\in L^1(\mathbb{R})$, we can verify DCT.
Hene, we can say that $g*f_2$ belongs to $PAA_0(\mathbb{R},\mathbb{R})$. $\square$

Corollary 2.3.1 *Suppose function $f:\mathbb{R}\to\mathbb{R}\in PKAA(\mathbb{R})$ and $\exists$ a bi-almost automorphic function $l(s,t)$ such that:*

$$|l(s,t)|\leq ce^{-\beta(s-t)}, \quad s\geq t,$$

then the function $\psi(s)=\int_{-\infty}^{s}l(s,t)f(t)\,dt$ belongs to the class $PKAA(\mathbb{R})$.

On V_∞, we use the following equivalence relation $\sim$ introduced :

$$(\mu_1\sim\mu_2) \iff \frac{\mu_2}{\mu_1}\in V_b.$$

Theorem 2.3.1 *Let $\mu_1,\mu_2\in V_\infty$. If $\mu_1\sim\mu_2$, then $WPAA(\mathbb{R},\mu_1)=WPAA(\mathbb{R},\mu_2)$.*

Proof. Let $\mu_1\sim\mu_2$. There exists $(p,q)\in\mathbb{R}^2$ such that

$$p\mu_1\leq\mu_2\leq q\mu_1.$$

Thus,

$$p\mu(r,\mu_1)\leq\mu(r,\mu_2)\leq q\mu(r,\mu_1),$$

and

$$\begin{aligned}\frac{p}{q}\frac{1}{\mu(r,\mu_1)}\int_{-r}^{r}\|\psi(s)\|_{\mu_1(s)}\,ds &\leq \frac{1}{\mu(r,\mu_2)}\int_{-r}^{r}\|\psi(s)\|_{\mu_2(s)}\,ds\\ &\leq \frac{p}{q}\frac{1}{\mu(r,\mu_1)}\int_{-r}^{r}\|\psi(s)\|_{\mu_1(s)}\,ds,\end{aligned}$$

Hence proof. $\square$

Lemma 2.3.7 *Let $g\in BC(\mathbb{R},Y)$. Then $g\in PAA_0(\mathbb{R},\eta)$ where $\eta\in U_b$ if and only if for every $\epsilon>0$,*

$$\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}mes(M_{r,\epsilon}(g))=0$$

where

$$M_{r,\epsilon}(g):=\{t\in[-r,r]\mid\|g(t)\|\geq\epsilon\}.$$

Proof.Necessity

With the help of contradiction, let $\exists\ \epsilon_0 > 0$ such that
$$\lim_{r\to\infty} \frac{1}{\mu(r,\eta)} \text{mes}(M_{r,\epsilon_0}(g)) \neq 0.$$
Then $\exists\ \delta > 0$ such that for every $n \in \mathbb{N}$,
$$\frac{1}{\mu(r_n,\eta)} \text{mes}(M_{r_n,\epsilon_0}(g)) \geq \delta \text{ for some } r_n > n.$$
We get
$$\frac{1}{\mu(r_n,\eta)} \int_{-r_n}^{r_n} \|g(s)\|\eta(s)\,ds = \frac{1}{\mu(r_n,\eta)} \int_{M_{r_n,\epsilon_0}(g)} \|g(s)\|\eta(s)\,ds$$
$$+\frac{1}{\mu(r_n,\eta)} \int_{[-r_n,r_n]-M_{r_n,\epsilon_0}(f)} \|g(s)\|$$
$$\geq \frac{1}{\mu(r_n,\eta)} \int_{M_{r_n,\epsilon_0}(f)} \|g(s)\|\eta(s)\,ds \geq$$
$$\epsilon_0 \frac{1}{\mu(r_n,\eta)} \int_{M_{r_n,\epsilon_0}(g)} \eta(s)\,ds \geq \epsilon_0\delta\gamma,$$
where $\gamma = \inf_{s\in\mathbb{R}} \eta(s)$. Contradiction of assumption.

Sufficient

Suppose
$$\lim_{r\to\infty} \frac{1}{\mu(r,\eta)} \text{mes}(M_{r,\epsilon}(g)) = 0.$$
Then $\forall\ \epsilon > 0$, $\exists\ r_0 > 0$ such that $\forall\ r > r_0$,
$$\frac{1}{\mu(r,\eta)} \text{mes}(M_{r,\epsilon}(g)) < \frac{\epsilon}{KM},$$
where $M := \sup_{t\in\mathbb{R}} \|g(t)\| < \infty$
and $K := \sup_{t\in\mathbb{R}} \eta(t) < \infty$.

Now we have
$$\frac{1}{\mu(r,\eta)} \int_{-r}^{r} \|g(s)\|\eta(s)\,ds = \frac{1}{\mu(r,\eta)} \int_{M_{r,\epsilon}(g)} \|g(s)\|\eta(s)\,ds$$
$$+\int_{[-r,r]-M_{r,\epsilon}(g)} \|g(s)\|\eta(s)\,ds$$
$$\leq MK \frac{1}{\mu(r,\eta)} \text{mes}(M_{r,\epsilon}(g)) + \epsilon \frac{1}{\mu(r,\eta)} \int_{[-r,r]-M_{r,\epsilon}(g)} \eta(s)\,ds$$
$$\leq 2\epsilon.$$
which show that
$$\lim_{r\to\infty} \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \|g(s)\|\eta(s)\,ds = 0,$$
,i.e., $g \in PAA_0(\mathbb{R},\eta)$. $\square$

2.4 Completeness of Weighted Pseudo Compact Almost Automorphic function

Theorem 2.4.1 *$\forall\ g \in PAA(Y, \eta)$, $\exists$ a decomposition $g = j_0 + h_0$, where $j_0 \in AA(Y)$ and $h_0 \in PAA_0(Y, \eta)$, such that*

$$\|j_0\| \leq \|g\|.$$

Proof. Suppose $g = j + h$, where $j \in AA(Y)$ and $h \in PAA_0(Y, \eta)$. Set

$$C_n = \left\{ t \in \mathbb{R} : \|j(t)\| > \|g\| + \frac{1}{n} \right\}, \quad n \in \mathbb{N}.$$

Note that $C_n \subset \left\{ t \in \mathbb{R} : \|h(t)\| \geq \frac{1}{n} \right\}$, it can be seen from previous lemma that every C_n is a μ-ergodic zero set.

Define

$$j_n(t) = \begin{cases} j(t), & t \notin C_n, \\ \left(\|g\| + \frac{1}{n}\right) \cdot \frac{j(t)}{\|j(t)\|}, & t \in C_n. \end{cases}$$

Now, we prove that j_n has the following properties:

$\forall\ n \in \mathbb{N}$, j_n is continuous on $\mathbb{R}$

Suppose $t_0 \in \mathbb{R}$ and $t_i \to t_0$.

Proof of $j_n(t_i) \to j_n(t_0)$ is divided into three cases.

Case (i): Suppose $\|j(t_0)\| > \|g\| + \frac{1}{n}$, then $\|j(t_i)\| > \|g\| + \frac{1}{n}$ for large i since j is continuous, and thus $t_i \in C_n$ for large i. Also, $t_0 \in C_n$.

Then, it gives

$$\|j_n(t_i) - j_n(t_0)\| = \left(\|g\| + \frac{1}{n}\right) \cdot \left\| \frac{j(t_i)}{\|j(t_i)\|} - \frac{j(t_0)}{\|j(t_0)\|} \right\| \to 0.$$

Case (ii): Suppose $\|j(t_0)\| < \|g\| + \frac{1}{n}$, then $\|j(t_i)\| < \|g\| + \frac{1}{n}$ for large i since j is continuous, and thus $t_i \notin C_n$ for large i.

Also, here, $t_0 \notin C_n$. Then, it follows

$$\|j_n(t_i) - j_n(t_0)\| = \|j(t_i) - j(t_0)\| \to 0.$$

Case (iii): If $\|j(t_0)\| = \|g\| + \frac{1}{n}$, then $t_0 \notin C_n$. We have $\|j(t_i)\| \to \|g\| + \frac{1}{n}$, we have

$$\|j_n(t_i) - j_n(t_0)\| = \|j(t_i) - j(t_0)\| \cdot [1 - \chi_{C_n}(t_i)] + (\|f\| + \frac{1}{n} \cdot \left\| \frac{j(t_i)}{\|j(t_i)\|} - j(t_0) \right\| \cdot \chi_{C_n}(t_i) \to 0,$$

where χ_{C_n} is the characteristic function on C_n, i.e., $\chi_{C_n}(t) = 1\ \forall\ t \in C_n$ and $\chi_{C_n}(t) = 0\ \forall\ t \notin C_n$.$\square$

Corollary 2.4.1 *$\forall\ g \in PAP(Y,\eta)$, $\exists$ a decomposition $g = l_0 + h_0$, where $l_0 \in AP(Y)$ and $h_0 \in PAA_0(Y,\eta)$, such that $\|l_0\| \le \|g\|$.*

Proof. This corollary proof works as theorem 2.4.1 proof. We need to change little bit proof previous theorem, i.e., prove that $\forall\ \epsilon > 0$,

$$\|l_n(s+\zeta) - l_n(s)\| < \epsilon, \quad s \in \mathbb{R}, \quad \tau \in P(\epsilon, l).$$

Fix $n \in \mathbb{N}$. $\forall\ s, s+\zeta \notin C_n$, we can see that

$$\|l_n(s+\zeta) - l_n(s)\| = \|l(s+\zeta) - l(s)\| < \epsilon.$$

$\forall\ s, s+\zeta \in C_n$, we have

$$\|l_n(s+\zeta) - l_n(s)\| \le \left\| \frac{l(s+\zeta)}{\|l(s)\|} - \frac{l(s+\zeta)}{\|l(s+\zeta)\|} \right\| \le 2\|l(s+\zeta) - l(s)\| < 2\epsilon.$$

$\forall\ s \notin C_n$ and $s+\zeta \in C_n$, we have

$$\|l(s)\| \le \|g\| + \frac{1}{n} < \|l(s+\zeta)\|,$$

and

$$\|l_n(s+\zeta) - l_n(s)\| = \left\| \frac{\|g\| + \frac{1}{n}}{\|l(s+\zeta)\|} l(s+\zeta) - l(s) \right\| \le \|l(s+\zeta) - l(s)\|$$
$$+ \left(\|g\| + \frac{1}{n} - \|l(s+\zeta)\| \right) \le 2\|l(s+\zeta) - l(s)\| < 2\epsilon.$$

$\forall\ s \in C_n$ and $s+\zeta \notin C_n$, the proof is similar. $\square$

Theorem 2.4.2 *$\forall\ \eta \in V_\infty$, $PAA(X,\eta)$ is a Banach space under sup norm.*

Proof. Suppose $\{g_n\}$ is a Cauchy sequence in $PAA(Y,\eta)$. Then, we have subsequence $\{g_{i_k}\}$ st

$$\|g_{n_{i+1}} - g_{n_i}\| \le \frac{1}{2^i}.$$

We can see that $\{g_n\}$ being a Cauchy sequence, we can say that

$$\lim_{n\to\infty} l_n = l$$

It proves that PAA(Y, η) is a Banach

$$g_{n_i+1} - g_{n_i} \le \frac{1}{2^i}.$$

Let $f_{n_1} = l_{n_1} + h_{n_1}$, where $l_{n_1} \in AA(Y)$ and $h_{n_1} \in PAA_0(Y,\eta)$.

Using previous theorem on $g_{n_2} - g_{n_1}$, there exist $l_0 \in AA(Y)$ and $h_0 \in PAA_0(Y,\eta)$ such that

$$g_{n_2} - g_{n_1} = l_0 + h_0$$

and

$$\|l_0\| \leq \|g_{n_2} - g_{n_1}\|.$$

Let $l_{n_2} = l_0 + l_{n_1}$ and $h_{n_2} = h_0 + h_{n_1}$. Then $l_{n_2} \in AA(Y)$, $h_{n_2} \in PAA_0(Y, \eta)$, $g_{n_2} = l_{n_2} + h_{n_2}$ and

$$\|l_{n_2} - l_{n_1}\| = \|l_0\| \leq \|g_{n_2} - g_{n_1}\| \leq \frac{1}{2}.$$

Continuing , we can obtain two sequences $\{l_{n_i}\} \subset AA(Y)$ and $\{h_{n_k}\} \subset PAA_0(Y, \eta)$ such that for all $i \in \mathbb{N}$, $g_{n_i} = l_{n_i} + h_{n_i}$ and

$$\|l_{n_{i+1}} - l_{n_i}\| \leq \|g_{n_{i+1}} - g_{n_i}\| \leq \frac{1}{2i}.$$

Hence, $\{l_{n_i}\}$ and $\{h_{n_i}\}$ are Cauchy sequences.

Suppose

$$\lim_{i\to\infty} l_{n_i} = l, \quad \lim_{i\to\infty} h_{n_i} = h,$$

and $g = l + h$.

Then, $l \in AA(Y)$, $h \in PAA_0(Y, \eta)$, $g \in PAA(Y, \eta)$, and $l_{n_k} \to l$.

$\{g_n\}$ is a Cauchy sequence, we conclude $\lim_{n\to\infty} g_n = g$.

Hence proof. □

Theorem 2.4.3 *For every $\eta \in V_\infty$, $PKAA(Y, \eta)$ is a Banach space under sup norm.*

Proof In theorem 2.4.2, we have proved that for every $\eta \in V_\infty$, $PAA(Y, \eta)$ is a Banach space under the supremum norm.

Since $PKAA(Y, \eta)$ is a closed subset of $PAA(Y, \eta)$.
We know that closed subset of banach space is banach space wrt sup norm.
Hence, $\forall\ \eta \in V_\infty$, $PKAA(Y, \eta)$ is a Banach space under the supremum norm. □

3 Applications in Abstract Differential Equation

3.1 Application of Weighted Pseudo Almost Automorphic function - Approach1

Consider differential equation
$u'(t) = Bu(t) + g(t, u(t)), \quad t \in \mathbb{R}$...........(3.1.1)
with conditions:

- **(A1)** $g(t,u)$ is almost automorphic.
- **(A2)** $g(t,u)$ is Lipschitz continuous,i.e., $\exists$ a $L_g \geq 0$
  $\|g(t,u) - g(t,v)\| \leq L_g \|u - v\|, \quad \forall t \in \mathbb{R}$ and $u, v \in U$
- **(A3)** $X(t,s), t \geq s$, is an exponentially stable evolution family on U.
- **(A4)** $\forall$ sequence of $\mathbb{R}$ $\{s_n\}_{n\in\mathbb{N}}$, $\exists$ a subsequence $\{\zeta_n\}_{n\in\mathbb{N}}$ and $\forall$ fixed $s \in \mathbb{R}$, $\delta > 0$, $\exists$ an $N \in \mathbb{N}$ such that, $\forall$ $n > N$, it follows that
  $$\|X(t + \zeta_n, s + \zeta_n) - X(t,s)\| \leq \delta e^{-\frac{\epsilon}{2}(t-s)},$$
  $\forall$ $t \geq s \in \mathbb{R}$; Moreover,
  $$\|X(t - \zeta_n, s - \zeta_n) - X(t,s)\| \leq \delta e^{-\frac{\epsilon}{2}(t-s)},$$
  $\forall$ $t \geq s \in \mathbb{R}$.

