A Study of Quantum Sequential Growth Dynamics, Observables in Causal sets and Renormalization

A thesis

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by

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Certificate

This is to certify that this dissertation entitled *A Study of Quantum Sequential Growth Dynamics, Observables in Causal sets and Renormalization* towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Ritesh Srivastava at Indian Institute of Science Education and Research under the supervision of Prof Sumati Surya, Theoretical Physics Department, Raman Research Institute (RRI) Bengaluru, during the academic year 2024-25.

Expert : Dr Sunil Mukhi

Supervisor: Prof. Sumati Surya

This thesis is dedicated to my Parents

Declaration

I hereby declare that the matter embodied in the report entitled *A Study of Quantum Sequential Growth Dynamics, Observables in Causal sets and Renormalization* are the results of the work carried out by me at the Theoretical Physics (TP) department of Raman Research Institute (RRI) Bengaluru, under the supervision of Prof. Sumati Surya and the same has not been submitted elsewhere for any other degree.

Ritesh

Ritesh Srivastava

May 14, 2025

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Contents

List of Symbols

$\tilde{\Omega}$ **:** Set of labeled past finite causets

$\tilde{\Omega}(n)$**:** Set if labeled causets with containing n elements

Ω**:** Set of unlabeled past finite causets

$\mathcal{N}$**:** Total number of n element causets

$\mathcal{J}(n)$**:** The set $\{1, 2, \cdots, \mathcal{N}\}$

$\mathcal{J}_n(n+m, i)$**:** The set of all the labels of $n+m$ element causets which are children of c_n^i.

$\mathcal{P}$**:** Poscau

$\mathcal{T}(\tilde{\Omega}(n) \to \tilde{\Omega}(n+m)$**:** The set of all transition amplitudes which takes measures of n element labeled causets as input and give out measures of $n+m$ element causets which can be generated from the input causet.

$\mathcal{T}_{\mathcal{B}}$**:** $\bigcup_{n=1}^{\infty} \bigcup_{m>n}^{\infty} \mathcal{T}(\tilde{\Omega}(n) \to \tilde{\Omega}(m))$

$\mathcal{T}$**:** This is the ring of transition amplitudes with identity generated by $\mathcal{T}_{\mathcal{B}}$

α_n^i**:** Represents a transition from c_n^i to c_{n+1}^j where $i \in \mathcal{J}(n)$ and $j \in \mathcal{J}_n(n+1, i)$

$\mathbb{R}$ Set of real numbers

$\mathbb{N}$ Set of natural numbers

$\mathbb{R}^+$ Set $[0, \infty)$.

$\exists$ There exists

$\forall$ For all

Abstract

Causal Set theory is a discrete framework for Quantum Gravity that assumes that space-time is discrete and causal structure is the property of space-time that needs to be quantized. The causal structure is a partial order and thus, causal sets are locally finite posets. In this thesis, we study the fundamental dynamics or causal sets known as Sequential Growth Dynamics (SGD) where we start with one element and keep on growing causal sets by adding one element at a time and assigning transition probability to these transitions. We study the quantum SGD where rather than assigning probability to transition, we assign vectors lying in some Hilbert space to causal sets and linear operators over that Hilbert space to transition amplitudes. We show that under the generalized assumptions of SGD the algebra of transition amplitudes is commutative if the transition amplitudes are invertible. We also study the observables in SGD by which we mean the set of questions we can ask, like probability of occurrence of certain causal structure. We construct measures of certain observables. Towards the end, we study the dynamics in case where the universe collapses and re-forms again and again in the SGD. We find the limit points of dynamics for more general type of collapse than those studied before.

Introduction

We know that Quantum field theory (QFT) and General relativity (GR) have been successful in their own domains and they have passed every experimental verification so far. QFT talks about the particles at the atomic and subatomic level and primarily explains the processes happening at that level. On the other hand, GR models gravity as curvature of space-time and explain the motion of objects as the effect of this curvature. The matter present in the space-time influences this matter as given by Einstein equations. These theories, though successful in their own domains, are not unified yet into a single theory.Consider the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \quad , \tag{1}$$

where $T_{\mu\nu}$ is the stress energy tensor. Here we come across our first problem that, if matter is quantum, $T_{\mu\nu}$ cannot be used directly in above question as it will be a quantum operator. We assume that the space-time has no quantum nature and thus, we can just use average value of $T_{\mu\nu}$ in the Einstein's equations. But $T_{\mu\nu}$ itself may have fluctuations and thus, taking gravity to be classical seems restrictive. This motivates the idea that gravity should also have a quantum nature. There have been several attempts to quantize gravity, like string theory, loop quantum gravity, causal set theory and other models of quantum gravity.

Causal set theory(CST) is a model of quantum gravity, where causal structure of space-time is taken to be the fundamental property of space-time that needs to be quantized. The main motivation for this comes from the theorem of Hawking-King-McCarthy-Malament [18], which states that if there is a chronological bijection between two d-dimensional future and past distinguishing space-times, then those two space times are conformally isometric for $d > 2$. The causal structure also contains information about the dimension of space time as shown in [12]. These things show that most of the information about the spacetime (M,g) like it's metric, topology, dimensions etc. are contained in causal structure itself. This causal structure forms a poset structure, because it is **acyclic** i.e. $x \prec y$ and $y \prec x$ iff $y = x$. It is transitive as we know $x \prec y$ and $y \prec z$ means that $x \prec z$. The hypothesis of causal set theory is that a space-time (M,g) is an approximation to a poset that is locally finite i.e. in any finite volume of space-time, there are only finitely many elements of causet. We implement this by saying that the volume of a part of space-time is related to the number of elements in that volume. This helps us solve the problem of UV divergences. It is important to know that not all causet can be embedded in a manifold. Now that we have a structure to work with i.e. causets, we will study the dynamics followed by them.

There are works on QFT over causets and formulation of GR using causets, but what we are concerned with in this thesis is a more fundamental dynamics of causets. We argued that causal structure is the basic property of any space-time and that this is the property that is to be quantized. This dynamics tries to accomplish this quantization. The key idea behind this is to start with an element and generate larger causets by adding one element at a time. A good analogy to this would be random walk over a discrete lattice, where each step is considered to be a new element. We look at all the possible ways in which the new elements are added making sure that the element is not added into the past of any existing element. The idea is to assign probabilities to each of the causets generated by this procedure. We do it by assigning transition amplitudes to each of the individual transition, where, by transition we mean the process of adding one element to the existing causet. The probability of the new causet $c(n+1)$ generated is then given by the action of this transition amplitude over the probability of the causal set $c(n)$ to which the element is added. These dynamics are called Sequential Growth Dynamics (SGD).[13]

What we are supposed to do with this? We first make a theory for transitions with positive classical measure. We later generalize it to the quantum case by generalizing this measure, changing the addition law for measures of disjoint set. This is done to ensure that quantum measure follows the probability addition law as followed by the probabilities given by path integral approach. This is talked about in the paper [17] with the help of 3 slit experiment. Thus, we can start with a singleton set $\{\cdot\}$ and assign it some measure. For the classical case, we take a positive measure which is easier to deal with and thus, helps us gain insights for the quantum case as well[13]. For the quantum case, we can use a vector valued measure[1] which is countably additive, and then generate the quantum measure by taking the inner product with itself as shown in [7][8]. Note that this process is analogous to path integrals in double slit experiment. We add the amplitudes and multiply by conjugate to get the probability density[19]. The transition amplitudes thus become operators over this Hilbert space. In this thesis, we have studied the algebra of these transition amplitudes and found out a general form for the transition amplitudes given certain properties that we expect the transitions to follow. One of the properties that we require the transition amplitude to have is a local causality condition called Bell causality. In this thesis, we looked at two possible definitions of this property and found that if we assume that the transition amplitudes to be invertible, then the algebra turns out to be commutative.

We have also studied certain aspects of classical sequential growth dynamics. One thing

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[1] The vector measure is a mapping from causets to a Hilbert space $\mathcal{H}$

that we studied are the observables in SGD. As causets themselves model space-time, there is no such thing as an external observer and thus, an observable refers to the fundamental property of causets i.e. the causal structure. This means that observable refers to the possible set of questions that we can ask about our causets in SGD. So, we can ask questions like: "What is the probability that a certain causal structure occurs?" and their answers will be the probability of occurrence if those causets. For that, we first have to define the possible set of questions that can be answered i.e. causal structures that can be assigned some probability. For that, we define a σ-algebra over the set of causets generated in SGD [4][19] and assign a probability measure over this σ-algebra. This measure can actually be generated from the measure that we defined for the causets generated in SGD[2]. In this thesis, I have calculated the measures of some basic observables like originary event, posts and break in classical sequential growth dynamics. We can do this for the quantum case as well, but there is a question of convergence of these measures which is difficult to answer.

Another question that can be asked about causets in SGD is how do the transition amplitudes renormalize if the universe collapses to a point and re-forms from that point again and again. Such events are called posts. We also want to answer the questions like :"If the universe collapses and re-forms multiple times, what will the dynamics look like as the number of collapses tends to infinity?" This has been studied in [11] where they showed that the transition amplitudes tend to a certain type of dynamics called the transitive percolation under certain conditions if there are multiple posts. Such a question can be asked for a more general type of collapse of universe called break. It is a generalization of posts as here, universe does not collapse to a point. The only condition required is that any element in the new universe is in the future of elements of the past universe. In this thesis, we tried answering that question. As it turns out, finding fixed points for break seems possible only when the past set is of a special type as discussed in chapter 4. For a general break, it seems impossible to find such limit points and thus, in case of multiple break, the dynamics may not have a limiting dynamics in general.

In conclusion, we have studied the algebra of transition amplitudes in quantum SGD and showed that it is commutative, calculated the measures of some observables in the classical SGD and studied the renormalization in causets undergoing multiple posts or breaks.

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[2] In fact, the σ-algebra is constructed in a way that the probability measures of causets can be extended to the full σ-algebra. The measures were assigned to the causets generated in SGD so that we can answer such questions related to the probability of occurrence of certain causal structures.

Chapter 1

Preliminaries

We are trying to model space-time using causal sets. But the very first question that can come to anyone's mind is, "Why causal sets?" We will try to motivate that in the next section.

1.1 Why Causal Sets?

We know that in General Relativity, we define space-time as a Lorentzian manifold. The metric provides us with a causal structure of space-time : two points which can be connected by timelike or null geodesics are causally related to one another. We will define this condition more rigorously later. We first need to be able to distinguish past from future i.e. we should be able to tell that x lies in the past of y or not. Such a condition puts a restriction on the space-time that the geodesics cannot form a loop. To rigorously impose these conditions on the space-time, we first define what a time orientable space-time and then we discuss the past and future distinguishing space-times.

We want our space-time to be **time orientable** [5] i.e. if it admits a continuous field of timelike vectors X. We can then talk about future directed and past directed curves. A vector $v \in T_pM$ is called future directed iff $g(X, v) < 0$. We thus define a future directed causal curve as a piecewise smooth path $\gamma : I \to M$ such that $\gamma'|_p$ is future directed vector $\forall p \in M \cap \gamma^*$ where γ^* is the image of γ. With that, we can define

Definition *A point x is called in the "chronological" past[respectively causal past] of y, if there exists a future directed piecewise smooth timelike[or causal i.e. timelike or null] paths* *from x to y. We write $x \prec\prec y$ if x is in "chronological" past of y and write $x \prec y$ if x is in causal past of y.*

We also assume some conditions on our space-time so that we can ask questions like; "Is x in the future of y?" For that, we need to assume that space-time is future and past distinguishing. A space-time (M, g) is called **future and past distinguishing** iff $\forall p \in M$ and forall open sets U containing p, there exists an open neighborhood U_1 of p such that $U_1 \subset U$ such that no future directed smooth causal curve[1] through p that leaves U_1 ever returns to U_1. This particular assumption allows us to ensure that no timelike curve form loops, or there is a well defined future and past for a point $p \in M$. With this we see that if we are given a causal structure over a manifold M, we can see that if x is in past of y and y is in the past of z, then x is in the past of z. Thus, causal relation is transitive. We also note that none of the points of space-time M can lie in the past of itself if the space-time is past and future distinguishing. We thus get that causal structure is acyclic in those space-times. We thus get that causal structure on a future and past distinguishing space-time is a partial order and thus, M is a poset(partially ordered set). We now talk about how we can get most of the information about space-time using the causal structure, thus establishing that causal structure is the fundamental property of a space-time.

A **chronological bijection** between two spacetimes (M, g) and $(N, \tilde{g})$ is a map f_b such that $f_b(x) \prec\prec f_b(y)$ iff $x \prec\prec y$ i.e. f_b preserves the causal structure. With this, we state the Hawking-King-McCarthy Malament (HKMM) theorem[18].

Theorem 1.1.1. ***Hawking-King-McCarthy Malament (HKMM) :*** *If a chronological bijection f_b exists between two d-dimensional spacetimes which are both future and past distinguishing, then these space-times are conformally isometric when $d > 2$.*

We also know that for future and past distinguishing space-times, we can find out the topology, given the causal structure[10]. We can also find the dimension of space-time, if the causal structure is given as shown in [12]. With all this, we find that causal structure is a fundamental property of space-time and if we know the causal structure, we can find most of the information about the space-time. Causal set theory thus assumes that causal structure is the fundamental property of space-time that is to be quantized. We will deal with the problem of quantization of causal structure in this thesis.

Another property that we demand from our theory is that the space-time is discrete. This particular assumption solves the problem of UV divergences. We thus assume that the continuum space-time is an approximation of causet which is a discrete poset. We will talk about this in later section.

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[1] i.e. a timelike or null geodesic

Talking about some of the achievements of Causal set theory, there had been works by Dowker, Benincasa and Glaser where they constructed the discrete Einstein-Hilbert action for causets, also known as Benincasa-Dowker (BD) action[2][3]. One of the predictions made using the causal set theory is the prediction of cosmological constant Λ by Sorkin in [15].

1.2 Causal Sets or Causets

For a poset P, we define

$$I(x,y) := \{e \in P : x \prec e \text{ and } e \prec y\} \tag{1.1}$$

We give the definition of a causet as

Definition *A* ***causet*** *or* ***causal set*** *is a poset C which is locally finite i.e.*

1. *Acyclic :* $x \prec y$ *and* $y \prec x \Rightarrow x = y$, $\forall x, y \in C$
2. *Transitive:* $x \prec y$ *and* $y \prec z \Rightarrow x \prec z$, $\forall x, y, z \in C$
3. *I(x,y) contains finitely many elements.*

Here this poset structure is actually the causal structure. But this class of causets is too general. What we want is that these causets can be embedded in some unique manifold. This is not true for all the causets given by above definition. The causets that can be embedded in a manifold are called manifold like causets. The main feature that these causets are required to have is that they are discrete. As discussed in the introduction, we want our causet to be discrete as that solves the problem of UV divergences. The causal set theory views this discrete structure as the fundamental structure of space-time while the continuum is an approximation to this discrete structure.
Before defining them rigorously, we need to answer the following question: "What do we mean by embedding causet in a manifold?"

1.2.1 The embedding of causet in a manifold (M,g)

We want the causet to be discrete and thus, we define a volume cutoff V_c which is the volume of a continuum space-time that contains only one element on average(note that V_c is arbitrary. We can chose it to be whatever we want based on the scale at which we expect space-time to be discrete). The idea is that space-time is fundamentally discrete but it looks like a continuum at volumes much larger than V_c. As V_c is arbitrary, we can chose it to be really small. A useful analogy in this case would be fluids which are discrete at the molecular level but can be treated as a continuous object when we deal with them macroscopically. In such cases, we generally use statistical mechanics to determine a lot of properties of the fluid. In essence, the properties that we calculate using statistical mechanics are “averaged” over several microscopic configurations of the particles of the fluid. The idea is similar in the case of causets, where we can chose several configurations that can approximate the same space-time. It the the average over these configurations that give us the values of quantities that we observe in the continuum space-time.

We also need to ensure that this discreteness is related to volume of space-time. More concretely, the number of space-time points in a volume V of space-time should only depend on the volume V and no other factor. We demand that so that the discreteness is not dependent on the volume we observe. This would obviously be absurd if the space-time behaves like a continuum around point ‘a’ and behaves like a discrete one around point ‘b’. This assumption that number of particles only depend on volume ensures that when we choose a scale, we can unambiguously say if we can approximate the causet by a continuum or not. Another reason is that causal structure gives us metric only upto a conformal factor. But knowing volume allows us to find out the conformal factor as well and gives us Lorentzian geometry. In CST, we by analogy want to represent the whole Lorentzian geometry as causal structure and number of space-time points(which is directly proportional to the volume). For this reason, a uniform distribution seems to be the way out. But this has some subtlety involved which we discuss below.

Another problem that we need to tackle while embedding a causet in space-time is that we cannot embed the causet like a uniform regular lattice as that will not preserve Lorentz invariance. “What do we mean by Lorentz invariance for causets and why do we need it?” Intuitively speaking, for a Minkowski space-time it means that we cannot distinguish two reference frames around the same point in space-time based on the causets that approximate them. This is required as this will ensure that we can directly deal with causets and the calculations done over them is covariant. The generalization to an arbitrary space-time is straightforward. We just need to ensure that causal sets do not pick out a specific reference frame. Now that we have an intuitive understanding of the idea, we will explain how this is done in practice.

Poisson sprinkling We realise from above discussion that uniform distribution is needed for unambiguously saying that a volume V contains n number of particles and thus choosing discreteness scale without any ambiguity. But we saw that for preserving Lorentz invariance, we need to embed causet in such a way that it does not depend on frame of reference. One such way to embed the causets in a space-time is the **Poisson sprinkling** which assumes that the number of elements in a volume V of space-time is given by Poisson distribution as

$$P_V(n) = \frac{1}{n!}\left(\frac{V}{V_c}\right)^n e^{-\frac{V}{V_c}} \tag{1.2}$$

The above distribution also ensures that the average density of causet elements in the continuum space-time is $\rho_c = 1/V_c$ and thus, it is a uniform distribution on average. The causal structure of space-time is used to assign the causal relations between the randomly chosen points of causet. This process produces a class of causets rather than a single causet and this class of causets is represented by $\mathcal{C}(M, 1/V_c)$. A more detailed discussion is given in [6]. This process ensures Lorentz covariance on average and thus, if we calculate some quantity over causets generated by Poisson sprinkling and average over them, we will get a covariant quantity.

Above discussion showed how to construct a class of causets $\mathcal{C}(M, 1/V_c)$ but what if we are given a causet? How do we figure out which space-time it embeds in? We say that a causet C is approximated by a space-time (M,g) if C is produced with a high probability in the Poisson sprinkling on (M,g). A **faithful embedding** of a causet C into manifold M is a map, $\Phi : C \to M$ which is a uniform distribution i.e. the probability of finding n particles in a volume V is same over the whole manifold. This means that we can assume that any volume contains $\rho_c V$ number of particles(which is the average number of particles in a volume V if we are doing Poisson sprinkling). Given a manifold (M,g), we can always construct a faithfully embedded causet in principle. What we need to do is to assume uniform distribution and assign causal structure between points of causet using the causal structure of (M, g).

With all these, we can say that a causet C represents a "universe" just like a manfiold (M,g) represents a universe. Thus, whenever we use the term "universe" in this thesis, it will mean a causet.

Remark Note that not all causets can be approximated by a manifold. There are causets like KR posets (see fig 1.1) which are not manifold like. Such causets are not desirable as the theory should approximate GR at larger scales. The sequential growth dynamics are a way to quantize the causal structure and these dynamics ensure that such non manifold like posets occur with a very small probability[13]. We will talk more about these dynamics in the next section.

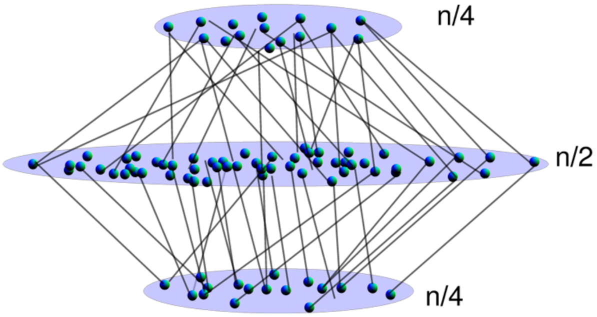


Figure 1.1: KR poset

The fundamental conjecture of CST

From above discussion, one can see that there is a subtlety involved in the above process. We do not have much information at scales smaller than V_c as such volumes cannot contain more than one element on average. Thus, it may happen that there are two different space-times that approximate the same causet. Thus, we need some notion of closeness between the space-times. In other words, we have to define a notion of "approximate" isometry as the space-times obtained might not be isometric at scales smaller than V_c but are almost similar/ isometric at larger scales as they are approximated by the same causet. This brings us to the fundamental conjecture of CST which states that

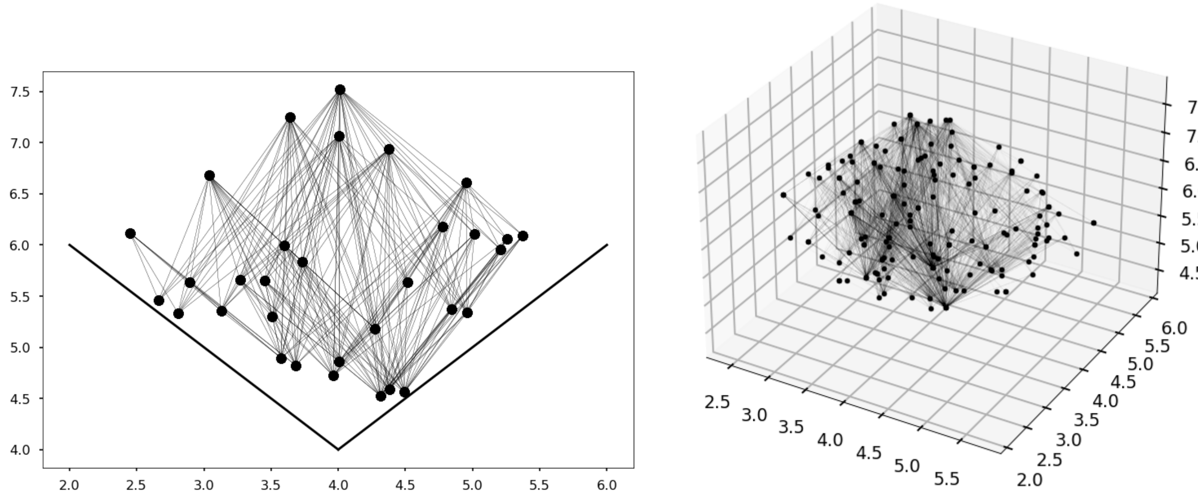
The Hauptvermutung of CST: *C can be faithfully embedded at density* ρ_c $(= 1/V_c)$ *into two distinct space-times, (M,g) and* $(\tilde{M}, \tilde{g})$ *if they are approximately isometric.*

It is difficult to define approximate isometry because there is no well defined notion of distance between two Lorentzian manifolds. Several works have been done in this area to prove the Hauptvermutung of CST but so far, it remains an open problem.

An example of Poisson sprinkling in Minkowski space-time

I have performed this Poisson sprinkling in 2D and 3D causal diamond in a Minkowski's space-time as shown in the figures 1.2. Here, the lines represent causal relation i.e. the points connected by lines are connected by a timelike or null geodesic while the points that are not connected are space like.

We get different causets each time we do such embedding. Any observable in GR is then



(a) 2D causal diamond

(b) 3D causal diamond

Figure 1.2: Causet embedded in causal diamonds in Minkowski space-time

calculated as an average over all these causets produced. We can calculate the proper time between two space-time points using this as given in [18]. The main idea is to take the two points and embed a causet in the causal diamod between them and find the length of longest chain and then average over all these causets[2]. Similarly, we can calculate other quantities averaging over the causets in $\mathcal{C}(M, 1/V_c)$.