We define a mapping G by

$$(Gu)(t) = \int_{-\infty}^{t} X(t,s) g(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Lemma 3.1.1 *If $u(s)$ is almost automorphic, then the function Gu is almost automorphic.*

Proof. We can see that G is bounded. By assumption (A1) and previous lemma, $g(t,u(t))$ is almost automorphic and hence bounded.

Assume $\exists$ $M_1 > 0$, st

$$\|g(t,u(t))\|_{\mathrm{AA}(U)} \leq M_1.$$

Thus,

$$\|(Gu)(t)\| \leq \int_{-\infty}^{t} \|X(t,s)\| \|g(s,u(s))\|\, ds \leq$$

$$\int_{-\infty}^{t} M e^{-\epsilon(t-s)} M_1\, ds \leq \frac{M_1 M}{\epsilon} < \infty.$$

Hence, $(Gu)(t)$ is bounded.

To Prove: $(Gu)(t)$ is almost automorphic wrt $t \in \mathbb{R}$.

From (A1), $g(t, u(t))$ is almost automorphic, i.e., $g(t, u(t)) \in \mathrm{AA}(R\times U, U)$. From (A2) and previous lemma, $g(t, u(t)) \in \mathrm{AA}(U)$, and we define $g(t, u(t)) = F(t)$.

Suppose $\{s_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of $\mathbb{R}$. Since $F(t) \in \mathrm{AA}(U)$, $\exists$ a subsequence $\{\zeta_n\}_{n\in\mathbb{N}}$ st

(H1) $f(t) = \lim_{n\to\infty} F(t + \zeta_n)$ is well defined $\forall\ t \in \mathbb{R}$
(H2) $\lim_{n\to\infty} f(t - \zeta_n) = F(t)\ \forall\ t \in \mathbb{R}$.

Now consider,

$$(Gu)(t+\zeta_n) = \int_{-\infty}^{t+\zeta_n} X(t+\zeta_n, s)F(s)\, ds = \int_{-\infty}^{t} X(t+\zeta_n, s+\zeta_n)F(s+\zeta_n)\, ds.$$

So we have

$$\|(Gu)(t + \zeta_n)\| \leq \int_{-\infty}^{t} \|X(t + \zeta_n, s + \zeta_n)\|\|F(s + \zeta_n)\|\, ds \leq$$
$$\int_{-\infty}^{t} Me^{-\epsilon(t-s)}\|F(s + \zeta_n)\|\, ds \leq \frac{M_1M}{\epsilon},$$

$\forall\ n = 1, 2, \ldots$.

For (H1), $\forall$ fixed $s \in \mathbb{R}$ and $\epsilon > 0$, $\exists$ an $N_1 \in \mathbb{N}$ such that, $\forall\ n > N_1$, it follows that

$$\|F(s + \zeta_n) - f(s)\| < \epsilon.$$

In addition, by condition (A4), for s and ϵ above, $\exists$ an $N_2 \in \mathbb{N}$ such that, $\forall\ n > N_2$, it follows that

$$\|X(t + \zeta_n, s + \zeta_n) - X(t, s)\| < \delta e^{-\frac{\epsilon}{2}(t-s)}.$$

Thus, taking $N = \max\{N_1, N_2\}$, for all $n > N$,

$$\|X(t + \zeta_n, s + \zeta_n)F(s + \zeta_n) - X(t, s)f(s)\| \leq \|X(t + \zeta_n, s + \zeta_n)$$
$$-X(t, s)\|\|F(s + \zeta_n)\| + \|X(t, s)\|\|F(s + \zeta_n) - f(s)\|.$$

Using the bounds, we get

$$\|X(t+\zeta_n, s+\zeta_n)F(s+\zeta_n)-X(t, s)f(s)\| \leq M_1\delta e^{-\frac{\epsilon}{2}(t-s)}+M\delta e^{-\epsilon(t-s)}.$$

As $n \to \infty$, we have

$$X(t + \zeta_n, s + \zeta_n)F(s + \zeta_n) \to X(t, s)f(s)$$

$\forall$ fixed $s \in \mathbb{R}$ and any $t \geq s$.

Notice that

$$\|X(t + \zeta_n, s + \zeta_n)F(s + \zeta_n)\| \leq Me^{-\epsilon(t-s)}\|F\|,$$

for $t \geq s$.

So, by the DCT, we get

$$(Gu)(t+\zeta_n) \to (Fu)(t) \text{ as } n \to \infty,$$

where

$$(Fu)(t) = \int_{-\infty}^{t} X(t,s)f(s)\,ds, \quad t \in \mathbb{R}.$$

Similarly,

$$(Fu)(t-\zeta_n) \to (Gu)(t) \text{ as } n \to \infty,$$

$\forall\, t \in \mathbb{R}$. This proves that $Gu \in AA(X)$. Hence proof. □

Theorem 3.1.1 *Suppose $g(t,u)$ and $X(t,s)$ satisfy **(A1)–(A4)**. Then Equation has an unique mild almost automorphic solution when $L_g < \frac{\epsilon}{M}$.*

Proof. From previous lemma, we get $Gu \in AA(U)$. For $u, v \in AA(U)$, we can see that

$$\|(Gu)(t)-(Gv)(t)\| = \left\| \int_{-\infty}^{t} X(t,s)g(s,u(s))\,ds - \int_{-\infty}^{t} X(t,s)g(s,v(s))\,ds \right\|$$

$$= \left\| \int_{-\infty}^{t} X(t,s)\left(g(s,u(s)) - g(s,v(s))\right)\,ds \right\| \leq$$

$$L_g\|u-v\|_{AA(U)} \int_{-\infty}^{t} Me^{-\epsilon(t-s)}\,ds$$

$$= \frac{M}{\epsilon} L_g \|u-v\|_{AA(U)}.$$

So we have

$$\|(Gu)(t)-(Gv)(t)\|_{AA(Y)} \leq \frac{M}{\epsilon} L_g \|u-v\|_{AA(U)}.$$

For $0 < \frac{M}{\epsilon} L_g < 1$ and by the Banach contraction principle, G has a unique fixed point $u \in AA(U)$ such that $Gu = u$.

Fix $s \in \mathbb{R}$, we have

$$u(t) = \int_{-\infty}^{t} X(t,r)g(r,u(r))\,dr.$$

Since $X(t,s) = X(t,r)X(r,s)$ for $t \geq r \geq s$, let

$$u(a) = \int_{-\infty}^{b} X(b,s)f(s,x(s))\,ds.$$

So

$$U(t,b)u(b) = \int_{-\infty}^{b} U(t,s)g(s,u(s))\,ds.$$

For $t \geq a$,

$$\int_{b}^{t} X(t,s)g(s,u(s))\,ds = \int_{-\infty}^{t} X(t,s)g(s,u(s))\,ds -$$

$$\int_{-\infty}^{b} X(t,s)g(s,u(s))\,ds = u(t) - X(t,b)u(b).$$

So that

$$u(t) = X(t,b)u(b) + \int_{b}^{t} X(t,s)g(s,u(s))\,ds.$$

Hence $u(t)$ is a mild solution of Equation (3.1.1).
Finally, we get $u(t)$ is the unique mild solution Equation (3.1.1). □

Now, we consider conditions (A3), (A4) and:

- **(B1)** $g(t,u)$ is weighted pseudo almost automorphic.
- **(B2)** $g(t,u)$ is Lipschitz continuous,i.e., $\exists$ a positive number L_g such that
$$\|g(t,u) - g(t,v)\| \leq L_g\|u - v\|,$$
for all $t \in \mathbb{R}$ and $u, v \in WPAA(\mathbb{R},\eta)$, $\eta \in X_\infty$.

We also define a mapping Y by

$$(Yu)(t) = \int_{-\infty}^{t} X(t,s)g(s,u(s))\,ds, \quad t \in \mathbb{R}.$$

Lemma 3.1.2 *If $u(t)$ is weighted pseudo almost automorphic, $Yu(t)$ is weighted pseudo almost automorphic.*

Proof. From (B1), $g(t,u(t))$ is weighted pseudo almost automorphic,i.e., $g(t,u(t)) \in WPAA(\mathbb{R},\eta)$. From (B2) and Lemma, $g(t,u(t)) \in WPAA(\mathbb{R},\eta)$. Suppose

$$g(t,u(t)) = h(t) + \psi(t),$$

where $h \in AA(U)$ and $\psi \in PAA_0(\mathbb{R},\eta)$.
Then

$$(Yu)(t) = \int_{-\infty}^{t} X(t,s)g(s,u(s))\,ds = \int_{-\infty}^{t} X(t,s)h(s)\,ds + \int_{-\infty}^{t} X(t,s)\psi(s)\,ds.$$

Let

$$y(t) = \int_{-\infty}^{t} X(t,s)h(s)\,ds \quad \text{and} \quad z(t) = \int_{-\infty}^{t} X(t,s)\psi(s)\,ds.$$

Similarly, it gives $y \in AA(U)$, so y is almost automorphic.

To show that $Y(u)(t) \in WPAA(\mathbb{R},\eta)$, we need to prove that $z(t) \in PAA_0(\mathbb{R},\eta)$, which means we show that

$$\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\int_{-r}^{r}\|z(t)\|\eta(t)\,dt=0.$$

We have

$$0\le\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\int_{-r}^{r}\|z(t)\|\eta(t)\,dt$$
$$\le\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\int_{-r}^{r}\int_{-\infty}^{t}Me^{-\epsilon(t-s)}\|\psi(s)\|\eta(s)\,ds\,dt.$$

Now let

$$P_1=\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\int_{-r}^{r}\int_{-\infty}^{-r}Me^{-\epsilon(t-s)}\|\psi(s)\|\eta(s)\,ds\,dt.$$

By using the Fubini theorem, we have

$$P_1=\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\int_{-\infty}^{-r}e^{\epsilon s}\|\psi(s)\|\,ds\int_{-r}^{r}Me^{-\epsilon t}\eta(t)\,dt.$$

This is equivalent to

$$P_1=\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\sup_{t\in\mathbb{R}}\|\psi(t)\|\|\eta\|_{L^1_{\text{loc}}(\mathbb{R})}\frac{M}{\epsilon}$$
$$\left(e^{-\epsilon r}-e^{\epsilon r}\right)\int_{-\infty}^{-r}e^{\epsilon s}\,ds.$$

Then,

$$P_1=\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\sup_{t\in\mathbb{R}}\|\psi(t)\|\|\eta\|_{L^1_{\text{loc}}(\mathbb{R})}\frac{M}{\epsilon^2}\left(e^{-2\epsilon r}-1\right).$$

Since $\psi(t)$ is bounded and $\lim_{r\to\infty}\mu(r,\eta)=\infty$, then $P_1=0$.

Also let

$$P_2=\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\int_{-r}^{r}\|\psi(t)\|\eta(t)\,dt\int_{-r}^{t}Me^{-\epsilon(t-s)}\,ds.$$

Thus,

$$=\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\frac{M}{\epsilon}e^{-\epsilon(t-s)}\Big|_{s=-r}^{s=t}\int_{-r}^{r}\|\psi(t)\|\eta(t)\,dt$$
$$=\lim_{r\to\infty}\frac{1}{\mu(r,\eta)}\frac{M}{\epsilon}\left(1-e^{-\epsilon(t+r)}\right)\int_{-r}^{r}\|\psi(t)\|\eta(t)\,dt.$$

Since $-r\le t\le r$ and $\epsilon>0$, then $\frac{M}{\epsilon}(1-e^{-\epsilon(t+r)})$ is bounded.

Now, $\phi\in PAA_0(\mathbb{R},\eta)$, thus $P_2=0$. Hence proof. $\square$

Theorem 3.1.2 *Let $g(t,u)$ and $X(t,s)$ satisfy (B1), (B2), (A3), (A4) and $0<\frac{M}{\epsilon}L_g<1$. Then Equation (3.1.1) has a unique mild weighted pseudo almost automorphic solution.*

Proof. From Lemma 3.1.2, we get $Y(u(t))$ maps $WPAA(\mathbb{R},\eta)$ into $WPAA(\mathbb{R},\eta)$.

Let $u, v \in WPAA(\mathbb{R},\eta)$, and observe

$$\|(Yu)(t)-(Yv)(t)\| = \left\|\int_{-\infty}^{t} X(t,s)g(s,u(s))\,ds - \int_{-\infty}^{t} X(t,s)g(s,v(s))\,ds\right\|$$
$$= \int_{-\infty}^{t} \|X(t,s)\|\|g(s,u(s)) - g(s,v(s))\|\,ds.$$

We have

$$L_g\|u-v\|\ {\scriptstyle \int_{-\infty}^{t} Me^{-\epsilon(t-s)}\,ds = \frac{M}{\delta} L_g\|u-v\|_{WPAA(\mathbb{R},\eta)}}.$$

So we have

$$\|(Yu)(t)-(Yv)(t)\| \le \frac{M}{\epsilon} L_g \|u-v\|_{WPAA(\mathbb{R},\eta)}.$$

For $0 < \frac{M}{\epsilon}L_g < 1$, Y is a contractive mapping.