1.3 Sequential Growth Dynamics (SQD)

We now look at the dynamics of causal sets. There are works on QFT over causets and formulation of GR using causets, but what we are concerned with in this thesis is a more fundamental dynamics of causets. We argued that causal structure is the basic property of any space-time and that this is the property that is to be quantized. This dynamics tries to accomplish this quantization. The key idea behind this is to take one element in causal sets and generate larger causets by adding one element at a time. We look at all the possible ways in which the new elements are added making sure that the element is not added into the past of any existing element. This property is called **internal temporality**. This growth of causets can be represented as a tree called poscau ($\mathcal{P}$), as shown in figure 1.3. Our aim is to assign a probability of occurrence to each of these causal structures generated. We can thus ask questions like "What is the probability of getting a particular causal structure?"

We have to eventually make a quantum theory, which we will see is a generalization of the measure theory. What this means is that, the probability measure assigned to the causal

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[2] We use this generalization because proper time is the largest geodesic distance between two points in

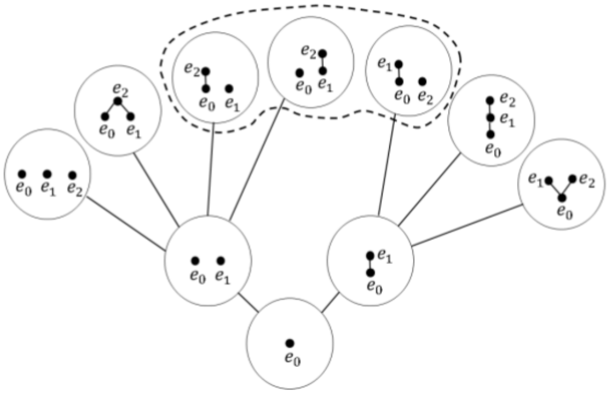

Figure 1.3: sequential growth dynamics

structures is not a classical measure. We will learn more about it later. We first learn about the classical case, where we have a classical probability measure. The measure space is defined keeping in mind that we have to ask questions about the probability of occurrence of different causets. The classical SGD($\mathbb{R}^{+}$SGD) is simpler to deal with as the measure is countably additive. Another reason to study these is to develop some intuition and gain some insights from this relatively simple case before going to the more complex quantum case. Before going any further, it is better to define some terminology related to SGD.

Some important definitions

As we discussed above, we start from a single element and add one element at a time forming all possible posets that we can ensuring that internal temporality is followed. We can label these elements with integers with first element labeled by 1, second element by 2 and so on. But we have already stated that it is the causal structure or poset structure is the physical property of causets that we care about. Any possible relabeling of causets is giving us exactly the same poset structure as before and thus, they are physically equivalent. while relabeling, we note one thing that $e_i \prec e_j$ if and only if $i < j$ by internal temporality. We will make these ideas precise by giving the following definitions.
In a given causet c, the past of an element is defined as

$$\text{Past(e)} = \{\alpha \in c : \alpha \prec e\} \tag{1.3}$$

The **precursor** set of an element $e \in c$ is the Past(e) and **spectator** set of e is the set $c/(\text{past(e)} \cap \{e\})$. A causet c is called **past finite** if $\forall e \in c$, the set $\{a \in c : a \prec e\}$ has finitely many elements.

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space-time where distance is induced by the metric of space-time

We define the set of naturally labeled, past finite causets as $\tilde{\Omega}$, where naturally labeled means that if $e_i \prec e_j$, then $i < j$. A **relabeling** of a causet is a function $L : \mathbb{N}_n \to \mathbb{N}_n$, which is a bijection and $e_i \prec e_j \Leftrightarrow e_{L(i)} \prec e_{L(j)}$. Here $\mathbb{N}_n = \{1, 2, \cdots, n\}$. If $\mathbb{N}_n = \mathbb{N}$, it is a relabeling of infinite causet. Two labeled causets a and b are called **isomorphic** if there exists a relabeling from a to b i.e. $\exists L$ such that $a_{L(i)} = b_i$ where a_i and b_i denotes the ith elements of a and b respectively. Such causets are written as $a \cong b$. It is important to note that isomorphic causets form an equivalence class.

We denote the set of unlabeled past finite causets as Ω which is constructed as $\Omega = \tilde{\Omega}/\sim$, i.e.where $\sim$ is an equivalence relation between causets related by relabelings. We call the set of n element causets c_n^i as the **stage** n, and it's cardinality is given as $\mathcal{N}$ and we define $\mathcal{J}(n) = \{1, 2, \cdots, \mathcal{N}\}$. For the children of a particular causet $c_n^i, i \in \mathcal{J}(n)$, we label them as $\mathcal{J}_n(n+m, i)$, which represent the labels of all the children of c_n^i at stage $n+m$. We represent the probability of transition $c_n^i \to c_{n+1}^{j(i)}$ as α_n^j, $j \in \mathcal{J}(n+1, i)$.

A **gregarious** transition is the one where $past(e) = \emptyset$. A **timid** transition $c_n^i \to c_{n+1}^{j(i)}$ is the one in which whole set c_n is in the past of e_{n+1}. A **stem** in a causet c is a subset that contains all it's past elements i.e. if b is a stem in c, $e \in b$, and if $\exists a \in c$ such that $a \prec c \Rightarrow a \in b$.

The Poscau $\mathcal{P}$

We will like to talk about the terminologies related to the poscau as they are useful in various cases (see fig 1.3). The nodes or vertices are labeled causets in this tree. We must note that $\mathcal{P}$ is a directed graph where the edged point from stage n to stage $n+1$. A **path** is is a finite or infinite sequence of edges directed in the same direction which joins a sequence of vertices where the vertices and edges are distinct. We will represent a path in this tree by γ_n^i where $i \in \mathcal{J}(n)$, which is a path from stage 0 to c_n^i. We can define a partial order over $\mathcal{P}$ as, $D \in \mathcal{P}$ and $C \in \mathcal{P}$. Then $D \prec C$ if and only if D is a stem in C. A subtree is a downward closed tree i.e. if $C \in \mathcal{P} \Rightarrow D \in \mathcal{P}$, $\forall D \prec C$.

1.3.1 Properties of Probability distribution in classical SGD or $\mathbb{R}^+$SGD

In this model, we assign a transition probability to the transitions from n element causet to all possible $n+1$ element causets. The actual probabilities that we are interested in i.e. the probability of occurrence of a given "labeled" causet of n elements is given by a product of these transition probabilities along the path which has the given causet as the end point in Poscau $\mathcal{P}$. This is a classical probability measure which takes up values in [0,1]. These probabilities have certain natural rules that they follow which are

1. **General covariance** says that all the relabelings of a given causal set has equal probability of occurring. This is assumed as relabeling actually doesn't change the poset structure and thus, it refers to the same space-time. This allows us to define the notion of observables as well.

2. **Markov rule** states that sum of transition probabilities α_n^j from $c_n^i \to c_{n+1}^{j(i)}$ is 1. This ensures that no finite element universes are created.

$$\sum_j \alpha_n^i = 1 \tag{1.4}$$

3. **Bell Causality** says that similar causal structures are produced in similar relative proportions because the elements that are not causally related to the new element cannot affect the transition probability(in classical case). Suppose that $\alpha(c_n^i \to c_{n+1}^j)$ represent the transition probability from $c_n^i \to c_{n+1}^j$ while $\beta(c_m^i \to c_{m+1}^j)$ represents the transitions from the union of precursor sets in transitions corresponding to labels j_1 and j_2.

$$\frac{\alpha(c_n^i \to c_{n+1}^{j_1})}{\alpha(c_n^i \to c_{n+1}^{j_2}))} = \frac{\beta(c_m^k \to c_{m+1}^{j_1})}{\beta(c_m^k \to c_{m+1}^{j_2}))} \tag{1.5}$$

   Where c_m^k is the union of precursor sets of e_{n+1} in the transitions $c_n^i \to c_{n+1}^{j_1}$ and $c_n^i \to c_{n+1}^{j_2}$.

This probability of a transition from causal set $c_n^i \in \tilde{\Omega}(n)$ such that the precursor set has m maximal elements and ϖ total elements, can be written in terms of parameters q_n as

$$\alpha_n^i = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{q_n}{q_{\varpi-k}} \tag{1.6}$$

or in terms of t_n as

$$\alpha_n^i = \frac{\sum_{k=m}^{\varpi} \binom{\varpi - m}{k - m} t_k}{\sum_{k=0}^{n} \binom{n}{k} t^k} \tag{1.7}$$

Where

$$\frac{1}{q_n} = \sum_k \binom{n}{k} t_k \tag{1.8}$$

An interesting example A good example to understand this model is transitive percolation where $q_n = q^n$. It can be written in another way as well. In this model we take some n element set, and the probability of any two elements having a link is given by q. Thus, it can also be used in above SGD model by just putting $q_n = q^n$. It is interesting because this model is actually very simple to deal with and it doesn't care if we make causets from past to future or future to past in any n element causet. The only thing that matters for providing the probability of getting a causet is the number of links.

We will now look at the σ-algebra of causets more carefully before doing anything more with the formalism, because we need to clearly state what are the possible questions that the model of SGD wishes to answer.

1.4 Observables in SGD

We know that every measurement occurs in space-time itself and thus we cannot have an external observer. So, we cannot have observables in the usual sense. Thus, we instead want to ask different type of questions. For that we will look at the intrinsic property of causets i.e. causal structure and ask questions like "What is the probability of getting a particular causal structure or poset structure?" We thus define a σ-algebra over the space Ω of unlabeled causet and measure over that algebra gives us the answers to our questions. Here, questions means elements of σ-algebra and answers means the measure and thus the probability of occurrence of that observable under SGD.

For Classical Sequential growth dynamics ($\mathbb{R}^+$SGD), we first assign a σ-algebra and a measure to labeled causets in $\tilde{\Omega}$ and then we use this to induce a measure on unlabeled causets in Ω.

The σ algebra of causal sets We first define cylindrical sets as

$$cyl(c_n) = \{c \in \tilde{\Omega} : c|_n = c_n\} \tag{1.9}$$

The σ-algebra generated by these cylindrical sets is denoted by $\tilde{\mathcal{R}}$. [3]. The measurable space that we get is $(\tilde{\Omega}, \tilde{\mathcal{R}})$

We then define the probability measure over the measure space $(\tilde{\Omega}, \mathcal{R})$ as $\tilde{\mu}(cyl(c_n))$ =Prob(getting

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[3] The σ-algebra on a set X generated by some collection of subsets of X is the smallest σ-algebra that contains those collection of subsets.

labelled set c_n).[4] The relabeling form a equivalence relation and thus, we can form the set of unlabeled causet as $\Omega = \tilde{\Omega}/\sim$. We define the sigma algebra over this as the set $\mathcal{R}$ s.t. $A \in \mathcal{R}$ only if $A \in \tilde{\mathcal{R}}$ and if $c \in \tilde{\Omega}$, then all relabeling of c lie in A. We define the map $p : \tilde{\Omega} \to \Omega$ which takes labeled set to unlabeled ones, and we define $\mu = \tilde{\mu} \circ p^{-1}$. The questions that we can ask should be unlabeled and thus we form another algebra of meaningful covariant questions. Suppose $b \in \Omega$ is a finite causet, and we define,

$$stem(b) = \{c \in \Omega : \text{ c contains a stem isomorphic to b}\} \tag{1.10}$$

The map p takes the cylinder sets to stem sets and these generate the algebras $\tilde{\mathcal{R}}$ and $\mathcal{R}(\mathcal{S})$ respectively. But we have a problem here that $\tilde{\mathcal{R}}$ separates points in $\tilde{\Omega}$ but $\mathcal{R}(\mathcal{S})$ does not separate points in Ω. Thus, even though $\mathcal{R}(\mathcal{S})$ forms a set of meaningful questions to ask, it isn't an exhaustive set of questions that can be asked. In Classical sequential growth dynamics, the algebra $\mathcal{R}(\mathcal{S})$ is sufficient as rest of the sets in $\mathcal{R}$ form a set of measure 0.