By Lemma $WPAA(\mathbb{R},\eta)$ is complete. Therefore, by the Banach fixed point theorem, Y has a unique fixed point $u \in WPAA(\mathbb{R},\eta)$ such that $Yu(t) = u$.
Fixing $s \in \mathbb{R}$, we have

$$u(t) = \int_{-\infty}^{t} X(t,r)g(r,u(r))\,dr.$$

Since $X(t,s) = X(t,r)X(r,s)$, for $t \ge r \ge s$, it gives $u(t)$ satisfies Equation (3.1.1). Hence $u(t)$ is the unique mild solution of Equation (3.1.1).
$\square$

3.2 Application of Weighted Pseudo Almost Automorphic function - Approach2

Consider differential equation [**10**]
$u'(t) = Bu(t) + g(t, u(t)), \quad t \in \mathbb{R}$ .... (3.2.1)
with:

- **H1.** $g(t, u)$ is uniformly continuous $\forall$ bounded subset $L \subset U$ uniformly in $t \in \mathbb{R}$.
- **H2.** $h(t, u)$ is uniformly continuous $\forall$ bounded subset $L \subset U$ uniformly in $t \in \mathbb{R}$.
- **H3** B is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space U st
$$\|T(t)\| \leq Ne^{-\eta t}, \quad t \geq 0.$$
- **H4** $g = h + \psi \in WPAA(\mathbb{R}, \eta)$ where $\eta \in X_\infty$.
- **H5** $\|g(t, u) - g(t, v)\| \leq L_g \|u - v\|, \quad \forall u, v \in U.$
- **H6** $\|h(t, u) - h(t, v)\| \leq L_h \|u - v\|, \quad \forall u, v \in U.$

Theorem 3.2.1 *Suppose $g = h + \psi \in WPAA(\mathbb{R}, \eta)$ where $\eta \in V_\infty$ and suppose H1 and H2 hold. Then $L(\cdot) := g(\cdot, h(\cdot)) \in WPAA(\mathbb{R}, \eta)$ if $g \in WPAA(\mathbb{R}, \eta)$.*

Proof. We know that $g = h + \psi$ where $h \in AA(\mathbb{R}, U)$ and $\psi \in PAA_0(\mathbb{R}, \eta)$, and $f = \nu + m$ where $\nu \in AA(\mathbb{R}, U)$ and $m \in PAA_0(\mathbb{R}, \eta)$.
Now
$$L(\cdot) = h(\cdot, \nu(\cdot)) + g(\cdot, h(\cdot)) - h(\cdot, \nu(\cdot)) =$$
$$h(\cdot, \nu(\cdot)) + g(\cdot, f(\cdot)) - g(\cdot, \nu(\cdot)) + \psi(\cdot, \nu(\cdot)).$$
By Lemma , $h(\cdot, \nu(\cdot)) \in AA(\mathbb{R}, U)$.

Consider now the function
$$\Psi(\cdot) := g(\cdot, f(\cdot)) - g(\cdot, \nu(\cdot)).$$
Clearly, $\Psi(t) \in BC(\mathbb{R}, U)$. For $\Psi(t)$ to be in $PAA_0(\mathbb{R}, \eta)$, it is sufficient to prove that
$$\lim_{r\to\infty} \frac{1}{\mu(r, \eta)} \text{mes}(M_{r,\delta}(\Psi)) = 0.$$
By Lemma , $m(\mathbb{R}) \subset f(\mathbb{R})$, which is a bounded.

Using Assumption H1 with $K = f(\mathbb{R})$, we have $\forall\ \delta > 0$, $\exists\ \epsilon > 0$ such that

$$u, v \in K, \quad \|u - v\| < \epsilon \quad \Rightarrow \quad \|g(t,u) - g(t,v)\| < \delta, \quad \forall t \in \mathbb{R}.$$

Thus we obtain

$$\frac{1}{\mu(r,\eta)}\text{mes}(M_{r,\delta}(\Psi(t))) = \frac{1}{\mu(r,\eta)}\text{mes}(M_{r,\delta}(g(t,f(t)) - g(t,m(t)))) \leq$$
$$\frac{1}{\mu(r,\eta)}\text{mes}(M_{r,\epsilon}(f(t) - m(t)))$$
$$= \frac{1}{\mu(r,\eta)}\text{mes}(M_{r,\epsilon}(\nu(t))).$$

Now since $\nu \in PAA_0(\mathbb{R},\eta)$, by Lemma,

$$\lim_{r\to\infty} \frac{1}{\mu(r,\eta)}\text{mes}(M_{r,\delta}(\nu(t))) = 0.$$

Consequently,

$$\lim_{r\to\infty} \frac{1}{\mu(r,\eta)}\text{mes}(M_{r,\delta}(\Psi(t))) = 0.$$

Thus, $\Psi \in PAA_0(\mathbb{R},\rho)$.

Now, we prove $\psi(t,m(t)) \in PAA_0(\mathbb{R},\eta)$. We have $\psi(t,m(t))$ is uniformly continuous on $[-r,r]$, and that $m([-r,r])$ is compact since m is continuous on $\mathbb{R}$ is almost automorphic function. Hence, $\forall\ \delta > 0$, $\exists\ \epsilon > 0$ such that

$$\mu([-r,r]) \subset \bigcup_{i=1}^{n} A_i \quad \text{where}$$
$$A_i = \{u \in U \mid \|u - u_i\| < \epsilon\} \quad \text{for some } u_i \in m([-r,r]),$$
$$\|\psi(t,m(t)) - \psi(t,u_i)\| < \frac{\delta}{2}, \quad m(t) \in A_i, \quad t \in [-r,r].$$

We have $X_i := \{t \in [-r,r] \mid m(t) \in A_i\}$ is open in $[-r,r]$ and that $[-r,r] = \bigcup_{i=1}^{n} X_i$. Define Y_i

$$Y_1 = X_1, \quad Y_k = X_k - \bigcup_{k=1}^{i-1} X_k, \quad 2 \leq i \leq n.$$

Then we can see that $Y_k \cap Y_l = \emptyset$, if $k \neq l$, $1 \leq k, l \leq n$. So we get

$$\xi := \left\{t \in [-r,r] \mid \|\psi(t,m(t))\| \geq \frac{\delta}{2}\right\} \subset$$
$$\bigcup_{i=1}^{n} \{t \in Y_i \mid \|\varphi(t,m(t)) - \psi(t,u_i)\| + \|\psi(t,u_i)\| \geq \delta\}.$$
$$\subset \bigcup_{i=1}^{n} \left[\left\{t \in Y_i \mid \|\psi(t,m(t)) - \psi(t,u_i)\| \geq \frac{\delta}{2}\right\} \cup\right.$$

$$\left\{ t \in Y_i \mid \|\psi(t, u_i)\| \geq \frac{\delta}{2} \right\}.$$

It gives

$$\left\{ t \in Y_i \mid \|\varphi(t, m(t)) - \psi(t, u_i)\| \geq \frac{\delta}{2} \right\}$$
$$= \emptyset, \quad i = 1, 2, \ldots, m.$$

Thus, we get

$$\frac{1}{\mu(r,\eta)} \mathrm{mes}(M_{r,\delta}(\psi(t, \beta(t)))) \leq \sum_{i=1}^{n} \frac{1}{\mu(r,\eta)} \mathrm{mes}(M_{r,\delta}(\psi(t, u_i))).$$

And since $\psi(t, u) \in PAA_0(\mathbb{R} \times U, \eta)$ and

$$\lim_{r \to \infty} \frac{1}{\mu(r,\eta)} \mathrm{mes}(M_{r,\frac{\delta}{2}}(\psi(t, u_i))) = 0,$$

it follows that

$$\lim_{r \to \infty} \frac{1}{\mu(r,\eta)} \mathrm{mes}(M_{r,\frac{\delta}{2}}(\psi(t, m(t)))) = 0,$$

i.e., $\psi(t, m(t)) \in PAA_0(\mathbb{R}, \eta)$. Hence proof. $\square$

Corollary 3.2.1 *Let $g = h + \psi \in WPAA(\mathbb{R}, \eta)$ where $\eta \in V_\infty$. Assume both g and h are Lipschitz in $u \in U$ uniformly in $t \in \mathbb{R}$. Then $L(\cdot) := g(\cdot, f(\cdot)) \in WPAA(\mathbb{R}, \eta)$ if $f \in WPAA(\mathbb{R}, \eta)$.*

Lemma 3.2.1 *Suppose $g = h + \psi \in WPAA(\mathbb{R}, \eta)$ where $\eta \in X_\infty$ and $(Q(t))_{t \geq 0}$ is an exponentially stable semigroup. Then*

$$G(t) := \int_{-\infty}^{t} Q(t-s) g(s)\, ds \in WPAA(\mathbb{R}, \eta).$$

Proof. Suppose $G(t) = H(t) + \Psi(t)$ where

$$H(t) := \int_{-\infty}^{t} Q(t-s) h(s)\, ds \quad \text{and} \quad \Psi(t) := \int_{-\infty}^{t} Q(t-s) \psi(s)\, ds.$$

Now, $H(t) \in AA(\mathbb{R}, U)$. We prove that $\Psi(t) \in PAA_0(\mathbb{R}, \eta)$.
We have

$$\frac{1}{\mu(r,\eta)} \int_{-r}^{r} \|\Psi(s)\| \eta(s)\, ds =$$
$$\frac{1}{\mu(r,\eta)} \int_{-r}^{r} \int_{-\infty}^{s} Q(s-\sigma)\psi(\sigma)\, d\sigma\, \eta(s)\, ds = P_1 + P_2,$$

where

$$P_1 := \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \int_{-\infty}^{-r} Q(s-\sigma)\psi(\sigma)\, d\sigma\, \eta(s)\, ds$$

and

$$P_2 := \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \int_{-r}^{s} Q(s-\sigma)\psi(\sigma)\, d\sigma\, \eta(s)\, ds.$$

We have

$$\begin{aligned}
P_1 &:= \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \int_{-\infty}^{-r} Q(s-\sigma)\psi(\sigma)\, d\sigma\, \eta(s)\, ds \leq \\
&\frac{1}{\mu(r,\eta)} \int_{-r}^{r} \left(\int_{-\infty}^{-r} \|Q(s-\sigma)\| \|\psi(\sigma)\|\, d\sigma \right) \eta(s)\, ds \\
&\leq N \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \left(\int_{-\infty}^{-r} e^{-\rho(s-\sigma)} \|\psi(\sigma)\|\, d\sigma \right) \eta(s)\, ds \\
&\leq N \frac{1}{\mu(r,\eta)} \left(\int_{-r}^{r} e^{-\rho s} \eta(s)\, ds \right) \left(\int_{-\infty}^{-r} e^{\rho\sigma} \|\psi(\sigma)\|\, d\sigma \right) \\
&\leq N \frac{1}{\mu(r,\eta)} \|\eta\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \left(\int_{-r}^{r} e^{-\rho s}\, ds \right) \left(\int_{-\infty}^{-r} e^{\rho\sigma} \|\psi(\sigma)\|\, d\sigma \right) \\
&\leq N \frac{1}{\rho} \frac{1}{\mu(r,\eta)} \|\eta\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \left(e^{-\rho r} - e^{\rho r} \right) \sup_{t\in\mathbb{R}} \|\psi(t)\| \left(\int_{-\infty}^{-r} e^{\rho\sigma}\, d\sigma \right) \\
&\leq N \frac{1}{\rho} \frac{1}{\mu(r,\eta)} \|\eta\|_{L^1_{\mathrm{loc}}(\mathbb{R})} e^{-\rho r} \sup_{t\in\mathbb{R}} \|\psi(t)\| \left(\int_{-\infty}^{-r} e^{\rho\sigma}\, d\sigma \right) \\
&\leq N \frac{1}{\rho^2} \frac{1}{\mu(r,\eta)} \|\eta\|_{L^1_{\mathrm{loc}}(\mathbb{R})} e^{-2\rho r} \sup_{t\in\mathbb{R}} \|\psi(t)\| \\
&\leq N \frac{1}{\rho^2} \frac{1}{\mu(r,\eta)} \|\eta\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \sup_{t\in\mathbb{R}} \|\psi(t)\|.
\end{aligned}$$

Since $\sup_{t\in\mathbb{R}} \|\psi(t)\| < \infty$ and $\lim_{r\to\infty} \mu(r,\eta) = \infty$, we conclude that

$$\lim_{r\to\infty} P_1 = 0.$$

Now, for P_2, we have

$$\begin{aligned}
P_2 &:= \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \int_{s}^{-r} Q(s-\sigma)\psi(\sigma)\, d\sigma\, \eta(s)\, ds \\
&\leq \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \left(\int_{s}^{-r} \|Q(s-\sigma)\| \|\psi(\sigma)\|\, d\sigma \right) \eta(s)\, ds \\
&\leq N \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \left(\int_{s}^{-r} e^{-\rho(s-\sigma)} \|\psi(\sigma)\|\, d\sigma \right) \eta(s)\, ds \\
&\leq N \frac{1}{\rho} \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \left(1 - e^{-\rho(s+r)} \right) \|\psi(s)\| \eta(s)\, ds \\
&\leq N \frac{1}{\rho} \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \|\psi(s)\| \eta(s)\, ds.
\end{aligned}$$

Since $\psi \in PAA_0(\mathbb{R}, \eta)$, we have

$$\lim_{r\to\infty} \frac{1}{\mu(r,\eta)} \int_{-r}^{r} \|\psi(s)\| \eta(s)\, ds = 0.$$

Thus, we conclude that

$$\lim_{r\to\infty} P_2 = 0.$$

□

Theorem 3.2.2 *Under H3–H6, Equation (3.2.2) has a unique mild solution in $WPAA(\mathbb{R}, \eta)$ when $NL_g/\rho < 1$.*

Proof. Define, $\Delta : WPAA(\mathbb{R}, \eta) \to WPAA(\mathbb{R}, \eta)$ st

$$(\Delta u)(t) := \int_{-\infty}^{t} T(t-\sigma) g(\sigma, u(\sigma))\, d\sigma, \quad t \in \mathbb{R},$$

is well-defined.
If $u, v \in WPAA(\mathbb{R}, \eta)$, we get

$$\|(\Delta u)(t) - (\Delta v)(t)\| = \left\| \int_{-\infty}^{t} T(t-\sigma) \left(g(\sigma, u(\sigma)) - g(\sigma, v(\sigma)) \right) d\sigma \right\|$$

$$\leq NL_g \int_{-\infty}^{t} e^{-\rho(t-\sigma)} \|u(\sigma) - v(\sigma)\|\, d\sigma \leq NL_g \frac{1}{\rho} \|u - v\|_\infty, \quad \forall t \in \mathbb{R}.$$

Thus,

$$\|\Delta u - \Delta v\|_\infty \leq \frac{NL_g}{\omega} \|u - v\|_\infty.$$

Apply principle of contraction to conclude the result.□

4 Mathematical Models

4.1 Nicholson Blowflies Model

In past few decades [**1**], we can notice that because of the progress of population models, the theory related to Nicholson's blowflies model has achived a great milestone.

By the develpoment of Nicholson model, researchers gain knowledge of the factors that effects population dynamics and influences population of blowflies.

Nicholson model is a mathematical model that helps us to understand the population dynamics of blowflies explains the certain patterns, fluctuations and changes in the population of blowflies.

$\dot{x}(s) = -\gamma x(s) + \delta x(s-\eta)e^{-\zeta x(s-\eta)}$ ....(4.1.1)

where γ, δ, η, and ζ are positive constants, and $x(s)$ is the size of the population at time s. The parameter δ stands for maximum per capita daily egg production rate, γ denotes the per capita daily adult death rate, $\frac{1}{\zeta}$ stands for the size at which the population reproduces at its maximum rate, and η stands for time it takes for the egg to become an adult blowfly.

This delay differential equation is called Nicholson's equation.