Remark We must note that the σ-algebra constructed above does not depend on the measure and thus, we can assign any measure over it, be it real valued or vector valued. This will be useful later on.

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[4] A similar measure can be assigned in the case of σ-algebra $(\tilde{\Omega}(n), \tilde{\mathcal{R}}_n$.

Chapter 2

Quantum Sequential Growth Dynamics

In this chapter, we will look at the generalization of classical SGD. Our aim is to obtain a more complete picture of the quantum dynamics of causets. The main idea is to take motivation from classical SGD and path integrals to generalize the classical measure to quantum measure. This involves replacing the property of countable additivity of measure by a weaker condition inspired from the path integral formalism[4]. We then revisit the complex SGD studied in [8],[19]. After that, we come to the main topic of this thesis, the quantum SGD. Here, we will deal with more general type of quantum measures, and assign properties similar to those assigned in classical SGD. We will then study the dynamics of these quantum measures under two different generalizations of Bell causality.

2.1 Generalization to Quantum Case

For making the quantum theory of sequential growth dynamics, we will use the path integral approach. We note that the probabilities there, say in two slit experiment, does not add up.[16][17] Thus, if we represent the probability as a measure μ over our given algebra, we can see that measure is not countably additive. In that sense, the measure is not classical. The value of this measure for different sets in algebra will be interpreted as some "generalized" probability. If we look at the three slit experiment, we find that this measure satisfies,

$$\mu(A\bigcup B\bigcup C) = \mu(A\bigcup B) + \mu(B\bigcup C) + \mu(A\bigcup C) - \mu(A) - \mu(B) - \mu(C) \tag{2.1}$$

Where A,B,C are pairwise disjoint. Thus, it was proposed by Sorkin that such a measure should follow the above addition law instead of being countably additive. Such a measure is called **Quantum measure**.

Remark An important thing to note here is that the interpretation of probability is different here. The measure is not countably additive and thus, we cannot simply say that the probability of occurrence of A is p(A). That is because, the probability of disjoint events are themselves not independent of each other entirely. Thus, making a statement like, A and B are disjoint events with probability p(A) and p(B) does not make much sense as $p(A \bigcup B)$ is not the sum of individual probabilities. There is some interdependence, maybe due to the new 'weaker' addition law.

2.1.1 Constructing Quantum Measure

We saw that the quantum measure is difficult to deal with. We will thus look at some ways in which a quantum measure can be constructed from objects that are simpler to deal with. One such object is called the **Decoherence functional**. First, lets try to understand in the context of path integrals and we later geberalize it to the case of causets. Lets say that the set of paths is given by Ω with the sigma algebra $\mathcal{A}$. (in general, we can take Ω to be some arbitrary set and the definition of decoherence functional given below still acts equally well. The actual set Ω and the sigma algebra $\mathcal{A}$ matters when we are trying to find the form of decoherence functional.).

Definition A **Decoherence functional** is a function $D : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ which has the following properties

1. For $\alpha, \beta \in \mathcal{A}$, $D(\alpha, \beta) = D(\beta, \alpha)*$
2. D is finitely bi additive i.e. $D(\bigcup_i \alpha_i, \beta) = \bigcup_i D(\alpha_i, \beta)$.
3. Normalised: $D(\Omega, \Omega) = 1$
4. D is **strongly positive** i.e. for any finite collection of $\alpha_i \in \mathcal{A}$, $M_{ij} = D(\alpha_i, \alpha_j)$ is a positive semidefinite matrix.

Then a **quantum measure** μ can be defined as $\mu(\alpha) = D(\alpha, \alpha)$. We can construct a Hilbert space $\mathcal{H}$ as given in $\cdots$, such that, there is a vector valued measure μ_V which is finitely additive such that $D(\alpha, \alpha) = \langle \mu_v(\alpha), \mu_v(\alpha) \rangle$. So, we want to deal with such vectro valued measures for now, as they are simple and finitely additive. We will first construct the vector measures over some abstract Hilbert space(which depends on the problem) and finally, for calculating probabilities, we fill find the inner product. Thus, we will have the required properties for this "generalized" probability.

2.1.2 Why Quantum Measure?

It is a good time to stop and ask, what are we planning to achieve by using a quantum measure. We can see that quantum measure is actually a mathematical way to write path integral, though it doesn't have all the properties of a path integral. But we have to see what properties of path integral we need in order to look at the evolution of the universe from a path integral perspective. What it means is that, all the branches of the Poscau that we generate are actually various paths that could have been followed. The only difference here is that, these paths are the evolution of universe itself and thus, these paths are not in some space-time background as it is usually the case. So, we have to talk about the path integrals of evolution of the universe itself. That is why we are taking some basic properties of quantum measure and making a measure. this measure can be interpreted as some sort of generalised probability. We will assign some extra properties to this measure which are analogue of properties of measures in classical sequential growth dynamics.

2.1.3 How to deal with quantum measure?

We can make use of vector valued measures μ_V to construct quantum measures. These vector valued measures are finitely additive and their range lies in some Hilbert space $\mathcal{H}$. The inner product of these vector with themselves gives us the quantum measure. The main idea is to construct this vector measure such that $\mu_V : \mathcal{A} \to \mathcal{H}$ where $\mathcal{H}$ is some suitable Hilbert space. Here, the inner product acts as the decoherence functional i.e.

$$\mu(\alpha) = \langle \mu_V(\alpha), \mu_V(\alpha) \rangle \tag{2.2}$$

A better way to represent the vector measure is to write $\mu_V(\alpha) = |\alpha\rangle$. A general construction of Hilbert space of paths is given in [7]. There is a subtlety here. The vector measure may not be defined over the whole σ-algebra generated by cylinder sets. We thus deal with the measures of cylinder sets and the Boolean algebra generated by them. The extension to σ-algebra is a question that is left unanswered for now. We first need to construct vector measures for cylinder sets before we even talk about the extension questions and this is the main focus of this chapter.

Another thing that we do to simplify our problems is to deal with transition amplitudes rather than vector measures themselves. By transition amplitudes, we mean operators that take a vector measure $\mu_V(c^i_n)$ corresponding to $cyl(c^i_n)$ to another vector measure $\mu_V(c^j_{n+m})$ corresponding to $cyl(c^j_{n+m})$. For simplicity, we assume that these transitions are linear operators.

The ring of transition amplitudes

Assumption *A transition amplitude is a linear operator on* $\mathcal{H}$
We want to assign some properties to these transition amplitudes analogous to the case of $\mathbb{R}^+SGD$. We first try to look at how these transitions are related to one another. For that we give the following definition.

Definition *The set of all transition amplitudes from* $\tilde{\Omega}(n) \to \tilde{\Omega}(n+m)$ *is represented by* $\mathcal{T}(\tilde{\Omega}(n) \to \tilde{\Omega}(n+m))$*. The set of all possible transition amplitudes* $\mathcal{T}_{\mathcal{B}}$ *is give by*

$$\mathcal{T}_{\mathcal{B}} = \bigcup_{n=1}^{\infty} \bigcup_{m>n}^{\infty} \mathcal{T}(\tilde{\Omega}(n) \to \tilde{\Omega}(m)) \tag{2.3}$$

From this we see that if $A \in \mathcal{T}_{\mathcal{B}}(\tilde{\Omega}(n) \to \tilde{\Omega}(n+m))$ and $B \in \mathcal{T}_{\mathcal{B}}(\tilde{\Omega}(n+m) \to \tilde{\Omega}(n+m+\ell)))$, then it is obvious that $BA \in \mathcal{T}_{\mathcal{B}}(\tilde{\Omega}(n) \to \tilde{\Omega}(n+m+\ell))$. Thus,$\mathcal{T}_{\mathcal{B}}$ is closed under product of operators of above form. But we want a little more general type of set as we may want to ask questions like, what is the measure of the event that the transition from a causet c_n^i to c_{n+1}^i is either timid or gregarious which requires a sum of transition amplitudes. Markov property requires this too. This tells us that we need more general type of elements than in $\mathcal{T}_{\mathcal{B}}$. From above discussion, we know that we want two essential properties in the elements i.e. addition and multiplication and we need an identity as well. All these properties are satisfied if we take the ring generated by $\mathcal{T}_{\mathcal{B}}$ and name it as $\mathcal{T}$. We know that scalar multiplication is not an operation that may always be allowed and thus, a ring with identity will suffice for our work.

Definition $\mathcal{T}$ *is a ring with identity which is generated by* $\mathcal{T}_{\mathcal{B}}$*.*

We can always generate an algebra from the ring $\mathcal{T}$, but for now, it is enough to look at the ring structure.

2.1.4 Complex SGD ($\mathbb{C}$SGD)

We first focus on a specific case, the complex sequential growth dynamics, i.e where the $\mathcal{H} \cong \mathbb{C}$. [8][19]. it's properties are given as

1. **General Covariance:** The measures assigned to a labeled causal set is same along any path.

2. **Markov Property:** To each transition, $c_n^i \to c_{n+1}^{j(i)}$, we have an operator $O(c_n^i \to c_{n+1}^{j(i)})$ such that $\sum_j O(c_n^i \to c_{n+1}^{j(i)}) = \mathbb{I}$ where $\mathbb{I}$ is the identity element of hilbert space. Here, these operator are complex numbers as $\mathcal{H} = \mathbb{C}$

3. **Bell Causality:** For complex percolation, we can define this property in the same way as we defined it for classical sequential growth dynamics. Thus,

$$\frac{O(c_n^i \to c_{n+1}^{j_1})}{O(c_n^i \to c_{n+1}^{j_2}))} = \frac{O(c_m^k \to c_{m+1}^{j_1})}{O(c_m^k \to c_{m+1}^{j_2}))} \tag{2.4}$$

   Where c_m^k is the union of precursor sets of the transitions $c_n^i \to c_{n+1}^{j_1}$ and $c_n^i \to c_{n+1}^{j_2}$.

We can thus see that, because all the properties are similar to classical case, the amplitudes $O(c_n^i \to c_{n+1}^{j(i)})$(not the probabilities) are given as

$$O(c_n^i \to c_{n+1}^{j(i)}) = q_n \sum_{k=m}^{\varpi} \binom{\varpi - m}{k - m} t_k = \frac{\lambda(\varpi, m)}{\lambda(n, 0)} \tag{2.5}$$

Where $\lambda(\varpi, m) = \sum_{k=m}^{\varpi} \binom{\varpi - m}{k - m} t_k$

Remarks This is not enough to formulate the above dynamics as such. We need to ensure that the measure extends to the full σ-algebra. The theorem that tells us when a measure is extendible is called the **Caratheodary-Hahn-Kluvnek** theorem. Now we move on to the actual problem that we are concerned with, i.e. to study the vector measures on higher dimensional Hilbert spaces.

2.2 Higher Dimensional Vector Measures

Now we come to the main result of this thesis. We study the vector measures under two different generalizations of Bell causality and we assume that the transition amplitudes are invertible. In other words, we assume $\mathcal{T}$ to be a ring with identity which is closed under inversion. We will eventually show that under both types of Bell causality, the ring $\mathcal{T}$ is cummutative.
We start with generalizing the General Covariance and Markov properties.

1. **General Covariance:** The measures assigned to a labeled causal set is same along any path.