A. J. Nicholson perform a lot of experiments with the Australian blowfly and collected some experimental data and form this model. It is a successful model

Mathematicians further investigated the Nicholson's model with a linear harvesting term:

$\dot{x}(s) = -\gamma(s)x(s) + \delta(s)x(s-\zeta(s))e^{-\eta(s)x(s-\zeta(s))} - H(s)x(s-\sigma(s))$ ....(4.1.2)

where the functions $\gamma, \delta, \eta, H : \mathbb{R} \to (0, \infty)$ are pseudo almost periodic continuous functions, and $\zeta, \sigma : \mathbb{R} \to [0, \infty)$ are almost periodic continuous functions.

4.2 Lasota Wazeska Model

Lasota and Wazewska [**1**] proposed a model for the survival of RBCs in animals with the help of delay differential equations.

Model is :

$\dot{x}(s) = -\gamma u(s) + \delta e^{-\zeta x(s-\eta))}$.......(4.2.1)

where x represents the number of red blood cells. γ denotes the rate of death of the cells. δ and ζ determine the rate of production of red blood cells η denote time required to produce the cells.

This delay differential equation is known as Lasota-Wazewska model. It is also a very important model in population dynamics. But in this thesis our main focus is on Nicholson Blowflies Model. We have done analysis of only Nicholson Blowflies Model.

5 Analysis of Nicholson Model

5.1 Existence and Uniqueness of Solution

We tell the non-autonomous equation of Nicholson Blowflies Model
$\dot{x}(t) = -\beta(t)x(t) + \sum_{i=1}^{n} H_k(t, x(t-\eta_k(t))) + a(t)F(x(t))$........
(5.1.1)

Here, $x(t)$ stands for the density of the population.
$\alpha(t)$ stands for the death rate of the population
$F_i(t,\psi)$ stands for the birth rate
$a(t)H(x(t))$ stands for the harvesting rate or immigration rate.
$\eta_i(\cdot)$ in the birth rate.

In population models, only non-negative solutions work. Hence, we consider subsets of C

$$C^+ := C([-\eta, 0], [0, \infty)), \quad C_0 := \{\varphi \in C^+ : \psi(0) > 0\}.$$

Necessary assumptions for our result: **[1]**

A1. $\beta(t)$ is positive almost automorphic on $\mathbb{R}$, $\eta_k(t)$ denote positive compact almost automorphic on $\mathbb{R}$ for $k = 1, \ldots, n$, and $\beta_k(t)$, $\lambda_k(t)$, and $a(t)$ denote positive pseudo compact almost automorphic functions.

A2. $\beta(\cdot)$, $\gamma_k(\cdot)$, $\xi_k(\cdot)$ are bounded away from zero.

A3. $F : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is Lipschitz continuous and $F(0) = 0$, i.e., $\exists$ a positive constant L_F st:

$$-L_F x < F(x) < L_F x, \quad \text{for} \quad x \in \mathbb{R}_0^+.$$

A4. $\forall\ 1 \le k \le n$, $f_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are Lipschitz continuous, non-negative, and their supremum value over $\mathbb{R}_0^+$, i.e., $f_k = f_k(w_k^*)$ with $w_k^* \in \mathbb{R}_0^+$, and f_k is a non-increasing function for $x > w_k^*$.

A5. $\exists$ two positive constants δ_1 and δ_2 st:

$$0 \le \frac{w^*}{\xi} < \delta_1 < \frac{\sum_{k=1}^{n} \underline{\gamma_k} f_k(\overline{\xi}\delta_2) - \overline{a}L_F\delta_2}{\overline{\beta}}, \quad \frac{\sum_{K=1}^{n} \overline{\gamma_k f_k} + \overline{a}L_F\delta_2}{\underline{\beta}} < \delta_2$$

where $w^* := \max_{1 \le k \le n}\{w_k^*\}$.

A6. $\forall\ 1 \le k \le n$, $\exists$ positive numbers L_{f_k} which are the Lipschitz constants of f_k on $[w^*, \infty)$.

$\forall$ bounded continuous function f on $\mathbb{R}$, suppose

$$\overline{f} = \sup_{t\in\mathbb{R}}\{f(t)\}, \quad \underline{f} = \inf_{t\in\mathbb{R}}\{f(t)\}.$$

Positive constants:

$$\eta := \max_{1\le k\le n}\{\eta_k\}, \quad \lambda := \min_{1\le k\le n}\{\lambda_k\}.$$

Theorem 5.1.1 *Assume (A1)–(A3) hold and $\overline{\beta} > \overline{a}$, then $\forall$ solution $x(t)$ of equation with $\psi \in C_0$ satisfies:*

$$0 \le \frac{\sum_{k=1}^{n}\overline{\gamma_k}\overline{f_k}}{\overline{\beta}+\overline{a}L_F} \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le \frac{\sum_{k=1}^{n}\overline{\gamma_k}\overline{f_k}}{\underline{\beta}-\overline{a}L_F}.$$

Proof. [**1**]Suppose $x(t)$ be a solution, then

$$\dot{x}(t) = -\beta(t)x(t) + \sum_{k=1}^{n}\gamma_k(t)f_k(\lambda_i(t)x(t-\eta_k(t))) + a(t)F(x(t))$$

$$\le -\underline{\beta}x(t) + \sum_{k=1}^{n}\overline{\beta_k f_k} + \overline{a}L_F x(t) \le -(\underline{\alpha} - \overline{a}L_F)x(t) + \sum_{k=1}^{n}\overline{\gamma_k}\overline{f_k}.$$

It follows by the comparison principle that

$$x(t) \le x(0)e^{-(\overline{\beta}-\overline{a}L_F)t} + \frac{\sum_{k=1}^{n}\overline{\gamma_k}\overline{f_k}}{\overline{\beta}-\overline{a}L_F} \quad \text{for } t \ge 0.$$

Using the assumption $\overline{\beta} - \overline{b}L_F > 0$, we obtain

$$\limsup_{t\to+\infty} x(t) \le \frac{\sum_{k=1}^{n}\overline{\beta_k f_k}}{\overline{\alpha}-\overline{a}L_F}.$$

Other side, we have

$$\dot{x}(t) = -\beta(t)x(t) + \sum_{k=1}^{n}\beta_k(t)f_k(\xi_k(t)x(t-\eta_k(t))) + a(t)F(x(t))$$

$$\ge -\underline{\beta}x(t) + \sum_{k=1}^{n}\underline{\beta_i f_k} - \overline{b}L_F x(t) \ge -(\underline{\beta} + \overline{b}L_F)x(t) + \sum_{k=1}^{n}\underline{\gamma_k f_k}.$$

From comparison principle,

$$x(t) \ge x(0)e^{-(\underline{\beta}+\overline{a}L_F)t} + \frac{\sum_{k=1}^{n}\underline{\gamma_k f_k}}{\underline{\beta}+\overline{a}L_F}(1-e^{-(\underline{\beta}+\overline{a}L_F)t}), \quad t \ge 0.$$

we obtain

$$\liminf_{t\to+\infty} x(t) \ge \frac{\sum_{k=1}^{n}\gamma_k f_k}{\beta + aL_F}.$$

We get $x(t) \ge 0$.

Now, define $Q : C(\mathbb{R}) \to C(\mathbb{R})$

$$(Qx)(s) = \sum_{k=1}^{n} \gamma_k(s) f_k(\xi_k(s)x(s - \eta_k(s))) + a(s)F(x(s)),$$

Now, we estimate the nonlinear terms for $(t, x) \in \mathbb{R} \times [\mu, \delta_2]$, as:

$$\sum_{k=1}^{n} \gamma_k f_k(\xi\delta_2) - aL_F\delta_2 \leq (Qx)(t) \leq \sum_{k=1}^{n} \gamma_k f_k + aL_F\delta_2,$$

where $(Qx)(t)$ is defined.

Lemma 5.1.1 *Suppose* $U_0 = \{\psi : \psi \in C^+, \delta_1 < \psi(t) < \delta_2, t \in [-\eta, 0]\}$. *Suppose that (A1)-(A5) hold. Then,* $\forall\ \psi \in C$, *the solution* $x(t; t_0, \psi)$ *of equation(5.1.1) satisfies*

$$\delta_1 < x(t; t_0, \psi) < \delta_2, \quad t \in [t_0, \zeta(\psi))$$

and the existence interval $\forall$ *solution of equation (2) can be extended to* $[t_0, +\infty)$.

Now, we define,

$$(Px)(t) = \int_t^{\infty} e^{-\int_t^s \beta(y)dy}(Qx)(s)\, ds$$

Theorem 5.1.2 *Assume (A1)–(A6) all assumptions hold. If* $\eta_k \in KAA(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R}^+)$ *and* $\eta_k'(t) \leq \eta^* < 1$, *for* $k = 1, \ldots, n$, *and*

$$\frac{\overline{a}L_F + \sum_{k=1}^{n} \overline{\gamma_k\xi_k} l_{f_k}}{\underline{\beta}} \leq 1$$

Proof [1] Claim : $\forall\ x \in$ PKAA$(\mathbb{R}, \mathbb{R}^+)$, we get $(Qx)(s)$ belongs to PKAA$(\mathbb{R}, \mathbb{R}^+)$.
We can see that, the functions $x(t - \eta_k(t)) \in$ PKAA$(\mathbb{R}, \mathbb{R}^+)$ since $\eta_k \in$ KAA$(\mathbb{R}, \mathbb{R}^+)$.
Using previous lemma, $\xi_k(t)x(t - \eta_k(t)) \in$ PKAA$(\mathbb{R}, \mathbb{R}^+)$.

Using composition Theorem of pseudo almost automorphic functions, $s \mapsto f(\xi_k(s)x(s - \eta_k(s))) \in$ PKAA$(\mathbb{R}, \mathbb{R}^+)$
Now, we obtain $s \mapsto \gamma_k(t)f(\delta_k(s)x(s - \eta_k(s)))$ is in PKAA$(\mathbb{R}, \mathbb{R}^+)$, for $k = 1, \ldots, n$.

Since PKAA$(\mathbb{R})$ is a vector space, $(Qx)(s) \in$ PKAA$(\mathbb{R}, \mathbb{R}^+)$.

Hence $(Qx)(s) = (Qx)_1(s) + (Qx)_2(s)$, where $(Qx)_1(s)$ and $(Qx)_2(s)$ are in $\text{KAA}(\mathbb{R}, \mathbb{R}^+)$ and $\text{PAA}_0(\mathbb{R}, \mathbb{R}^+)$, respectively.

We note that

$$e^{-\int_s^t \beta(t)dt} \leq e^{-\underline{\beta}(t-s)},$$

next, P maps the space $\text{PKAA}(\mathbb{R}, \mathbb{R})$ into itself.

With the help of lemma 5.1.1, the closed subset

$$U_0 = \{x : x \in \text{PKAA}(\mathbb{R}, \mathbb{R}^+), \delta_1 \leq x(t) \leq \delta_2, \forall t \in \mathbb{R}\}$$

is such that $Q(U_0) \subset U_0$.

Hence, $\forall\ x, y \in \Omega$, we get

$$\begin{aligned} |(Px)(t) - (Py)(t)| &= \left| \int_{-\infty}^t e^{-\int_s^t \beta(\xi)d\xi} \left[(Qx)(s) - (Qy)(s)\right] ds \right| \\ &\leq \int_{-\infty}^t e^{-\underline{\beta}(t-s)} \left|(Qx)(s) - (Qy)(s)\right| ds. \end{aligned}$$

With the help of supremum norm, we obtain

$$||(Px) - (Py)||_\infty \leq \frac{\overline{a} L_F + \sum_{k=1}^n \overline{\gamma_k \xi_k} l_{f_k}}{\underline{\beta}} ||x - y||_\infty.$$

Hence, we get P is a contraction.

After using fixed point theorem, P has a unique fixed point in U_0. □

5.2 Exponential Stability of solution using Halanaya's Inequality

Lemma 5.2.1 *Let $x(s) > 0$, $t \in K$, and $T \in [0, \infty)$, $s_0 \in \mathbb{R}$. Suppose*

$$x'(t) \leq -bx(t) + a \sup x(t), \quad s > s_0.$$

If $b > a > 0$, then $\exists$ constants $\delta > 0$ and $\lambda > 0$ such that

$$x(s) \leq \lambda e^{-(s-s_0)}, \quad s > s_0.$$

Hence, $x(s) \to 0$ as $s \to \infty$.

Theorem 5.2.1 *Suppose $u(t)$ is a nonnegative function st*

$$u'(t) \leq -b(t)u(t) + a(t)\big(\sup_{t-T(t) \leq s \leq t} u(s)\big), \quad t > t_0.$$

Additionally,

$$u(s) \leq |\phi(s)|, \quad s \in [t_0 - T^*, t_0].$$

where $T(t)$ is nonnegative, continuous, and bounded function defined for $t \in \mathbb{R}$ and

$$T^* = \sup T(t).$$

The function $\phi(s)$ *is continuous and defined for* $s \in [t_0 - T^*, t_0]$. $b(t)$ *and* $a(t)$*, for* $t \in \mathbb{R}$*, are nonnegative, continuous, and bounded. Suppose*

$$b(t) - a(t) \geq a, \quad t \in \mathbb{R},$$

where

$$b = \inf(b(t) - a(t)) > 0.$$

Then $\exists$ *a* $\alpha \geq 0$ *st*

$$u(t) \leq \left(\sup_{s \in [t_0 - T^*, t_0]} u(s) \right) e^{-\alpha(t-t_0)}, \quad t > t_0.$$

Proof: [**12**] We define the function F as follows for $t > t_0$:

$$F(t, m) = -b(t) + m + a(t)e^{-mT^*}, \quad m \in \mathbb{R}, \quad t \in \mathbb{R}$$

Since $b(t)$ and $a(t)$ are continuous and bounded for $t \in \mathbb{R}$, we consider the function G defined by

$$G(m) = \sup_{t \in \mathbb{R}} F(t, m).$$

Clearly, G is a continuous function of $m \in \mathbb{R}$. Using assumption, we obtain from that

$$G(0) = \sup_{t \in \mathbb{R}} F(t, 0) = \sup_{t \in \mathbb{R}}(-b(t)+a(t)) = -\inf_{t \in \mathbb{R}}(b(t)-a(t)) = -b < 0.$$

Moreover, due to the boundedness of $b(t)$ and $a(t)$, we have

$$G'(m) = -b' + m + a'e^{-mT^*}, \quad m \in \mathbb{R} \leq G(m) \leq G''(m) = -b'' + m + a''e^{-mT^*}, \quad m \in \mathbb{R}$$

where

$$b' = \inf_{t \in \mathbb{R}} b(t), \quad b'' = \sup_{t \in \mathbb{R}} b(t),$$
$$a' = \inf_{t \in \mathbb{R}} a(t), \quad b'' = \sup_{t \in \mathbb{R}} a(t).$$

We can see that $G'(m) \to \infty$ and $G''(m) \to \infty$ monotonically as $m \to \infty$, and hence we get

$$G(m) = \sup_{t \in \mathbb{R}} F(t, m) \to \infty \text{ monotonically as } \mu \to \infty.$$

Now let $0 < \alpha < \delta$, from continuity of $G(m)$ that $\exists\ \alpha > 0$ such that

$$G(\alpha) = \sup_{t \in \mathbb{R}} F(t,) < 0, \quad \text{where } 0 < \alpha < \delta.$$

Now we define

Define

$$\phi^*(t) = \sup_{t_0 - T^* \leq s \leq t_0} |\phi(s)|.$$

We obtain

$$\overline{u}'(t) \leq [-b(t) + \alpha]\overline{u}(t) + a(t)e^{-\alpha T^*} \sup_{t - T^* \leq s \leq t} \overline{u(s)}, \quad t > t_0.$$

Since $\psi(t)$ is continuous and defined for $t \in [t_0 - T^*, t_0]$, we let

$$\sup_{t_0 - T^* \le t \le t_0} |\psi(t)| = M, \quad M > 0.$$

Let $\epsilon > 1$ be arbitrary. We have

$$\overline{x(t)} \le \epsilon M, \quad \text{for } t \in [t_0 - T^*, t_0].$$

We claim that

$$\overline{u(t)} \le \epsilon M, \quad \text{for } t > t_0$$

Suppose above equation does not hold. Let $t_1 > t_0$ be the first time for which

$$\overline{u(t)} \le \epsilon M, \quad t_0 - T^* \le t < t_1.$$

Then we have

$$\overline{u(t_1)} = \epsilon M, \quad \text{and} \quad u'(t_1) \ge 0.$$

From previous equation, we have

$$\overline{u}'(t_1) \le [-b(t_1) + m]u(t_1) + a(t_1)e^{-mT^*} \sup_{t_1 - T^* \le s \le t_1} u(s).$$

Since $\overline{u(t_1)} = \epsilon M$, we obtain

$$\overline{u}'(t_1) \le -a\epsilon M < 0,$$

Since $\overline{u(t_1)} = \epsilon M$, we obtain

$$\overline{u}'(t_1) \le -a\epsilon M < 0,$$

This leads to a contradiction. Hence, the claim holds. Since $\epsilon > 1$ is arbitrary, by allowing $\epsilon \to 1^+$, we get

$$\overline{u(t)} \le M, \quad t > t_0.$$

It then follows and from u we have

$$u(t) \le Me^{-\alpha(t - t_0)}, \quad t > t_0.$$

Thus, the assertion is satisfied, completing the proof. □

Lemma 5.2.2 *Let $s_0 \in \mathbb{R}$ and η be a non-negative number. If $u : [s_0 - \eta, \infty) \to \mathbb{R}^+$ satisfies **[1]***

$$\frac{d}{ds}u(s) \le -\beta u(s) + \gamma \sup_{t \in [s - \eta, s]} u(t); \quad s \ge s_0,$$

where β and γ are constants with $\beta > \gamma > 0$, then

$$u(s) \le \|u(s_0)\| e^{-\zeta(s - s_0)} \quad \textit{for } s \ge s_0,$$

where ζ is the unique positive solution of

$$\zeta = \beta - \gamma e^{\zeta\eta}.$$

Theorem 5.2.2 *Suppose assumptions (A1)–(A6) hold, and above results hold, then the unique pseudo compact almost automorphic* *solution of the system in U_0 is globally exponentially stable.*

Proof.[1] The uniqueness and existence pseudo compact almost automorphic solution has been done.

Now, suppose $x(t)$ be a solution of equation st $\delta_1 \leq x(t) \leq \delta_2 \ \forall \ t \geq 0$.

$x^*(t)$ is the pseudo compact almost automorphic solution of the equation.

We consider

$$u(t) := x(t) - x^*(t),$$

clearly

$$\dot{u}(t) = -\alpha(t)u(t) + (Qx)(t) - (Qx^*)(t),$$

Consequently, $\forall \ t \geq 0$ we get

$$u(t) = u(0)e^{-\int_0^t \beta(\alpha)d\alpha} + \int_0^t e^{-\int_s^t \beta(\alpha)d\alpha} \left[(Qx)(s) - (Qx^*)(s)\right] ds.$$

It follows that

We have:

$$|u(t)| \leq |u(0)e^{-\int_0^t \beta(\alpha)\,d\alpha}| + \int_0^t |e^{-\int_s^t \beta(\alpha)\,d\alpha}||(Qx)(s) - (Qx^*)(s)|\,ds,$$

since $x(t)$ and $x^*(t)$ belong to U_0, and by $A6$, we obtain

$$|u(t)| \leq \|u\|_\zeta e^{-\underline{\beta}t} + \int_0^t e^{-\underline{\beta}(t-s)} \sum_{k=1}^{n} \overline{\gamma_k \xi_k f_k} + \overline{a} L_F \sup_{\sigma \in [s-\zeta, s]} |u(\sigma)|\,ds.$$

It implies that if

$$\underline{\beta} > \sum_{k=1}^{n} \overline{\gamma_k \xi_k f_k} + \overline{a} L_F > 0,$$

then by Halanay's inequality, we get $\exists$ positive constants η and M st:

$$|u(t)| \leq M\|u\|_\eta e^{-\zeta t},$$

where ζ is the real solution of the characteristic equation

$$\zeta = -\beta + (\sum_{k=1}^{k} \overline{\gamma_k \xi_k f_k} + \overline{b} L_F)e^{\zeta\eta}.$$

Therefore, we have

$$|x(t) - x^*(t)| \leq M\|x - x^*\|_\zeta e^{-\zeta t}, \quad t \geq 0.$$

Thus, x^* is asymptotically exponentially stable. $\square$

6 Result

6.1 Solving Nicholson Blowflies Model using Weighted Pseudo Compact Almost Automorphic function

First, we give some results that helps us in proving our result.

Lemma 6.1.1 *If $l(\cdot) \in WPKAA(\mathbb{R}, \mathbb{R})$, $\zeta(\cdot) \in KAA(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R}^+)$, and $\zeta(t) \leq \zeta^* < 1$, then we have $t \mapsto l(t - \zeta(t)) \in PKAA(\mathbb{R}, \mathbb{R})$.*

Proof. Since, we have $l(\cdot) \in PKAA(\mathbb{R}, \mathbb{R})$, we get $l(\cdot) = l_1(\cdot) + l_2(\cdot)$, where $l_1(\cdot) \in KAA(\mathbb{R}, \mathbb{R})$ and $l_2(\cdot) \in PAA_0(\mathbb{R}, \mathbb{R})$.

Now,we get that

$$l(t - \zeta(t)) = l_1(t - \zeta(t)) + l_2(t - \zeta(t)).$$

We get $l_1(t - \zeta(t)) \in KAA(\mathbb{R}, \mathbb{R})$.

Claim: $l_2(t - \zeta(t)) \in \mathrm{PAA}_0(\mathbb{R}, \eta)$. Using Assumption $\zeta(t)$, we have $\exists \zeta_+ \geq 0$ such that

$$\zeta_+ := \sup_{t \in \mathbb{R}} \{\zeta(t)\}.$$

We consider $t = s - \zeta(s)$ and $dt = (1 - \zeta'(s))ds$, that gives

$$\lim_{b \to \infty} \frac{1}{\mu(b, \eta)} \int_{-b}^{b} |l_2(s - \eta(s))| (\eta(s))\, ds =$$

$$\lim_{b \to \infty} \frac{1}{\mu(b, \eta)} \int_{-b-\zeta(-T)}^{b-\zeta(T)} |l_2(t)| (\eta(s)) \frac{1}{1 - \zeta'(t)}\, dt$$

$$\leq \frac{1}{1 - \zeta^*} \lim_{b \to \infty} \frac{\mu(b, \eta)}{\mu(b, \eta + \zeta_+)} \frac{1}{\mu(a, \eta + \zeta_+)} \int_{-b-\zeta_+}^{b+\zeta_+} |l_2(t)|\, dt.$$

Since the above estimation, $l_2 \in \mathrm{PAA}_0(\mathbb{R}, \eta)$, and $\frac{\mu(b,\eta)}{\mu(b,\eta+\zeta_+)}$ is finite, we obtain that

$$\lim_{b \to \infty} \frac{1}{\mu(b, \eta)} \int_{-T}^{T} |l_2(s - \zeta(s))| (\eta(s))\, ds = 0.$$

Lemma 6.1.2 *Suppose l and g in $WPKAA(\mathbb{R}, \mathbb{R})$. Then product $f \cdot g$ is also in $WPKAA(\mathbb{R}, \mathbb{R})$.*

Proof Since, l and g $\in (WPKAA(\mathbb{R},\mathbb{R})$. We have $l = l_1 + l_2$ and $g = g_1 + g_2$ where $l_1, g_1 \in KAA(\mathbb{R},\mathbb{R})$ and $l_2, g_2 \in PAA_0(\mathbb{R},\eta)$.

We get $l_1.g_1 \in KAA(\mathbb{R},\mathbb{R}))$. Now, we claim that $l_2.g_2 \in PAA0(\mathbb{R},\eta)$. Since $g_2(.) \in PAA_0(\mathbb{R},\eta) \in BC(\mathbb{R},\mathbb{R})$. So, we have supremum of $g_2(.)$, let we denote it by $g^* \leq \infty$.

$$\lim_{b\to\infty} \frac{1}{\mu(b,\eta)} \int_{-b}^{b} |l_2(t).g_2(t)|(\eta(t))\,dt \leq \lim_{b\to\infty} \frac{1}{\mu(b,\eta)}|g^*| \int_{-b}^{b} |l_2(s)|(\eta(t))ds$$

Since, $l_2(.) \in PAA0(R,\eta)$.

$\lim_{b\to\infty} \frac{1}{\mu(b,\eta)}|g^*| \int_{-b}^{b} |l_2(t)|(\eta(s))dt = |g^*| \lim_{a\to\infty} \frac{1}{m(b,\eta)} \int_{-b}^{b} |l_2(t)|(\eta(s))dt$
$= |g^*|.0 = 0$.
Hence prove.□

Theorem 6.1.1 *Let $l : [0,+\infty) \to [0,+\infty)$ be a Lipschitz continuous function, i.e.,*

$$\frac{|l(m) - l(n)|}{|m-n|} \leq L_l, \quad m,n \in [0,+\infty).$$

Then $l(x(t)) \in WPKAA(\mathbb{R},\mathbb{R}^+)$, provided $x(t) = x_1(t) + x_2(t) \in WPKAA(\mathbb{R},\mathbb{R}^+)$, where $x_1(t) \in KAA(\mathbb{R},\mathbb{R}^+)$ and $x_2(t) \in PAA_0(\mathbb{R},\mathbb{R}^+)$.

Proof. Since $x = x_1 + x_2 \in \text{WPKAA}(\mathbb{R},\mathbb{R}^+)$, we can decompose $l(x(t))$

$$l(x(t)) = l(x_1(t)) + (l(x(t)) - l(x_1(t)))\,.$$

Suppose we denote:

$$Q_1(t) = l(x_1(t)), \quad Q_2(t) = l(x(t)) - l(x_1(t)).$$

Claim: $Q_1(t) \in \text{KAA}(\mathbb{R},\mathbb{R}^+)$. (done in theorem 2.3.4)
Claim:$\text{Q}_2(t) \in PAA_0(\mathbb{R},,\eta)$.
We can see that

$$\frac{1}{\mu(b,\eta)} \int_{-b}^{b} Q_2(s)\,ds = \frac{1}{\mu(b,\eta)} \int_{-b}^{b} (l(x(s)) - l(x_1(s)))\;ds$$

$$\leq L_l \frac{1}{\mu(b,\eta)} \int_{-b}^{b} |x(s) - x_1(s)|\,ds = L_l \frac{1}{\mu(b,\eta)} \int_{-b}^{b} |x_2(s)|\,ds,$$

and since $x_2(s) \in \text{PAA}_0((\mathbb{R},\mathbb{R}^+),\eta)$ the claim follows. □

Lemma 6.1.3 *Suppose $g : \mathbb{R} \to \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}$ belong to $WPKAA(\mathbb{R},\mathbb{R})$ and $L^1(\mathbb{R})$, respectively, then $g{*}x$ belongs to $WPKAA(\mathbb{R})$.*

Proof. We have $g \in PKAA(\mathbb{R}, \mathbb{R})$, it follows that $(x * g)(s) = (x * g_1)(s) + (x * g_2)(s)$, where $g_1 \in KAA(\mathbb{R}, \mathbb{R})$ and $g_2 \in PAA_0(\mathbb{R}, \eta)$. We need to prove that: (a) $x * g_1 \in KAA(\mathbb{R}, \mathbb{R})$. (b) $x * g_2 \in PAA_0(\mathbb{R}, \mathbb{R})$.
a) has been done in lemma 2.3.6.
b) Finally, we prove that $x * g_2 \in \mathrm{PAA}_0(\mathbb{R}, \mathbb{R})$, i.e.

$$\lim_{b \to +\infty} \frac{1}{\mu(b,\eta)} \int_{-b}^{b} |k * g_2(t)|\, dt = 0.$$

Using Fubini's Theorem, we get

$$\frac{1}{\mu(b,\eta)} \int_{-b}^{b} |x * g_2(s)|\, dt \le \frac{1}{\mu(b,\eta)} \int_{-b}^{b} \int_{-\infty}^{\infty} |x(s-t)||g_2(t)|\, dt\, ds =$$

$$\int_{-\infty}^{\infty} |x(\xi)| \left(\frac{1}{\mu(b,\eta)} \int_{-b}^{b} |g_2(t-\xi)|\, dt \right) d\xi.$$

Since $g_2 \in \mathrm{PAA}_0(\mathbb{R})$ and using the property of invariance by translation of the mean, it follows.
From $x \in L^1(\mathbb{R})$, we can verify DCT. Thus, we conclude that $x * g_2$ belongs to $\mathrm{PAA}_0(\mathbb{R})$.

Theorem 6.1.2 *If the function* $x : \mathbb{R} \to \mathbb{R} \in WPKAA(\mathbb{R}, \mathbb{R})$ *and* $\exists$ *a almost automorphic function g(s, t) wrt t such that:*

$$|g(s,t)| \le c e^{-\beta(s-t)}, \quad s \ge t,$$

then $\psi(t) = \int_{-\infty}^{s} g(s,t) x(t) dt$ *belongs to the class* $\mathrm{WPKAA}(\mathbb{R}, \mathbb{R})$.