2. **Markov Property:** To each transition, $c_n^i \to c_{n+1}^{j(i)}$, we have an operator $O(c_n^i \to c_{n+1}^{j(i)})$ such that $\sum_j O(c_n^i \to c_{n+1}^{j(i)}) = \mathbb{I}$ where $\mathbb{I}$ is the identity element of hilbert space. Here, these operator are complex numbers as $\mathcal{H} = \mathbb{C}$

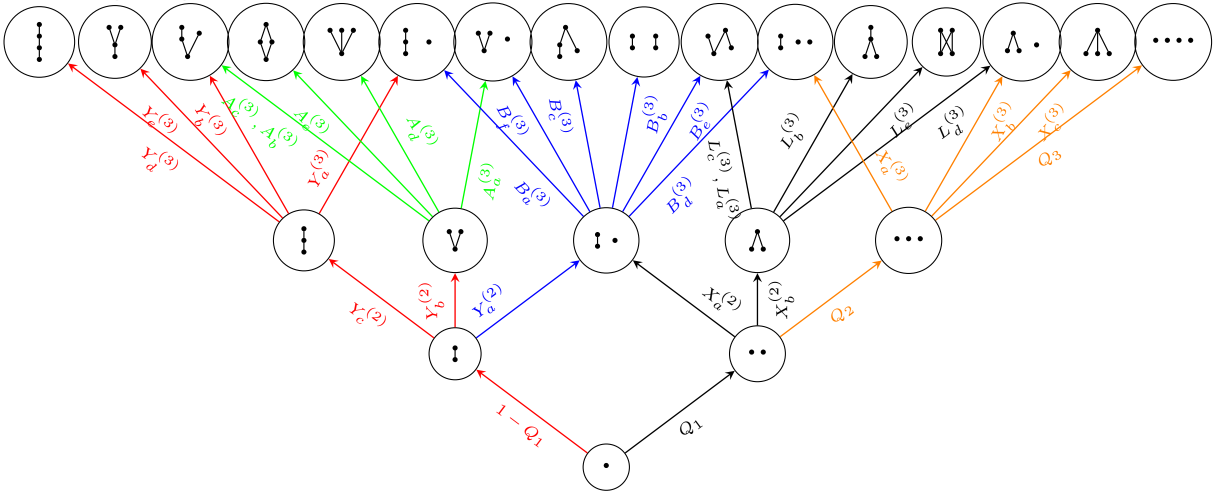
However, it is not so obvious as to how we can generalize the Bell Causality. There are a few ways in which Bell causality is defined. First we consider some transitions which have the same precursor sets $c_n^i \to c_{n+1}^{j_1}$, $c_n^i \to c_{n+1}^{j_2}$ and $c_m^i \to c_{m+1}^{j_1'}$, $c_m^i \to c_{m+1}^{j_2'}$ where m¡n and the corresponding transition amplitudes are A_n, B_n, A_m, B_m.
We will look at two of the possible ways in which Bell causality can be generalized.

Time ordered Bell Causality or type-I BC If $n > m$, $A_n B_m = B_n A_m$

Type II Bell causality We assume that all transition amplitudes are invertible and the Bell causality

$$A_n B_n^{-1} = A_m B_m^{-1} \tag{2.6}$$

We will first deal with type-II Bell causality and show that we get a cummutative ring. The following diagram will be used in our subsequent calculations.



2.2.1 The explicit calculation for amplitudes in type-II Bell causality

With this, we can explicitly calculate the transition amplitude for stage 1,2 and 3 for type II Bell causality. This will be helpful in motivating proofs and giving us a general idea. For stage 1, the transition amplitudes are Q_1 and $\nVdash - Q_1$. We will do it stage by stage

Stage 2

$$\begin{aligned} X_a^{(2)} Q_2^{-1} &= (1-Q_1) Q_1^{-1} \text{ (By Bell Causality (BC))} \qquad (2.7) \\ X_a^{(2)} &= \Gamma_1 Q_2 \text{ (Where } \Gamma_1 = Q_1^{-1} - 1) \qquad (2.8) \end{aligned}$$

By Markov property, we have

$$X_b^{(2)} = [Q_2^{-1} - 2Q_1^{-1} + 1] Q_2$$

For the $Y^{(2)}$ transitions, we have

$$\begin{aligned} Y_a^{(2)} (1-Q_1) &= \Gamma_1 Q_2 Q_1 \text{ (By General Covariance(GC))} \qquad (2.9) \\ Y_a^{(2)} &= \Gamma_1 Q_2 \Gamma_1^{-1} \qquad (2.10) \end{aligned}$$

And for Y_b^{-1}, we use BC to show,

$$Y_b^{(2)} = (\Gamma_1) \Gamma_1 Q_2 \Gamma_1^{-1} \qquad (2.11)$$

And by Markov property,

$$Y_c^{(2)} = \left(\Gamma_1 Q_2^{-1} \Gamma_1^{-1} - Q_1^{-1} \right) \Gamma_1 Q_2 \Gamma_1^{-1} \qquad (2.12)$$

Stage 3

We first see that we have labeled the equal transition amplitudes by the same label.

For X transitions $X_a^{(3)}$ and $X_b^{(3)}$. There are actually three of these and we have to keep that into account while using Markov property.

$$\begin{aligned} X_a^{(3)} Q_3 &= (1-Q_1)(Q_1^{-1}) \text{ By BC} \qquad (2.13) \\ X_a^{(3)} &= \Gamma_1 Q_3 \qquad (2.14) \end{aligned}$$

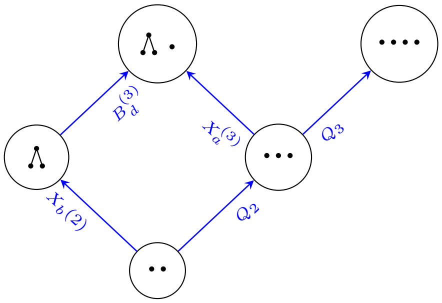
Again by BC,

$$X_b^{(3)} = X_b^{(2)} Q_2^{-1} Q_3 = (Q_2^{-1} - 2Q_1^{-1} + 1) Q_3 \qquad (2.15)$$

And finally by Markov property

$$\begin{aligned} X_c^{(3)} &= 1 - 3X_a^{(3)} - 3X_b^{(3)} - Q_3 \qquad (2.16) \\ &= \left(Q_3^{-1} - 3Q_2^{-1} + 3Q_1^{-1} - 1 \right) Q_3^{-1} \qquad (2.17) \end{aligned}$$

For B transitions We look at the leaves



$$B_d^{(3)} = \Gamma_1 Q_3 \Gamma_1^{-1} \tag{2.18}$$

Now we will use BC to give all the non timid transitions.

$$\begin{aligned} B_f^{(3)}(B_d^{(3)})^{-1} &= (1 - Q_1)Q_1^{-1} \\ \Rightarrow B_f^{(3)} &= \Gamma_1(\Gamma_1 Q_3 \Gamma_1^{-1}) \end{aligned} \tag{2.19}$$

$$\begin{aligned} B_a^{(3)}(B_d^{(3)})^{-1} &= Y_c^{(2)}(Y_a^{(2)})^{-1} \\ \Rightarrow B_a^{(3)} &= \left[\Gamma_1 Q_2^{-1}\Gamma_1^{-1} - Q_1^{-1} + 1\right]\Gamma_1 Q_3 \Gamma_1^{-1} \end{aligned} \tag{2.20}$$

$$\begin{aligned} B_e^{(3)}(B_d^{(3)})^{-1} &= X_b^{(2)} Q_2^{-1} \\ B_e^{(3)} &= \left[Q_2^{-1} - 2Q_1^{-1} + 1\right]\Gamma_1 Q_3 \Gamma_1^{-1} \end{aligned} \tag{2.21}$$

$$\begin{aligned} B_b^{(3)} &= \Gamma_1 B_d^{(3)} \\ B_b^{(3)} &= \Gamma_1(\Gamma_1 Q_3 \Gamma_1^{-1}) \end{aligned} \tag{2.22}$$

Now, by Markov property, we derive the timid child $B_c^{(3)}$

$$B_c^{(3)} = \left[\Gamma_1 Q_3^{-1}\Gamma_1^{-1} - \Gamma_1 Q_2^{-1}\Gamma_1^{-1} - Q_2^{-1} + Q_1^{-1}\right]\Gamma_1 Q_3 \Gamma_1^{-1} \tag{2.23}$$

The L transitions First relating the gregarious child to Q_3, we have

$$L_d^{(3)} X_b^{(2)} = X_b^{(3)} \text{ (by GC)} \tag{2.24}$$

$$\& X_b^{(3)} Q_3^{-1} = X_b^{(2)} Q_2^{-1} \text{ (by BC)} \tag{2.25}$$

$$\Rightarrow L_d^{(3)} = X_b^{(3)} Q_3 (X_b^{(3)})^{-1} \tag{2.26}$$

But $X_b^{(3)} = X_b^{(2)} Q_2^{-1} Q_3 = (Q_2^{-1} - 2Q_1^{-1} + 1)Q_3$ and thus, we get that

$$L_d^{(3)} = [Q_2^{-1} - 2Q_1^{-1} + 1]Q_3[Q_2^{-1} - 2Q_1^{-1} + 1]^{-1} \tag{2.27}$$

Now we find the other elements using BC

$$\begin{aligned} L_e^{(3)}(L_d^{(3)})^{-1} &= X_b^{(2)} Q_2^{-1} \\ \Rightarrow L_e^{(3)} &= [Q_2^{-1} - 2Q_1^{-1} + 1](L_d^{(3)}) \end{aligned} \tag{2.28}$$

$$\begin{aligned} L_c^{(3)}(L_d^{(3)})^{-1} &= \Gamma_1 \\ \Rightarrow L_c^{(3)} &= \Gamma_1(L_d^{(3)}) \end{aligned} \tag{2.29}$$

$$L_a^{(3)} = L_c^{(3)} \tag{2.30}$$

$$\begin{aligned} L_b^{(3)} &= 1 - L_a^{(3)} - L_c^{(3)} - L_d^{(3)} - L_e^{(3)} \\ L_b^{(3)} &= [(L_d^{(3)})^{-1} - 2Q_1^{-1} + 2 - 1 - Q_2^{-1} + 2Q_1^{-1} - 1]L_d^{(3)} \\ \Rightarrow L_b^{(3)} &= [(L_d^{(3)})^{-1} - Q_2^{-1}]L_d^{(3)} \end{aligned} \tag{2.31}$$

The A transitions By Bell causality and general covariance, the form of gregarious child $A_a^{(3)}$ is

$$A_a^{(3)} = \Gamma_1^2 Q_3 \Gamma_1^{-2} \tag{2.32}$$

And the other terms are found by Bell causality.

$$\begin{aligned} A_c^{(3)} &= Y_c^{(2)}(Y_a^{(2)})^{-1} A_a^{(3)} \\ A_c^{(3)} &= \left(\Gamma_1 Q_2^{-1} \Gamma_1^{-1} - Q_1^{-1}\right) A_a^{(3)} \end{aligned} \tag{2.33}$$

$$A_b^{(3)} = A_c^{(3)} \tag{2.34}$$

$$A_d^{(3)} = \Gamma_1 A_a^{(3)} \tag{2.35}$$

$$\begin{aligned} A_e^{(3)} &= 1 - A_a^{(3)} - A_b^{(3)} - A_c^{(3)} - A_d^{(3)} \\ A_e^{(3)} &= [\Gamma_1^2 Q_3^{-1} \Gamma_1^{-2} - 1 - 2\Gamma_1 Q_2^{-1} \Gamma_1^{-1} + 2Q_1^{-1} - Q_1^{-1} + 1]\Gamma_1^2 Q_3 \Gamma_1^{-2} \\ A_e^{(3)} &= [\Gamma_1^2 Q_3^{-1} \Gamma_1^{-2} - 2\Gamma_1 Q_2^{-1} \Gamma_1^{-1} + Q_1^{-1}]\Gamma_1^2 Q_3 \Gamma_1^{-2} \end{aligned} \tag{2.36}$$