Proof Since,we have x $\in \mathrm{WPKAA}(\mathbb{R}, \mathbb{R})$. So, we can decompose x as x $= x_1 + x_2$ where $x_1 \in \mathrm{KAA}(\mathbb{R}, \mathbb{R})$ and $x_2 \in PAA_0(\mathbb{R}, \eta)$.
Claim : $\psi(s) = \int_{-\infty}^{s} g(s,t) x(t) dt \in WPKAA(\mathbb{R}, \mathbb{R})$.
We can write, $\psi(s) = \int_{-\infty}^{s} g(s,t) x_1(t) + \int_{-\infty}^{s} g(s,t) x_2(t)$. We have $x_1(t) \in KAA(\mathbb{R})$.
So, $\forall$ sequence of $\mathbb{R}$ (t_n), $\exists$ a subsequence (t_{n_k}) st

$$x_1(t) := \lim_{k \to \infty} x_1'(t + t_{n_k})$$

$$\lim_{k \to \infty} x_1'(t - t_{n_k}) = x_1(t)$$

$$g(s,t) := \lim_{k\to\infty} g'(s, t+t_{n_k})$$

$$\lim_{k\to\infty} g'(s, t-t_{n_k}) = g(s)$$

$$\lim_{k\to\infty} \psi(s+t_{n_k}) = \lim_{k\to\infty} \int_{-\infty}^{s} g(s, t+t_{n_k}) u_1(s+t_{n_k})$$

Since g $\in L^1(\mathbb{R})$. Using DCT, we have
$\int_{-\infty}^{s} \lim_{k\to\infty} g(s, t+t_{n_k}) x_1(s+t_{n_k}) = \int_{-\infty}^{s} g'(s,t) x_1'(t)$. Let $\psi'(s - t_{n_k}) = \int_{-\infty}^{s} g'(s, t-t_{n_k}) x_1'(s-t_{n_k})$
Similarly, $\lim_{k\to\infty} \psi'(s-t_{n_k}) = \lim_{k\to\infty} \int_{-\infty}^{s} g'(s, t-t_{n_k}) x_1'(s-t_{n_k}) = \int_{-\infty}^{s} \lim_{k\to\infty} g'(s, t-t_{n_k}) x_1'(s-t_{n_k}) = \int_{-\infty}^{s} g(s,t) x_1(t)$.

$$\lim_{k\to\infty} x(s-t_{n_k}) = l(s)$$

And $\psi(s)$ is uniformly continuous.
We have $x_2 \in PAA_0(\mathbb{R}, \eta)$. So,
$\lim_{b\to+\infty} \frac{1}{\mu(b,\eta)} \int_{-b}^{b} \|\psi(s)\| \, \eta(s)\, ds =$

$$\lim_{b\to+\infty} \frac{1}{\mu(b,\eta)} \int_{-b}^{b} \| \int_{-\infty}^{s} g(s,t) x(t) dt(s) \| \, \eta(s)\, ds = 0$$

Hence proof. □

Model and Assumptions

Since we are looking for result of Nicholson model, we first modify our equation (5.1.1) with mixed delay.
$x'(t) = \beta(t)x(t) + \sum_{k=1}^{m} \gamma_k(t) x(t-\zeta_k) e^{-\xi_k x(t-\zeta_k)} - F(t)x(t) + p(t) \int_{-\zeta}^{0} K(t,s) x(t+s) e^{-x(t+s)}\, ds$ ....(6.1.1)

In this equation (6.1.1) we consider type of birth term and generalize our equation, birth term is

$H_k(t, x(t-\zeta_k(t))) := \gamma_k(t) g_k(\xi_k(t) x(t-\zeta_k(t))) = \beta_k(t) x(t-\zeta_k) e^{-\xi_k x(t-\zeta_k)}$.

Now, we introduce the following scalar delay differential equation:
$x'(t) = \beta(t)x(t) + \sum_{k=1}^{m} \gamma_k(t) x(t-\zeta_k) e^{-\xi_k x(t-\zeta_k)} - F(t)x(t) + p(t) \int_{-\zeta}^{0} K(t,s) x(t+s) e^{-x(t+s)}\, ds$ ....(6.1.2)

Assumptions require to get our result.

- **A1.** $\beta(t)$ is positive almost automorphic on $\mathbb{R}$, $\zeta_k(t)$ are positive compact almost automorphic on $\mathbb{R}$ for $k = 1, \dots, n$, while $\gamma_k(t)$, $\xi_k(t)$, and $a(t)$, $p(t)$ are positive weighted pseudo compact almost automorphic functions.
- **A2.** $\beta(\cdot)$, $\gamma_k(\cdot)$, and $\xi_k(\cdot)$ are bounded away from zero.
- **A3.** $F : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is Lipschitz continuous and $H(0) = 0$, i.e., $\exists$ a positive constant L_F such that:

$$-L_H x < H(x) < L_H x, \quad \text{for } x \in \mathbb{R}_0^+.$$

Notation: $\forall$ bounded continuous function g defined on $\mathbb{R}$.
Let

$$\overline{g} = \sup_{t\in\mathbb{R}}\{g(t)\}, \underline{g} = \inf_{t\in\mathbb{R}}\{g(t)\},$$

we have positive constants:

$$\overline{\zeta} = \sup_{1\le k\le n}\{\zeta_k\}, \underline{\zeta} = \inf_{1\le k\le n}\{\xi_k\}.$$

- **A4.** $\forall\ 1 \le k \le n$, the functions $g_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are Lipschitz continuous, non-negative, their maximum value over $\mathbb{R}_0^+$, i.e., $\overline{g_k} = g_i(m_i^*)$ with $m_i^* \in \mathbb{R}_0^+$ and $g_i$is a non-increasing function for $x > m_i^*$.
- **A5.** $\exists$ two positive constants δ_1 and δ_2 st:
$\frac{\left(\sum_{k=1}^{n} \overline{\gamma_k g_k} + \overline{a} L_F \delta_2 + \overline{p}\zeta M\right)}{\underline{\beta}} \le \delta_2.$

  $\frac{\left(\sum_{i=1}^{n} \underline{\alpha_i g_i} - \overline{a} L_F \lambda_1\right)}{\overline{\beta}} \ge \lambda_1$
- **A6.** $\forall\ 1 \le k \le n$, $\exists$ positive numbers l_{g_k} which are the Lipschitz constants of g_k on $[\overline{m^*}, \infty)$.
- **A7.** The kernel K(t, s) satisfies following conditions:

  (i) $\forall\ t \in \mathbb{R}^+$ and $s \in [t - s, t]$, $K(t, s) \ge 0$.

(ii) $\exists$ constants $k > 0$ and $c > 0$ st for all $t \in \mathbb{R}^+$ and $s \in [t-s, t]$,

$$K(t,s) \le ke^{x(t-s)}.$$

(iii) The function $t \mapsto K(t,s)$ is almost automorphic $\forall s$.

Define $Q : C(\mathbb{R}) \to C(\mathbb{R})$ by

$$(Qx)(s) = \sum_{k=1}^{n} \beta_i(s) g_k(\lambda_k(s)x(s-\zeta_k(s))) + a(s)H(x(s)) +$$

$$p(t)\int_{-\zeta}^{0} K(t,s)x(t+s)e^{-x(t+s)}\,ds$$

$$(Px)(t) = \int_{-\infty}^{t} e^{-\int_s^t \beta(\alpha)\,d\alpha}(Qx)(s)\,ds,$$

where $x \in WPKAA(\mathbb{R},\mathbb{R})$ and (Qx) is defined .

Theorem 6.1.3 *Assume (A1)–(A7) hold. If*

$\zeta_k \in KAA(\mathbb{R},\mathbb{R}^+) \cap C^1(\mathbb{R},\mathbb{R}^+)$ *and* $\zeta_k(t) \le \zeta^* < 1$, *for* $k = 1, \ldots, n$,

and $\frac{\overline{a}L_F + \sum_{k=1}^{n}\overline{\gamma_k\xi_k}l_{g_k} + \overline{p}M\zeta}{\underline{\beta}} < 1$
are satisfied, then equation has unique weighted pseudo compact almost automorphic solution in

$$U = \{x : x \in WPKAA(\mathbb{R},\mathbb{R}^+),\ \delta_1 \le x(t) \le \delta_2,\ \forall t \in \mathbb{R}\}.$$

Proof. Claim: $\forall\, x \in WPKAA(\mathbb{R},\mathbb{R}^+)$ it gives $(Qx)(s) \in WPKAA(\mathbb{R},\mathbb{R}^+)$.
$x(t-\zeta_k(t)) \in WPKAA(\mathbb{R},\mathbb{R}^+)$ since $\zeta_k \in KAA(\mathbb{R},\mathbb{R}^+)$ from lemma 6.1.1.
It gives $\delta_k(t)x(t-\zeta_k(t)) \in WPKAA(\mathbb{R},\mathbb{R}^+)$ from lemma 6.1.2. From the property of weighted pseudo almost automorphic functions, $s \mapsto g_k(\xi_k(s)x(s-\zeta_k(s)))$ belongs to $PKAA(\mathbb{R},\mathbb{R}^+)$ and we obtain that $s \mapsto \gamma_k(t)g_k(\xi_k(s)x(s-\zeta_k(s)))$ is in $WPKAA(\mathbb{R},\mathbb{R}^+)$,

for $k = 1, \ldots, n$ from theorem 6.1.1.

Since, F(x(s)) is lipschitz and x(s) ∈ WPKAA and a(s) ∈ WPKAA. So, from previous theorems, a(s).F(x(s)) ∈ WPKAA.
In consequence, since $WPKAA(\mathbb{R})$ is a vector space, the function $(Qx)(s)$ is in $WPKAA(\mathbb{R}, \mathbb{R}^+)$, hence $(Qx)(s) = (Qx)_1(s) + (Qx)_2(s)$, where $(Qx)_1(s)$ and $(Qx)_2(s)$ are in $KAA(\mathbb{R}, \mathbb{R}^+)$ and $PAA_0((\mathbb{R}, \mathbb{R}^+), \eta)$, respectively.

$$e^{-\int_t^s \beta(\alpha)\,d\alpha} \leq e^{-\int_t^s \underline{\beta}\,d\alpha} = e^{-\underline{\beta}(t-s)},$$

then it maps Q maps the space $WPKAA(\mathbb{R}, \mathbb{R})$ into itself.
The closed subset

$$U = \{x : x \in WPKAA(\mathbb{R}, \mathbb{R}^+),\ \delta_1 \leq x(t) \leq \delta_2,\ \forall t \in \mathbb{R}\}$$

Claim: $P(U) \subset P$.
Proof.
$P(x(t) = \int_{-\infty}^t e^{-\int_s^t \beta(\alpha)\,d\alpha} (\sum_{k=1}^n \gamma_k(s) f_i(\xi_k(s)x(s-\zeta_k(s))) + a(s)F(x(s)) + p(t)\int_{-\zeta}^0 K(t,s)x(t+s)e^{-x(t+s)})\,ds \leq \frac{(\sum_{k=1}^n \overline{\gamma_k f_k} + \overline{a} L_F \delta_2 + \overline{p}\zeta M)}{\underline{\alpha}} \leq \gamma_2$

$P(x(t) = \int_{-\infty}^t e^{-\int_s^t \beta(\alpha)\,d\alpha} (\sum_{k=1}^n \gamma_k(s) f_i(\xi_k(s)x(s-\zeta_k(s))) + a(s)F(x(s)) + p(t)\int_{-\zeta}^0 K(t,s)x(t+s)e^{-x(t+s)})\,ds \geq \frac{(\sum_{i=1}^n \underline{\alpha_i g_i} - \overline{a} L_F \lambda_1)}{\overline{\beta}} \geq \lambda_1$
Now, $\forall\ u, v \in U$, we get

$$|(Pu)(t) - (Pv)(t)| = \left| \int_{-\infty}^t e^{-\int_s^t \beta(\alpha)\,d\alpha} ((Qu)(s) - (Qv)(s))\,ds \right|$$

$$\leq \int_{-\infty}^t e^{-\underline{\beta}}(t-s)\,|(Qu)(s) - (Qv)(s)|\,ds \leq$$

$$\int_{-\infty}^t e^{-\underline{\beta}(t-s)}\,ds \left(\sum_{k=1}^n \overline{\gamma_k g_k} + \overline{a} L_F + \overline{p}\zeta M \right) \|u - v\|$$

Using the supremum norm, we have
$\|(Pu) - (Pv)\|_\infty \leq \frac{(\sum_{k=1}^n \overline{\gamma_k g_k} + \overline{a} L_F + \overline{p}\zeta M)}{\underline{\beta}} \|u - v\|_\infty$
Thus, by using assumption and fixed point theorem, P has a unique fixed point in U.

Theorem 6.1.4 *If all assumptions (A1) to (A7) are satisfied, and also assumption of previous theorem holds, then weighted pseudo compact almost automorphic solution in U is globally exponentially stable.*

Proof. We have done the uniqueness and existence of weighted pseudo compact almost automorphic solution of equation.

Suppose $x(t)$ is a solution of equation and $x^*(t)$ is the weighted pseudo compact almost automorphic solution of equation. Consider

$$g(t) := x(t) - x^*(t),$$

and clearly

$$\dot{g}(t) = -\beta(t)g(t) + (Nx)(t) - (Nx^*)(t),$$

where it holds. $\forall\ t \geq 0$, we have

$$g(t) = g(0)e^{-\int_0^t \beta(\alpha)\,d\alpha} + \int_0^t e^{-\int_s^t \beta(\alpha)\,d\alpha} \left((Qx)(s) - (Qx^*)(s)\right) ds.$$

It gives

$$|g(t)| \leq |g(0)e^{-\int_0^t \beta(\alpha)\,d\alpha}| + \int_0^t |e^{-\int_s^t \beta(\alpha)\,d\alpha}|\,|(Nx)(s) - (Nx^*)(s)|\,ds.$$

Since $x(t)$ and $x^*(t)$ in Ω, and using assumption (A6), we obtain

$$|v(t)| \leq \|v\|_\zeta e^{-\beta t} + \int_0^t e^{-\beta(t-s)} \left(\sum_{k=1}^n \overline{\gamma_k \xi_k f_i}\right.$$
$$\left. + \overline{a} L_F + \overline{p} M\zeta \sup_{\sigma\in[s-\zeta,s]} |v(\sigma)|\,ds\right.$$

We have

$\frac{\overline{a}L_F + \sum_{k=1}^n \overline{\gamma_k \xi_k f_k} + \overline{p}M\zeta}{\beta} < 1$

Then, by using Halanay's inequality, we get $\exists$ positive constants ω and C st

$$|g(t)| \leq C\|g\|_\zeta e^{-\omega t},$$

where ω is the real solution of the characteristic equation 6.1.1.

$\omega = -\underline{\beta} + (\sum_{k=1}^n \overline{\gamma_k \xi_k f_k} + \overline{a}L_F + \overline{p}M\zeta)e^{\omega\zeta}$

Therefore, we have

$$|x(t) - x^*(t)| \le C\|x - x^*\|_\tau e^{-\omega t}, \quad t \ge 0.$$

This concludes the result.

6.2 Stability Analysis using Lyapnov Stability

[**6**] Suppose $g : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous function.
Consider

$$u'(t) = g(t, u_t).$$

g is continuous here and the solution $u(x, \psi)(t)$ through (x, ψ) is continuous in (x, ψ, t) within the domain of the function.

$$g : (b, \infty) \times U \to \mathbb{R}^n,$$

where U is an open set in C.

Definition 6.2.1 Suppose $g(t, 0) = 0 \; \forall \; t \in \mathbb{R}$. Solution $u = 0$ of equation is *stable* if, $\forall \; x \in \mathbb{R}$ and $\delta > 0$, $\exists \; \epsilon = \epsilon(\delta, x)$ st $\psi \in B(0, \epsilon)$ gives

$$u_t(x, \psi) \in B(0, \delta) \quad \text{for } t \ge x.$$

Definition 6.2.2 Suppose $g(t, 0) = 0 \; \forall \; t \in \mathbb{R}$. Solution $u = 0$ of equation is *asymptotically stable* if it is stable and $\exists \; \epsilon_0 = \epsilon_0(u) > 0$ st $\psi \in B(0, \epsilon_0)$ implies

$$u(x, \psi)(t) \to 0 \quad \text{as } t \to \infty.$$

Definition 6.2.3 Suppose $g(t, 0) = 0 \; \forall \; t \in \mathbb{R}$. Solution $u = 0$ of equation is *uniformly stable* if, $\forall \; x \in \mathbb{R}$ and $\delta > 0$, $\exists \; \epsilon = \epsilon(\delta)$ st $\psi \in B(0, \epsilon)$ gives

$$u_t(x, \psi) \in B(0, \delta) \quad \text{for } t \ge x.$$

Definition 6.2.4 Suppose $g(t, 0) = 0 \; \forall \; t \in \mathbb{R}$. Solution $u = 0$ of equation is *uniformly asymptotically stable* if it is uniformly stable and $\exists \; \epsilon_0 > 0$ such that for every $\zeta > 0$, $\exists \; t_0(\zeta)$ such that

$$\psi \in B(0, \epsilon_0) \implies u_t(x, \psi) \in B(0, \eta) \quad \text{for } t \ge x + t_0(\zeta), \quad \text{for every } x \in \mathbb{R}.$$

Lyapnov Function: A *Lyapunov function* for an autonomous dynamical system

$$\begin{cases} g : \mathbb{R}^n \to \mathbb{R}^n, \\ \dot{x} = g(x) \end{cases}$$

with an equilibrium point at $x = 0$ is a scalar function

$$U : \mathbb{R}^n \to \mathbb{R}$$

that is continuous, has continuous first derivatives, is strictly positive for $x \neq 0$, and for which the time derivative

$$\dot{U} = \nabla U \cdot g$$

is non-positive.

Theorem 6.2.1 *Suppose* $g : \mathbb{R} \times C \to \mathbb{R}^n$ *maps* $\mathbb{R}\times$ *(bounded sets of* C*) into bounded sets of* $\mathbb{R}^n$*, and let* $x, y, z : \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous, nondecreasing functions such that* $x(t)$ *and* $y(t)$ *are positive for* $t > 0$*, with* $x(0) = y(0) = 0$*.*
If $\exists$ *a continuous function* $U : \mathbb{R} \times C \to \mathbb{R}$ *such that*

$$x(|\psi(0)|) \leq U(t, \psi) \leq y(|\psi|)$$

$$U(t, \psi) \leq -z(|\psi(0)|),$$

then $u = 0$ *of Equation is uniformly stable.*

Lyapnov Stability of equation

Here, we first give some notations for this section.
• refers to differentiation wrt t, for eg.   is du/dt.

$$K' = inf(K(t)), K'' = sup(K(t))$$

$$M = sup(x(t)), m = inf(x(t))$$

$$p' = inf(p(t)), p'' = sup(p(t))$$

$$\lambda' = inf(\lambda(t)), \lambda" = sup(\lambda(t))$$

$$\beta' = inf\beta(t), \beta" = sup\beta(t)$$

$$\alpha' = inf\alpha(t), \alpha" = sup\alpha(t)$$

$$b' = inf(b(t)), b" = sup(b(t))$$

$$\dot{u(t)} = \alpha(t)u(t) + \sum_{i=1}^{n} \beta_i(t)u(t-\tau_i)e^{-\lambda_i u(t-\tau_i)} - H(t)u(t) +$$

$$p(t)\int_{-\tau}^{0} K(t,s)u(t+s)e^{-u(t+s)}\,ds$$

Let $u(t_0)$ is the solution of the above differential equation.
We can translate t as $t_0 - t$.
$\dot{u(t+t_0)} = \alpha(t+t_0)u(t+t_0) + \sum_{i=1}^{m} \beta_i(t_0+t)u(t+t_0-\tau_i)e^{-\lambda_i u(t+t_0-\tau_i)} - H(t+t_0)u(t+t_0) + p(t+t_0)\int_{-\tau}^{0} K(t_0+t+\theta)u(t+t_0+s)e^{-u(t+t_0+s)}\,ds$
Put

$$u(t+t_0) = x(t).$$

Lyapnov function is

$$V(x,t) = x^2(t) - \mu\int_{-\infty}^{T}\sum_{i=1}^{n}\beta_i(\theta)x(\theta)d(\theta) + \int_{-T}^{T}\sum_{i=1}^{n} K(t_0+t+\theta)x^2(\theta)e^{-x(\theta)}\,d\theta$$

- **Assumption 1**

$$\frac{MK'(2T)}{e} \le \mu m\beta''$$

- **Assumption 2**

$$\beta^2\mu^2 \le \frac{4K'(T)(\mu)}{en}$$

- **Assumption 3**

$$d(K(t)/d(t) = -\eta$$

  where

$$\eta \ge 0$$

Claim :

$$-\mu\int_{-\infty}^{T}\sum_{i=1}^{n}\beta_i(\theta)x(\theta)d(\theta) + \int_{-T}^{T}\sum_{i=1}^{n} K(t_0+t+\theta)x^2(\theta)e^{-x(\theta)}\,d\theta \le 0$$

Using properties of sup and inf, we get

$$\frac{nMK'(2T)}{e} - \mu mn\beta'' \le 0$$

$$\frac{MK'(2T)}{e} \leq \mu m \beta''.$$

This is Assumption 1.

Hence,

$$V(x,t) \leq x^2(t)$$

Claim :

$$\frac{x^2}{2} \leq x^2(t) - \mu \int_{-\infty}^{T} \sum_{i=1}^{n} \beta_i(\theta) x(\theta) d(\theta) + \int_{-T}^{T} \sum_{i=1}^{n} K(t_0 + t + \theta) x^2(\theta) e^{-x(\theta)} \, d\theta$$

$$\frac{x^2}{2} \geq \mu \int_{-\infty}^{T} \sum_{i=1}^{n} \beta_i(\theta) x(\theta) d(\theta) - \int_{-T}^{T} \sum_{i=1}^{n} K(t_0 + \theta + t) x^2(\theta) e^{-x(\theta)} \, d\theta$$

$$\frac{m^2}{2} - \mu \beta'' n m + \frac{2nK'TM}{e} \geq 0$$

$$\beta^2 \mu^2 \leq \frac{4K'(T)(\mu)}{en}$$

This is assumption 2.

Hence, $V(x,t) \geq \frac{x^2}{2}$

$$V(x,t) = x^2(t) - \mu \int_{-\infty}^{T} \sum_{i=1}^{n} \beta_i(\theta) x(\theta) d(\theta) + \int_{-T}^{T} \sum_{i=1}^{n} K(t_0 + \theta) x^2(\theta) e^{-x(\theta)} \, d\theta$$

To Prove: $V(t, \dot{x}(t)) \leq 0$

Assumption 3 :

$$d(K(t)/d(t) = -\eta$$

where

$$\eta \geq 0$$

$$V(t, \dot{x}(t)) = \ x^2(t)\left(-2\alpha(t+t_0) - l_i b(t)\right) + \int_{-T}^{T} \sum_{i=1}^{n} \dot{K}(t_0{+}t{+}\theta)x^2(\theta){-}e^{x(\theta)} d\theta$$

$$2\frac{\sum_{i=1}^{n} \beta_i(t+t_0)}{e\lambda_i} + \frac{2p(t+t_0)\int_{-\tau}^{0} K(t_0+t+\theta)}{e})x(t)$$

Since the coefficient of $x^2(t)$ is negative, to prove our desired result, we use the quadratic discriminant condition $b^2 - 4ac \leq 0$:

$$\left(2\sum_{i=1}^{n} \beta_i e^{\lambda_i}(t_0+t) + 2p(t+t_0)\int_{-\tau}^{0} Ke(t_0+t+\theta)\right)^2$$
$$-4\left(-2\alpha(t+t_0) - lib(t)\right)\left(\int_{-T}^{T} \sum_{i=1}^{n} \dot{x}(t_0+t+\theta)x^2(\theta)e^{-x(\theta)}\right) \leq 0$$

Finally using properties of sup and inf:
We will get desired result.

7 Examples

Example 1: Suppose $g : \mathbb{R} \to \mathbb{R}$ is define as

$$g(t) = \cos\left(\frac{1}{1 + \sin(t) + \sin(\sqrt{3}t)}\right)$$

Then g is almost automorphic.

Example 2: $F : \mathbb{Z} \to \mathbb{R}$ defined as

$$F(n) = \cos(2\pi n\phi),$$

with ϕ is an irrational number, is almost automorphic sequence but not almost periodic.

Example 3 : Suppose $F : \mathbb{R} \to \mathbb{R}$ be a function st

$$F(t) = \cos\left(\frac{1}{2} + \cos(t) + \cos(\sqrt{3}t)\right) + \frac{1}{1 + t^2}.$$

This function is pseudo almost automorphic.

Example 4: **(A mixed blowflies model) [1]**
This example is about a model for *Lucilia cuprina*, with a harvesting function and two non linear functions: one of Nicholson type and one Lasota–Wazewska type.
In hypothesis , we change Nicholson model by adding eggs and harvesting adult flies every day. The rate of addition of the eggs corresponds to the term

$$\gamma_2(t)g_2(\xi_2(t)x(t - \zeta_2(t)))$$

which is a Lasota–Wazewska type term, while the harvest rate is represented by $a(t)F(u)$.
Thus, the model is:

$$\dot{x}(t) = -\beta(t)x(t) + \gamma_1(t)g_1(\xi_1(t)x(t - \zeta_1(t)))$$

$$+\gamma_2(t)g_2(\xi_2(t)x(t - \zeta_2(t))) - a(t)F(x).$$

Blowflies Model with Linear Harvesting

We consider the following blowflies model with linear harvesting:

$$\dot{x}(t) = -\beta(t)x(t) + \sum_{k=1}^{2} \gamma_k(t) g_k(\xi_k(t)x(t - \zeta_k(t))) - a(t)F(x)$$

where

$$\zeta_1(t) = \zeta_2(t) = 2, \quad \beta(t) = 15 + 21|\sin(\sqrt{2}t)|,$$

$$\gamma_1(t) = 12e^{1.1}, \quad g_1(x) = -xe^x, \quad \xi_1(t) = 1.2,$$

$$\gamma_2(t) = e, \quad g_2(x) = -e^x, \quad a(t) = 5001e^{0.5\cos^2(t)+0.5\cos^2(\sqrt{5}t)+1+\frac{1}{t^2}},$$

and

$$\xi_2(t) = 1.2, \quad F(x) = x.$$

After Computation:

$$\zeta = 2, \quad \beta = 15.5, \quad \underline{\beta} = 15, \quad \gamma_1 = 12e^{1.1}, \quad \gamma_2 = e,$$

$$\xi_1 = 1.2, \quad \xi_2 = 1.2, \quad a = 250e, \quad \underline{a} = 0, \quad L_F = 1.$$

$\forall\ 1 \leq k \leq 2$, the non-negative functions $g_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are Lipschitz continuous with $L_{g_k} = 1$. The functions g_k attain their maximum values for $\mu_1 = 1$ and $\mu_2 = 0$, where $g_1 = e^{-1}$ and $g_2 = 1$, i.e.,

$$\mu^* = 1, \quad \underline{g}_1 = e^{-2}, \quad \underline{g}_2 = e^{-1}.$$

We have a population model with harvesting, we get

$$\sum_{k=1}^{2} \gamma_k \underline{g}_k = 12e^{0.1} + e^{15} \approx 1.065355, \quad \mu^*\xi = 1 \cdot 1.2 = 1.2 \approx 0.833333.$$

Now, we fix $\delta_2 = 1.07$, so

$$\delta_2 \approx 0.875461 > \mu^*\xi.$$

We set $\delta_1 = 0.84 \in (\xi, \delta_2)$, and it follows that condition (A5) holds. Finally, we obtain:

$$aL_F + \sum_{k=1}^{2} \alpha_k \xi_k \underline{g}_k \beta \approx 0.4552306 < 1.$$

All the conditions of the theorem given in 'Solving and Stability Analysis of Nicholson Blowflies Model' section hold, and hence, model has a unique positive pseudo almost automorphic solution

$$u^*(t) \in \Omega = \left\{ u(t) \in PKAA(\mathbb{R}, \mathbb{R}^+) : 0.84 \leq u(t) \leq 1.07, \ \forall t \in \mathbb{R} \right\}.$$

Example 5: **A Mixed Circulating Blood Cells Model [1]**

Model is :

$$\dot{x}(t) = -\beta(t)x(t) + \sum_{k=1}^{2} \gamma_k(t) g_k(\xi_k(t)x(t - \zeta_k(t))) + a(t)F(x(t)).$$

The parameters are defined as follows:

$$\zeta_1(t) = \zeta_2(t) = 9, \quad \beta(t) = 20,$$

$$\gamma_1(t) = 25.5, \quad g_1(x) = \frac{1 + x^4}{x}, \quad \xi_1(t) = 0.9 + \frac{40\sin(\sqrt{3}t) + 40}{1 + t^2},$$

$$\gamma_2(t) = e^2, \quad g_2(x) = e^{-x}, \quad F(x) = x^{100}\left(\cos^2(t) + 1\right),$$

$$\xi_2(t) = 0.9, \quad a(t) = 100e^{0.5\cos^2(\sqrt{7}t) + 1 + \frac{1}{t^2}}.$$