The Y transitions First finding the gregarious child, using the relations of GC and BC, we get

$$\begin{aligned}
Y_a^{(3)} Y_c^{(2)} &= B_a^{(3)} Y_a^{(2)} && (2.37)\\
B_a^{(3)} (B_d^{(3)})^{-1} &= Y_2^{(2)} (Y_a^{(2)})^{-1} && (2.38)\\
\Rightarrow Y_a^{(3)} &= \left[\Gamma_1 Q_2^{-1} \Gamma_1^{-1} - Q_1^{-1} + 1\right] \Gamma_1 Q_3 \Gamma_1^{-1} \left[\Gamma_1 Q_2^{-1} \Gamma_1^{-1} - Q_1^{-1} + 1\right]^{-1} && (2.39)
\end{aligned}$$

$$Y_b^{(3)} = \Gamma_1 Y_a^{(3)} \tag{2.40}$$

$$Y_e^{(3)} = \left(\Gamma_1 Q_2^{-1} \Gamma_1^{-1} - Q_1^{-1}\right) Y_a^{(3)} \tag{2.41}$$

$$\begin{aligned}
Y_d^{(3)} &= 1 - Y_a^{(3)} - Y_b^{(3)} - Y_e^{(3)}\\
Y_d^{(3)} &= [(Y_a^{(3)})^{-1} - 1 - \Gamma_1 - \Gamma_1 Q_2^{-1} \Gamma_1^{-1} + Q_1^{-1}] Y_a^{(3)}\\
Y_d^{(3)} &= [(Y_a^{(3)})^{-1} - \Gamma_1 Q_2^{-1} \Gamma_1^{-1}] Y_a^{(3)} && (2.42)
\end{aligned}$$

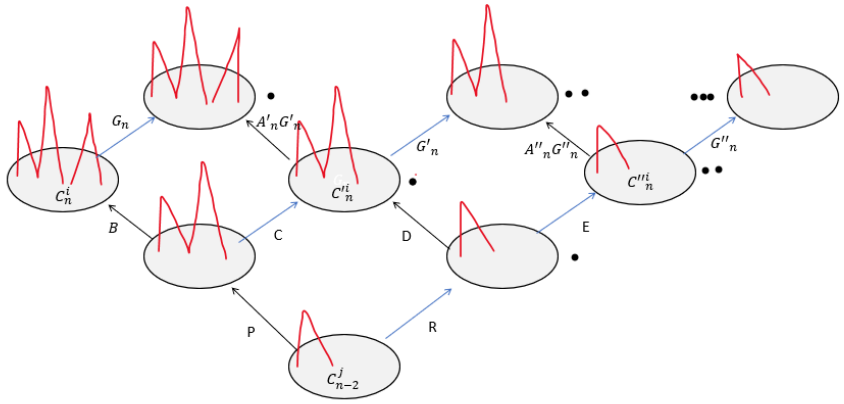
2.2.2 The general expression of transition amplitudes in type II Bell Causality

Now that we have seen some explicit calculations above, we will try to get insights from those to get a general result. As we will eventually see, the ring of operators turns out to be cummutative. If we generate an algebra from this ring, it will be cummutative as well but for now, lets focus on the ring of operators. The first thing we do is to relate the gregarious transitions at stage n, to Q_n(the gregarious transition to antichain). We represent the gregarious transitions to c_n^i by G_n^i. We show that G_n^i are related to Q_n by $G_n^i = \Gamma_n^i Q_n (\Gamma_n^i)^{-1}$ where Γ_n^i are the elements of the ring $\mathcal{T}$.

Theorem 2.2.1. *If* $|c_n^i\rangle = A_{n-1} A_{n-2} \cdots\ A_1 |\cdot\rangle$ *for one of the paths, we define* $\Gamma_n^j(c_n^i) = A_{n-1}(G_{n-1}^i)^{-1} \cdots A_1 G_1^{-1}$*, where* G_k^i *the corresponding gregarious transition to* c_k^i *and the j index in* $\Gamma_n^j(c_n^i)$ *represents the path followed to* c_n^i*. Then,* G_n^i *is related to* Q_n *by*

$$G_n^i = \Gamma_n^i Q_n (\Gamma_n^i)^{-1} \tag{2.43}$$

Proof. As shown in the following figure 2.2.2, we chose the leaves that have the transitions related to the transitions A_m for some m in the path $A_{n-1} A_{n-2} \cdots\ A_1 |\cdot\rangle$ by Bell causality. The first leave we chose is related to $B = A_{n-1}$ by Bell causality, the second leaf is related to $P = A_{n-2}$ by Bell causality.



Now, we are going to relate the gregarious transitions G_n, G'_n and G''_n. By GC, we have

$$G_nB = A'_nC \tag{2.44}$$

And by BC, we have

$$A'_n(G'_n)^{-1} = BC^{-1} \tag{2.45}$$

Thus, we get that $G_nB = BC^{-1}G'_nC \Rightarrow G_n = BC^{-1}G'_nCB^{-1}$. Similarly, we get that for c'^i_n that $G'_n = DE^{-1}G''_nED^{-1}$. But by BC, $DE^{-1} = PR^{-1}$. Thus, we have

$$\begin{aligned} G'_n &= PR^{-1}G''_nRP^{-1} && (2.46) \\ \Rightarrow G_n &= BC^{-1}PR^{-1}G''_nRP^{-1}CB^{-1} && (2.47) \end{aligned}$$

But remember that we chose the leaves such that $PR^{-1} = A_{n-2}(G^i_{n-2})^{-1}$ and $BC^{-1} = A_{n-1}(G^i_{n-1})^{-1}$ and thus, we have

$$G_n = (A_{n-1}(G^i_{n-1})^{-1}A_{n-2}(G^i_{n-2})^{-1})G''_n(A_{n-1}(G^i_{n-1})^{-1}A_{n-2}(G^i_{n-2})^{-1})^{-1} \tag{2.48}$$

Continuing this way, we can get the next term as $A_{n-3}(G^i_{n=3})^{-1}$ and so on. Thus we see that

$$\begin{aligned} G_n &= (A_{n-1}(G^i_{n-1})^{-1}\cdots A_1(G^i_1)^{-1})Q_n(A_{n-1}(G^i_{n-1})^{-1}\cdots A_1(G^i_1)^{-1})^{-1} \\ \Rightarrow G_n &= \Gamma^i_nQ_n(\Gamma^i_n)^{-1} && (2.49) \end{aligned}$$

■

Remarks We have shown the relation of Q_n to G^i_n for one such paths. For different paths, we may get different Γ^i_n and thus, it is yet to be shown that the dynamics can be made consistently. We will eventually show that this dynamics is consistent as the operators cummute and that ensures that all the gregarious child have the same transition amplitude.

Commutation relations in the algebra

In this section, we will prove that the ring of operators is commutative. To do that, we will first find the form of transition amplitudes for an antichain. It is easy to write those dynamics just in terms of Q_n as we just need Markov rule and Bell causality to find out all transition amplitudes. We then use one other important fact about these transitions. Any two transitions over an antichain can be carried out in any order and they will produce the same causet. This gives us exactly two possible paths to make the causet. The second transitions in both of these paths will not be over antichains but is related to transitions over antichain by Bell causality. We can thus get two different $\Gamma^i_n(c^i_n)$ for this set and thus, we get a relation between these two Γ operators. This new relation eventually allows us to prove that $[Q_n, Q_m] = 0 \; \forall n, m \in \mathbb{N}$. From this discussion, we are motivated to define the following type of transition amplitudes

Definition *If there are two transitions A and B such that A can only occur after B, then they are called **time ordered** or **ordered**. Else they are called **unordered**.*
We first prove an important lemma that will be used to prove further results.

Lemma 2.2.2. *Lets assume that we have a vector space V and the sum of elements in a finite subset X of V is represented by S(X). We can then show that for $X_i \subset V$, $i \in \{1, \cdots k\}$,*

$$S\left(\bigcup_i^k X_i\right) = \sum_i^k S(X_i) - \sum_{i<j} S(X_i \bigcap X_j) + \cdots + (-1)^{k-1} S\left(\bigcap_i^k X_i\right) \tag{2.50}$$

Proof. We will prove this by induction. For two sets X_1, X_2, we want to calculate $S(X_1 \cup X_2)$. Note that there are elements appearing in both X_1 and X_2 and thus, they are counted twice. They lie precisely in $X_1 \cap X_2$. Thus, we have

$$S(X_1 \cup X_2) = S(X_1) + S(X_2) - S(X_1 \cap X_2) \tag{2.51}$$

Now assume this to be true for stage $n-1$. We need to show this is true for stage n as well. We see that at stage n, we can split the sets as

$$\bigcup_i^k X_n = X_n \bigcup \left(\bigcup_i^{n-1} X_i\right) \tag{2.52}$$

We can by above arguments write, taking X_n as one set and $\cup_i^{n-1} X_i$ as the other set.

$$S\left(\bigcup_i^n X_i\right) = S(X_n) + S\left(\bigcup_i^{n-1} X_i\right) - S(X_n \bigcap (\bigcup_i^{n-1} X_i)) \tag{2.53}$$

As $X_n \cap (\cup_i^{n-1} X_i) = \cup_i^{n-1}(X_i \cap X_n)$, we can write the above summation as

$$\begin{aligned} S\left(\bigcup_i^n X_i\right) &= \sum_i^n S(X_i) - \sum_{i<j}^{n-1} S(X_i \bigcap X_j) + \cdots + (-1)^{n-2} S\left(\bigcap_i^{n-1} X_i\right) \\ &\quad - \sum_i^{n-1} S(X_i \bigcap X_n) + \sum_{i<j}^{n-1} S\left((X_i \bigcap X_n) \bigcap (X_j \bigcap X_n)\right) + \cdots \\ &\quad + (-1)^{n-1} S\left(\bigcap_i^{n-1} (X_i \bigcap X_n)\right) \\ \Rightarrow S\left(\bigcup_i^n X_i\right) &= \sum_i^n S(X_i) - \sum_{i<j}^{n} S\left(X_i \bigcap X_j\right) + \cdots + (-1)^{n-1} S\left(\bigcap_i^{n} X_i\right) \qquad (2.54) \end{aligned}$$

And this completes the proof. ■

Now we will find the form of transition amplitudes of transitions to an antichain.

Lemma 2.2.3. *The transition amplitude of a timid child to an n-antichain is*

$$\Lambda_n^{(n)} = \left[\sum_{k=0}^n (-1)^k \binom{n}{k} Q_{n-k}^{-1}\right] Q_n \qquad (2.55)$$

Proof. First we see that the child to one antichain can be written as $(Q_1^{-1} - 1)Q_1$. Now, we will prove the theorem by the principle of strong induction[1]. We assume that all the transitions upto stage $n-1$ are of the form $A_k Q_k$ $k < n$ and find the transition amplitude of timid transitions at stage n. Non timid ones can be found by Bell causality and thus, just looking at timid ones are enough.

$$\Lambda_n^{(n)} = \mathbb{I} - (\text{all other transitions})$$

We will represent the set of all transition amplitudes after removing one of $n-1$ maximal element labeled by m_i[2] by ζ_i, that by removing two maximal elements m_i, m_j by ζ_{ij} and so on. Then, $\cup_i \zeta_i$ will be the set that will have all the non timid child of c_n^i. Thus, the quantity that we want to calculate is $S(\cup_i \zeta_i)$ and then subtract it from identity to get the timid transition. We will use lemma 2.2.2 for this purpose.

$$S\left(\bigcup_i^n \zeta_i\right) = \sum_i^n S(\zeta_i) - \sum_{i<j}^n S(\zeta_i \bigcap \zeta_j) + \cdots + (-1)^n S\left(\bigcap_i^n \zeta_i\right) \qquad (2.56)$$

---

[1] The principle of strong induction is a variant of induction where we assume that all statements P(k) are true for $k \leq n-1$ and show that P(n) is true. This is actually equivalent to principle of induction in which only $P(n-1)$ is assumed to be true. For more information, look Wikipedia page for "Mathematical Induction".