After computation, we get:

$$\zeta = 9, \quad \beta = 20, \quad \gamma_1 = 25.5, \quad \gamma_2 = e^2,$$

$$\xi_1 = 0.95, \quad \xi_1 = 0.9, \quad \xi_2 = 0.92, \quad \xi_2 = 0.9,$$

$$a = 100, \quad \underline{a} = 0, \quad L_F = 1.$$

$\forall\ 1 \leq k \leq 2$, the non-negative functions $g_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are Lipschitz continuous with $L_{g_k} = 1$. The functions g_k attain their maximum values respectively for:

$$\mu_1 = 3^{-1/4} \approx 0.75983, \quad \mu_2 = 0.$$

The function values at these points are:

$$g_1 = \frac{3^{3/4}}{4} < 0.57, \quad g_2 = 1.$$

Thus, $\mu^* \in \mathbb{R}_0^+$ and g_k are non-increasing functions for $x > \mu_k$. It follows that:

$$\mu^* = 3^{-1/4}, \quad \underline{g}_1 = 169, \quad \underline{g}_2 = e^{-(3^{-1/4})}.$$

We have:

$$\sum_{k=1}^{2} \gamma_k \underline{g}_k = 25.5 \times 169 + e^2 e^{-(3^{-1/4})} \approx 4300.5 + 7.389 \times 0.721 = 4306.7.$$

Stability Conditions
It follows that:

$$\sum_{k=1}^{n} \gamma_k g_k + aL_F \delta_2 < \delta_2.$$

Since $aL_F < \beta$, we obtain the lower bound for δ_1 and δ_2:

$$\mu^* \xi = \frac{3^{-1/4}}{0.9} \approx 0.8442618 < \delta_1, \quad \sum_{k=1}^{2} \frac{\gamma_k g_k}{\beta} - a^* \approx 1.0445 < \delta_2.$$

Next, setting $\delta_2 = 1.0445$, we find:

$$\hat{\delta}_2 \approx 0.74224172 < \mu^* \xi.$$

Thus, assumption A5 is not satisfied. Furthermore, we estimate the lower bound for δ_2 as follows:

$$\sum_{k=1}^{n} \frac{\gamma_k g_k(\mu^*)}{\beta} + aL_F \delta_2 \approx 0.90000907 < \delta_2.$$

Now, after fixing $\delta_2^* = 0.91$, we obtain:

$$\hat{\delta}_2^* := \sum_{k=1}^{2} \frac{\gamma_k g_k(\xi \delta_2)}{\beta} \approx 0.862857533 > \mu^* \xi.$$

We choose $\delta_1 = 0.845$, with $\delta_1 \geq 0$ and $\delta_2 \geq 0$. Finally:

$$aL_F + \sum_{k=1}^{m} \frac{\gamma_k \xi_k g_k}{\beta} \approx 0.8459967 < 1.$$

Then, it follows that the model has a unique positive pseudo-compact almost automorphic solution:

$$x^*(t) \in U = \left\{ x(t) \in PKAA(\mathbb{R}, \mathbb{R}^+) : 0.845 \leq x(t) \leq 0.91, \forall t \in \mathbb{R} \right\}.$$

Immigration Function

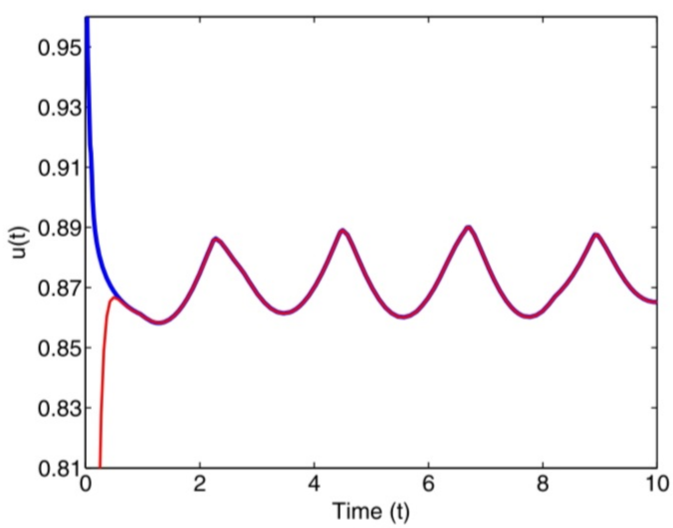
For immigration, we choose:

$$a(t)F(x(t)) = \frac{0.5\cos^2(\sqrt{7}t) + 1 + \frac{1}{t^2}}{100}\sin^2(x),$$

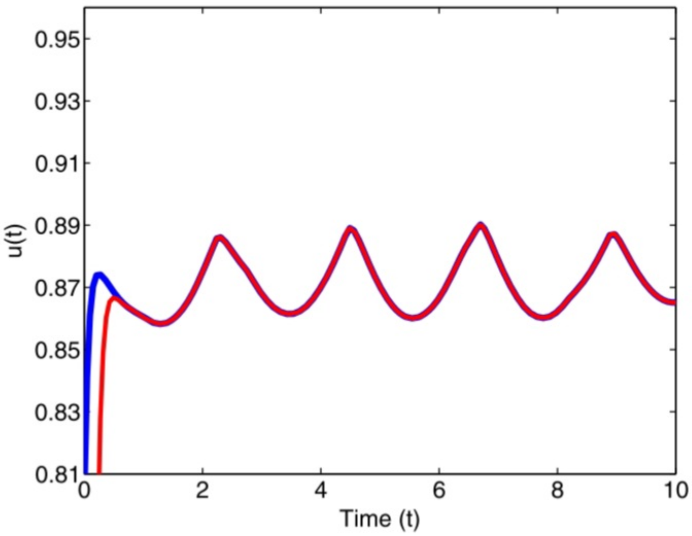
and

$$a(t)F(x(t)) = \frac{0.5\cos^2(\sqrt{7}t) + 1 + \frac{1}{t^2}}{100}|x|,$$

which are always positive and the solution is pseudo-compact almost automorphic.



(a)



(b)

Example 6: **Mixed Mackey-Glass and Nicholson Model** [1]

System is:

$$\dot{x}(t) = -\beta(t)x(t) + \sum_{k=1}^{2} \gamma_k(t) g_k(\xi_k(t)x(t - \zeta_k(t))) - a(t)F(x(t)).$$

The differential equation under consideration is:

$$\dot{x}(t) = -\beta(t)x(t) + \sum_{k=1}^{2} \gamma_k(t) g_k(\xi_k(t)x(t - \zeta_k(t))) - a(t)F(x),$$

where

$$\beta(t) = 15 + 21|\sin(\sqrt{2}t)|, \quad \zeta_1(t) = 0.4 + 0.4\sin(\sqrt{5}t),$$
$$\zeta_2(t) = 1 + t^2, \quad \gamma_1(t) = 15e^{1.1}, \quad \gamma_2(t) = e,$$
$$g_1(x) = xe^{-x}, \quad g_2(x) = 1 + x^3,$$
$$\xi_1(t) = 2 + 150(\sin^2(t) + \cos(\sqrt{3}t) + 1 + t^2),$$
$$\xi_2(t) = 2 + 40\cos^2(t),$$
$$a(t) = 500e^{-0.5\cos^2(t) - 0.5\cos^2(\sqrt{5}t) + 1 + t^2}.$$

After computation, we obtain the following bounds:

$$F(x) = x, \quad \zeta = 1.8, \quad \beta = 15.5, \quad \beta = 15, \quad \gamma_1 = 15e^{1.1}, \quad \gamma_2 = e.$$

$$\xi_1 = 2.02, \quad \xi_1 = 2, \quad \xi_2 = 2.05, \quad \xi_2 = 2, \quad a = 250e, \quad a = 0, \quad L_F = 1.$$

$\forall\, 1 \leq k \leq 2$, the non-negative functions $g_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are Lipschitz continuous with $L_{g_k} = 1$. The functions g_k attain their maximum values at $\mu_1 = 1$ and $\mu_2 = 2^{-1/3}$, with

$$g_1(\mu_1) = e^{-1}, \quad g_2(\mu_2) = 0.4424.$$

Since g_k are non-increasing for $x > \mu_k$, it follows that

$$\mu^* = 1, \quad \sup g_1 = e^{-2}, \quad \sup g_2 = \frac{4}{9}.$$

We have

$$\sum_{i=1}^{2} \frac{\beta \gamma_k g_k}{\xi_k} \approx 1.185, \quad \mu^* \xi \approx 0.5.$$

Now, fixing $\delta_2 = 1.185$, we obtain

$$\hat{\delta}_2 \approx 1.146 > \mu^* \xi,$$

where $\hat{\delta}_2$ is defined. Finally, we have

$$aL_F + \sum_{k=1}^{m} \frac{\gamma_k \xi_k g_k}{\beta} \approx 0.95466 < 1.$$

Harvesting Function

The graph corresponding to negative $a(t)F(x)$. From the graph, we can see that the solution remains positive.

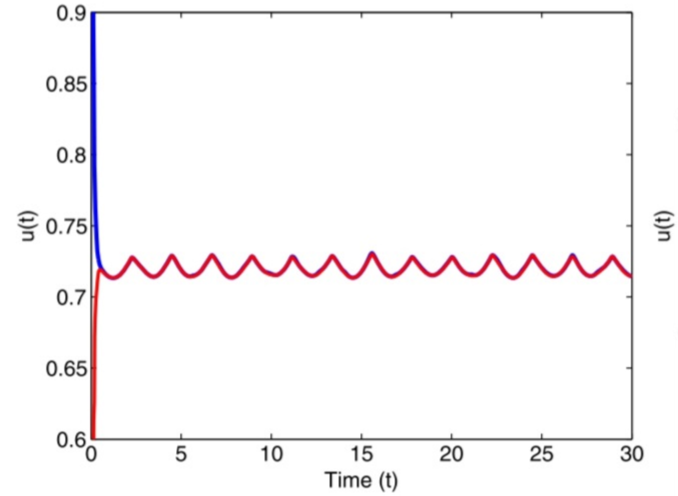
The figures illustrate two different cases of harvesting:

- (a) corresponds to:

$$a(t)F(x) = -5001e^{0.5\cos^2(t)+0.5\cos^2(\sqrt{5}t)+1+\frac{1}{t^2}} \sin^2(u).$$

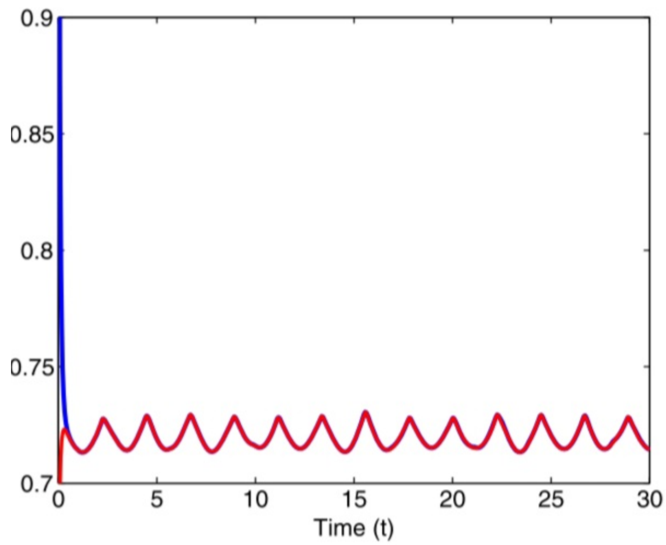
- (b) corresponds to:

$$a(t)F(x) = -5001e^{0.5\cos^2(t)+0.5\cos^2(\sqrt{5}t)+1+\frac{1}{t^2}} |u|.$$



0.9
0.85
0.8
0.75
0.7
0.65
0.6
u(t)
0
5
10
15
20
25
30
Time (t)
u(t)

(a)



0.9
0.85
0.8
0.75
0.7
0
5
10
15
20
25
30
Time (t)

(b)

References

[1] Syed Abbas , Soniya Dhama , Manuel Pinto , Daniel Sepúlveda. 'Pseudo compact almost automorphic solutions for a family ofdelayed population model of Nicholson type'.In: Journal of Mathematical Analysis and Applications.
doi:www.elsevier.com/locate/jmaa

[2] Zhe-Ming Zheng, Hui-Sheng Ding.'On completeness of the space of weighted pseudoalmost automorphic functions'. In:Journal of Functional Analysis.
doi: www.elsevier.com/locate/jfa

[3] Hui-Sheng Ding, Jin Liang, Ti-Jun Xiao 'Weighted pseudo almost automorphic functions and WPAAsolutions to semilinear evolution equations'. In:Journal of Mathematical Analysis andApplications, doi:www.elsevier.com/locate/jmaa.

[4] Stavros N. Busenberg, Kenneth L. Cooke 'Stability Conditions for Linear Non-Autonomous Delay Differential Equations'. In: Quarterly of Applied Mathematics
doi: https://www.jstor.org/stable/43637263

[5] László Hatvani.'On the Asymptotic Stability for Nonautonomous Functional Differential Equations by Lyapunov Functionals'. In: American Mathematical Society.
doi: https://www.jstor.org/stable/3072925

[6] Jack K. Hale, Sjoerd, M. Verduyn Lunel. 'Introduction to Functional Differential Equations'. In : Springer Science+Business Media, LLC.

[7] S. Mohamad, K. Gopalsamy 'Continuous and Discrete Halanay-Type Inequalities'. In: Bull. Austral. Math. Soc., Vol. 61 (2000).

[8]Jing-huai Liu , Xiao-qiu Song. 'Almost automorphic and weighted pseudo almostautomorphic solutions of semilinear evolution equations'. In: Journal of Functional Analysis 258 (2010) 196–207.

[9] R.M. Nisbet W.S.C. Gurney S.P. Blythe. "Nicholson's blowflies revisited, Nature 287 (1980) 17–21," in: Nature 1980 (2021). doi: https://doi.org/10.1038/287017a0.

[10] J. Blot a, G.M. Mophoub, G.M. N'Guérékata c, D. Pennequina 'Weighted pseudo almost automorphic functions and applications to abstract differential equations'. In: Nonlinear Analysis. doi: www.elsevier.com/locate/na

[11] Pablo Amster and Lev Idels. "Periodic solutions in general scalarnon-autonomous models with delays". In: Nonlinear Differential Equations and Applications NoDEA

[12] E. Ait Dads, F. Boudchich, B. Es-Sebbar. 'Compact almost automorphic solutions for some nonlinear integral equations with time-dependent and state-dependent delays'.In: Adv. Differ. Equ. 1. doi: https://doi.org/10.1186/s13662-017-1364-2.

[13] Jehad O. Alzabut.'Almost periodic solutions for an impulsive delay Nicholson's blowflies model'. In: Journal of Computational and Applied Mathematics. doi: www.elsevier.com/locate/cam

[14] Amster, L. Idels, Periodic solutions in general scalar non-autonomous models with delays, Nonlinear Differ. Equ. Appl.20 (5) (2013) 1577–15969 https://doi.org/10.1007/s00030-013-0222-5.

[15] W. Chen, X. Lu, Z. Guan, W. Zheng, Asymptotic stability in a neutral delay differential system with variable delays,SIAM J. Math. Anal. 37 (5) (2006) 1522–1534, https://doi.org/10.1137/S00361410 04441167