[2] That means the set of all transitions to $c_n^i/\{m_i\}$

Now, note that $\zeta_i \cap \zeta_j = \zeta_{ij}$ and in general $\zeta_1 \cap \cdots \zeta_k = \zeta_{12\cdots k}$. We can thus sum up the transitions in these sets to get the probability. Also note that any sum $S(\zeta_{m_{i_1} m_{i_2} \cdots m_{i_k}})$ represents the sum of all the transition amplitudes to the set $c_n^i / \{m_{i_1}, m_{i_2}, \cdots, m_{i_k}\}$. We can sum up the transitions in these sets to get the probability. For that, we note is that $\sum_{j \in \mathcal{J}(n-k+1,i)} A^j_{n-k} Q_n^{-1}$ (where the sum is over all possible transitions) can be found from markov sum rule. (sum is taken over everything except gregarious child, which will be added later)

$$\sum_{j \in \mathcal{J}(n-k+1,i) \setminus \{g\}} A^j_{n-k} Q_{n-k} = \mathbb{I} - Q_{n-k} \tag{2.57}$$

Thus, $\sum_{j \in \mathcal{J}(n-k+1,i) \setminus \{g\}} A^j_{n-k} = (1 - Q_{n-k}) Q^{-1}_{n-k}$. Also note that by Bell causality, the non timid transitions at stage n are given as $A^j_n = A^j_{n-k} Q_{n-k} Q^{-1}_{n-k} Q_n = A^j_{n-k} Q_n$ where k is the number of maximal elements removed. The sum is thus given by

$$\begin{aligned}
S(\zeta_{m_{i_1} m_{i_2} \cdots m_{i_k}}) &= \sum_{j \in \mathcal{J}(n-k+1,i) \setminus \{g\}} A^j_{n-k} Q_n \\
S(\zeta_{m_{i_1} m_{i_2} \cdots m_{i_k}}) &= \sum_{j \in \mathcal{J}(n-k+1,i) \setminus \{g\}} A^j_{n-k} Q_n + Q_n \\
S(\zeta_{m_{i_1} m_{i_2} \cdots m_{i_k}}) &= (1 - Q_{n-k}) Q^{-1}_{n-k} Q_n + Q_n \\
\Rightarrow S(\zeta_{m_{i_1} m_{i_2} \cdots m_{i_k}}) &= Q^{-1}_{n-k} Q_n \qquad (2.58)
\end{aligned}$$

Thus, we get the probability by putting in the equation 2.56.

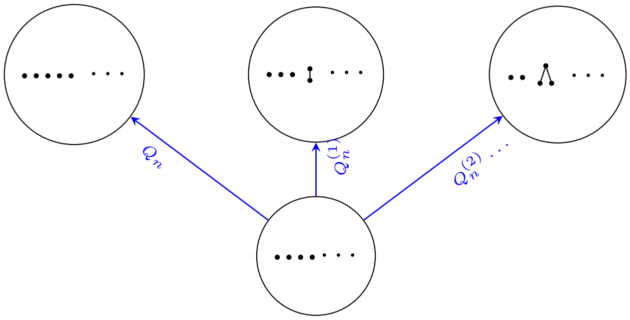
$$\Lambda^{(n)}_n = \left[\sum_{k=0}^{n} (-1)^k \binom{n}{k} Q^{-1}_{n-k} \right] Q_n \tag{2.59}$$

■

We actually do not need the exact form of transition amplitude to prove that the algebra is commutative. We can use the following theorem instead of the exact form of transition amplitudes found above.

Theorem 2.2.4. *The timid transition to n antichain are polynomials in* $Q_k, Q_k^{-1}\ \forall k \leq n$.

Proof. Look at the following diagram.



In this diagram, we define $Q_n^{(k)}$ to be the transition with k maximal elements. It is easy to see that there are $\binom{n}{k}$ such elements as this is the number of ways of choosing k elements from n elements. Thus, we get that

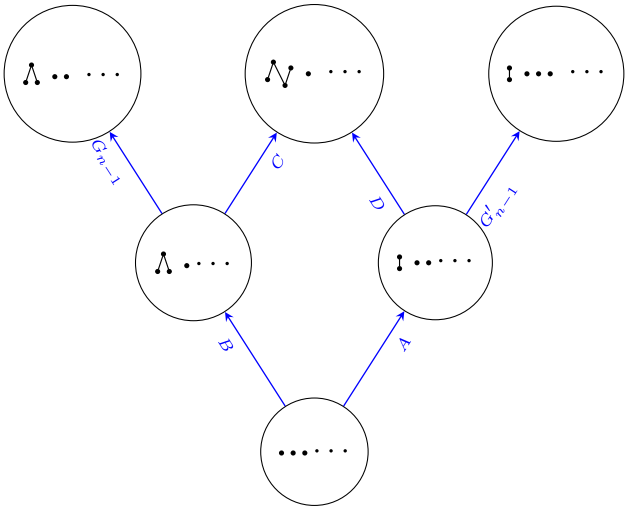
$$Q_n^{(n)} = \mathbb{I} - \sum_{k=0}^{n-1} \binom{n}{k} Q_n^{(k)} \tag{2.60}$$

Also, $Q_n^{(k)} = Q_k^{(k)} Q_k^{-1} Q_n$. We also know that $Q_2^{(2)} = \Lambda_2$ is a polynomial in Q_1, Q_2, Q_1^{-1}. From here, using mathematical induction we can see that $Q_n^{(n)}$ is a function of Q_k, Q_k^{-1} $\forall k \in \mathbb{N}$. ■

This much information is enough to prove the latter theorems without explicitly calculating the form of $Q_n^{(n)}$. We can simply take Q_n as common from here and use it to prove the following theorems.

Lemma 2.2.5. *If A and B are unordered transitions with some antichain as the precursor set, then,* $[AQ_n^{-1}, BQ_n^{-1}] = 0$.

Proof. Suppose that A and B are two such transitions as shown in the figure and that we are looking at transitions form an n-antichain. It is easy to see by Bell causality that $AQ_n^{-1}G_{n+1} = C$ and $BQ_n^{-1}G'_{n+1} = D$.



Then, by general covariance

$$AQ_n^{-1}G_{n+1}B = BQ_n^{-1}G'_{n+1}A \tag{2.61}$$

But $G_{n+1} = BQ_n^{-1}Q_{n+1}Q_nB^{-1}$ and $G'_{n+1} = AQ_n^{-1}Q_{n+1}Q_nA^{-1}$ and from here, we get

$$\begin{aligned} AQ_n^{-1}BQ_n^{-1}Q_{n+1}Q_nB^{-1}B &= BQ_n^{-1}AQ_n^{-1}Q_{n+1}Q_nA^{-1} \\ \Rightarrow AQ_n^{-1}BQ_n^{-1} &= BQ_n^{-1}AQ_n^{-1} \end{aligned} \tag{2.62}$$

■

With this, we get a useful way to check the commutation relation between Q_n. Say, we first look at $[Q_1, Q_2]$. We have seen in above calculations that

$$Y_b^{(3)}Y_c^{(2)} = A_c^{(3)}Y_b^{(2)} \tag{2.63}$$

We can see that for these transitions, we must have

$$\left[\Gamma_1 Q_2^{-1}\Gamma_1^{-1} - Q_1^{-1}, \Gamma_1\right] = 0 \tag{2.64}$$

$$\Rightarrow \left[\Gamma_1 Q_2^{-1}\Gamma_1^{-1}, Q_1^{-1}\right] = 0 \tag{2.65}$$

$$\Rightarrow [Q_1, Q_2] = 0 \tag{2.66}$$

Now that we have all the lemmas that we needed, all that is left is to show that $[Q_n, Q_m] = 0$. The lemma 2.2.5 allows us an easy way to do that by comparing two transition amplitudes to an n antichain.

Theorem 2.2.6. $[Q_n, Q_m] = 0 \ \forall n, m \in \mathbb{N}$.

Proof. We will prove this by strong induction. We already showed this for Q_1, Q_2. Now, assume this to be true till stage $n-1$ i.e. $[Q_1, Q_{n-1}] = 0$. Then, we consider the timid transition to n antichain. The form can be given by ansatz. Say that this transition is A. Another transition is B, which is $\Gamma_1 Q_n$. Then we know that these are unordered and by previous lemma, we have

$$[A, \Gamma_1] = 0 \tag{2.67}$$

By using theorem 2.2.3, we know that above commutation relation can be written as

$$\begin{aligned} [Q_n^{-1}, Q_1^{-1} - 1] &= 0 && (2.68) \\ \Rightarrow [Q_n, Q_1] &= 0 && (2.69) \end{aligned}$$

Similarly, we can show that $[Q_2, Q_n] = 0 \ \forall n \in \mathbb{N}$. We then assume that this condition is true till $n-1$. We will show that $[Q_n, Q_{n+k}] = 0 \ \forall k \in \mathbb{N}$. Lets write the timid transition to n antichain as τ_n and then, we can write the relation between transition amplitudes to n antichain as $[\tau_n Q_n^{-1}, \tau_m Q_m^{-1}] = 0$ from lemma 2.2.5. As the transitions in picture are over an antichain, from the 2.2.3, we have

$$\tau_n Q_n^{-1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} Q_{n-k}^{-1} \tag{2.70}$$

We will use strong induction again. First for $k = 1$, we use above formula and get $[Q_n, Q_{n+1}] = 0$. Assume that $[Q_n, Q_{n+k}] = 0 \ \forall k \leq m-1$. We need to show that $[Q_n, Q_m] = 0$. Again from above formula, because of our assumption, we get that

$$\begin{aligned} [\tau_n Q_n^{-1} Q_{n+m} Q_{n+m}^{-1}, \tau_{n+m} Q_{n+m}^{-1}] &= 0 \\ \Rightarrow [Q_n, Q_{n+m}] &= 0 \text{ by multilinearity of commutation brackets} \end{aligned} \tag{2.71}$$

With that we showed that, for all $n, m \in \mathbb{N}$, we have $[Q_n, Q_m] = 0$. ■

Theorem 2.2.7. *The ring of operators $\mathcal{T}$ is commutative and the timid child with m maximal elements has transition probability*

$$A_n(c_n^i \to c_{n+1}^{j(i)}) = \left[\sum_{k=0}^{m} (-1)^k \binom{m}{k} Q_{n-k}^{-1} \right] Q_n \tag{2.72}$$

Proof. We have already shown that till level 3, we have all the transitions as polynomials in Q_k. Now, we use strong induction to prove it for all the transitions. Suppose that till stage $n-1$, the the timid transitions at stage ℓ with m maximal elements are given by

$$A_\ell^i = \left[\sum_{k=0}^{m} (-1)^k \binom{m}{k} Q_{\ell-k}^{-1} \right] Q_\ell \tag{2.73}$$

Non timid transition to c_n^i at stage n are given by Bell causality. Assume that the gregarious child to c_n^i is G_n^i. Say we have one such transition A_n, with ϖ elements in precursor set with m maximal elements.We then get by Bell causality that

$$A_n = \left[\sum_{k=0}^{m}(-1)^k \binom{m}{k} Q_{\varpi-k}^{-1}\right] Q_n \tag{2.74}$$

For some path forming c_n^i, assume that $\Gamma_n^i = A_n \cdots A_1$. But we know from above equations that A_k and thus Γ_n^i it is a polynomial in Q_k, Q_k^{-1} with $k \leq n-1$. Also, $G_n^i = \Gamma_n^j(c_n^i) Q_n (\Gamma_n^j(c_n^i))^{-1}$. From this and the fact that $[Q_n, Q_m] = 0$

$$G_n^i = Q_n \tag{2.75}$$

Thus all gregarious transitions at stage n are equal to Q_n. Any non timid transition with ϖ elements in precursor set and m maximal elements is

$$A_n^j = \left[\sum_{k=0}^{m}(-1)^k \binom{m}{k} Q_{\ell-k}^{-1}\right] Q_n \tag{2.76}$$

The timid transition can be found using the lemma 2.2.2. Again, we will represent the set of all transition amplitudes after removing one of m_i by ζ_i, that by removing two maximal elements m_i, m_j by ζ_{ij} and so on. Then, $\cup_i \zeta_i$ will be the set that will have all the non timid child of c_n^i. Thus, the quantity that we want to calculate is $S(\cup_i \zeta_i)$ and then subtract it from identity to get the timid transition. We use the same steps as in lemma 2.2.3.

$$S\left(\bigcup_i^n \zeta_i\right) = \sum_i^n S(\zeta_i) - \sum_{i<j}^n S(\zeta_i \bigcap \zeta_j) + \cdots + (-1)^n S\left(\bigcap_i^n \zeta_i\right) \tag{2.77}$$

We can now sum up the transitions in these sets to get the probability. For that, we note is that $\sum_{j \in \mathcal{I}(m+1,i)} A_m^j Q_n^{-1}$ (where the sum is over all possible transitions) can be found from markov sum rule. Say, we are looking at the $S(\zeta_{k_1 k_2 \cdots k_\ell})$. We can calculate it as follows. Suppose that this is the set of all transitions to the set $c_{n-\ell}^\alpha$ $\alpha \in \mathcal{I}(n-\ell)$. Then, the elements of the $\zeta_{k_1 k_2 \cdots k_\ell}$ are of the form $A_{n-\ell}^\beta Q_{n-\ell}$ where $\beta \in \mathcal{J}(n-\ell+1, \alpha)$. We then note that

$$\begin{aligned} \sum_{\beta \in \mathcal{J}(n-\ell+1,\alpha)} A_{n-\ell}^\beta Q_{n-\ell}^{-1} Q_{n-\ell} &= \mathbb{I} \\ \Rightarrow \sum_{\beta \in \mathcal{J}(n-\ell+1,\alpha)} A_{n-\ell}^\beta Q_{n-\ell}^{-1} &= Q_{n-\ell}^{-1} \end{aligned} \tag{2.78}$$

Thus, we get the probability by putting in the equation 2.56.

$$A_n(c_n^i \to c_{n+1}^{j(i)}) = \left[\sum_{k=0}^{n}(-1)^k \binom{n}{k} Q_{n-k}^{-1}\right] Q_n \tag{2.79}$$

Thus, we proved the equation 2.72. Non timid transitions can be found by Bell causality. We thus showed that the transition amplitudes can be written as polynomials in Q_k, Q_k^{-1} and thus, they all commute. Thus the ring $\mathcal{T}$ is commutative. ■

With this, we can give an important corollary.

Corollary 2.2.7.1. *Any transition at stage n, with* ϖ *elements in precursor set and m maximal elements is given by*

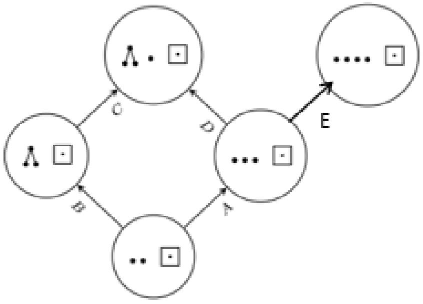
$$A_n^j Q_n = \left[\sum_{k=0}^{m} (-1)^k \binom{m}{k} Q_{\varpi-k}^{-1} \right] Q_n \tag{2.80}$$

2.2.3 Time Ordered Bell Causality

In this section, we will work with the other type of Bell causality. We assume that all transition amplitudes are invertible. In this case also, we will eventually show that the ring of operators is commutative and the proof involves looking at similar type of transition as in the case of type II Bell causality. We will first show that the gregarious transitions at stage n have the same amplitude Q_n (i.e. the transition amplitude of gregarious child to n antichain). We can then look at the transitions that are unordered and relate them using Bell causality. This will allow us to prove that Q_ns commute.

Theorem 2.2.8. *All the transition amplitudes to gregarious transitions at stage n are* Q_n*.*

Proof. Note that following picture is depicting just a few elements of the causal sets to establish a relation between different transitions, while the $\boxdot$ represents the remaining elements of causet.



By Bell causality,

$$EB = DA \tag{2.81}$$

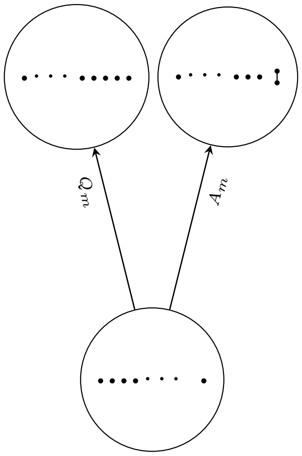
And by General covariance

$$DA = CB \tag{2.82}$$

Thus we get that, $EB = CB \Rightarrow E = C$. Thus, we get from here that the gregarious transitions at the same level have same transition amplitudes and hence, all transitions at stage n have transition amplitude Q_n. ■

Theorem 2.2.9. *The gregarious transitions to antichain are commutative i.e.* $[Q_n, Q_m] = 0$ $\forall n, m \in \mathbb{N}$.

Proof. Let the transition amplitude be as shown in the following figure.



$$A_m Q_1 = Q_m(1 - Q_1) \tag{2.83}$$

$$\Rightarrow A_m = Q_m(Q_1^{-1} - 1) \tag{2.84}$$

Also, we have that

$$\begin{aligned} A_n Q_m &= Q_n A_m && (2.85)\\ \Rightarrow Q_n(Q_1^{-1} - 1)Q_m &= Q_n Q_m (Q_1^{-1} - 1) && (2.86)\\ \Rightarrow Q_1^{-1} Q_m &= Q_m Q_1^{-1} && (2.87)\\ \Rightarrow [Q_m, Q_1] &= 0 && (2.88) \end{aligned}$$

Similarly, we can show that $[Q_2, Q_n] = 0$ $\forall n \in \mathbb{N}$ if we take the $\{\Lambda\}$ type of transition in place of $\{\,\vdots\,\cdot\,\}$. We will see that the timid transition to two antichain will be a polynomial in Q_1, Q_2, Q_1^{-1}.

Now we use strong induction. We assume that till $n-1$ $[Q_k, Q_m] = 0$ $\forall k, \ell < n$ and $\forall m \in \mathbb{N}$ and that timid transition to $k < n$ antichains are polynomials in Q_k, Q_k^{-1}. We can then see that by Bell causality, the transition A_k to any k antichain ($k < n$) can be related to a timid transition A_ℓ to ℓ antichain.

$$Q_k A_\ell = A_k Q_\ell$$

$$\Rightarrow A_k = Q_k A_\ell Q\ell^{-1} \tag{2.89}$$

Here A_ℓ is a polynomial in Q_k, Q_k^{-1} by the hypothesis and thus all transitions upto stage $n-1$ commute. At stage n, all non timid transitions can related to timid transitions to smaller antichains as above(just take $k = n$ in equation 2.89). The timid transition at stage n is

$$\mathbb{I} - (\text{ sum of all non timid transitions}) \tag{2.90}$$

Thus, it is a polynomial in Q_k, Q_k^{-1} for $k \leq Q_n$ and this tells us that all the transitions till stage n can be written as polynomials in Q_k, Q_k^{-1}. But by our hypothesis, $[Q_k, Q_m] = 0$ $\forall k \leq n-1$. Thus, we find that all the transition amplitudes commute till stage n. We can thus write the time ordered Bell causality as type II Bell causality on the antichain till stage n.

$$\begin{aligned} A_n B_m &= B_n A_m & (2.91) \\ \Rightarrow A_n B_n^{-1} &= A_m B_m^{-1} & (2.92) \end{aligned}$$

We can thus use lemma 2.2.3 to write the transition amplitudes. We call

$$\Xi_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} Q_{n-k}^{-1} \tag{2.93}$$

And thus the transition can be written as $A_n = Q_n \Xi_n = \Xi_n Q_n$. Putting this in equation 2.89, we can write $A_m Q_n = Q_m \Xi_n Q_n$ or $A_m = Q_m \Xi_n$ $\forall m > n$ $m \in \mathbb{N}$. Lets take $m > \ell > n$ for the similar transition. We can thus write

$$\begin{aligned} Q_m \Xi_n Q_\ell &= Q_m Q_\ell \Xi_n \\ Q_m \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} Q_{n-k}^{-1} Q_\ell + Q_m Q_n^{-1} Q_\ell &= Q_m Q_\ell \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} Q_{n-k}^{-1} + Q_m Q_\ell Q_n^{-1} \\ \Rightarrow Q_\ell Q_n &= Q_n Q_\ell & (2.94) \end{aligned}$$

For all $\ell \in \mathbb{N}$. Thus, we showed that $[Q_n, Q_m] = 0$. ■

Now, it just remains to show that the ring of operators is commutative. For that we have the following theorem.

Theorem 2.2.10. *The ring of operators $\mathcal{T}$ is commutative*

Proof. We will show that any transition can be written as a polynomial in Q_n, Q_n^{-1} and this will prove commutativity of the ring by previous theorem. We have already shown that gregarious transitions at stage n are equal to Q_n. We now look at the other transitions. At stage 1, non gregarious transition is $\mathbb{I} - Q_1$. Assume that the transitions are a polynomial in

Q_k, Q_k^{-1} till stage $n-1$. Then, at stage n, any non timid transition can be related to some transition in stage $n-1$ by Bell causality. We know that we have

$$A_n = Q_n A_{n-1} Q_{n-1}^{-1} \tag{2.95}$$

We thus know that this is a polynomial in Q_k, Q_k^{-1} and thus, all non timid transitions at stage n commute and are a polynomial in Q_k, Q_k^{-1}. Now, we go to the timid transitions which are just $\mathbb{I}-$non timid transitions. Thus, they too are polynomial in Q_k, Q_k^{-1} and thus commute with every other transition at stage n. As the transition till stage $n-1$ are polynomials in Q_k, Q_k^{-1} as well, all the transition amplitudes till stage n commute and are polynomials in Q_k, Q_k^{-1}. Thus, by induction, we showed that the ring of operators is commutative.

■

With this we can give a form to the transition amplitudes in the following corollary.

Corollary 2.2.10.1. *The transition from a set c_n^i with n elements and m elements as maximal element of the transition, the transition amplitude can be given as*

$$A_n(c_n^i) = \left[\sum_{k=0}^{m} (-1)^k \binom{m}{k} Q_{n-k}^{-1}\right] Q_n \tag{2.96}$$

2.3 Concluding Remarks

In this chapter, we generalized the classical measure to get a quantum theory. This generalized measure called quantum measure can be constructed from vector measures. We then generalized the properties of transition amplitudes. While generalizing Bell causality, we tried two possible generalizations, and in both the cases, we found that the ring generated by transition amplitudes is commutative. In both these cases, the property that allowed us to prove the commutativity was that all the transitions to antichains are commutative. Another fact was that in any version, Bell causality discards the gregarious transitions and thus, two different transitions to antichain can be related easily. This points towards a possibility that in any possible generalization of Bell causality, the ring $\mathcal{T}$ will come out to be commutative. But we haven't proven this result yet.

Chapter 3

Constructing measures of observables

We first need to look again at the σ-algebra of causets in $\tilde{\Omega}$. It was discussed in chapter 1 and here, we will discuss that algebra in further details. We note that the class of most natural physical questions can be represented by the σ-algebra $\mathcal{R}(\mathcal{S})$ where $\mathcal{S}$ is the set of all possible stem sets and $\mathcal{R}(\mathcal{S})$ is the σ-algebra generated by $\mathcal{S}$. These are not exhaustive set of questions. An example of question not in $\mathcal{R}(\mathcal{S})$ is the set containing a maximal element.

$$M = \{c \in \Omega |\ \text{c contains a maximal element}\} \tag{3.1}$$

This set cannot be represented by stem sets. The reason is that for any two stem sets a and b, any $c \in M$ is such that either $c \in a$ and $c \in b$ or $c \notin a \bigcup b$. From here, we can see that not all physical questions can be asked in terms of stem sets alone. We see that the following types of sets will cause problems

$$\Theta = \{c \in \Omega | c \in a \text{ and } c \in b \text{ or } c \notin a \bigcup b\} \tag{3.2}$$

It is obvious from above example that $\mathcal{R}(\mathcal{S})$ does not separate points in Ω and thus, Θ is non empty. As it was shown in [4] that

Theorem 3.0.1. *$\mu(\Theta) = 0$ in $\mathbb{R}^{+}SGD$*

Theorem 3.0.2. *For every set $A \in \mathcal{R}$, there is a set $B \in \mathcal{R}(\mathcal{S})$ such that $A \Delta B \subset \Theta$ where $A \Delta B = (A - B) \bigcup (B - A)$.*

The above theorems tell us that we can actually work with $\mathcal{R}(\mathcal{S})$ in case of $\mathbb{R}^{+}SGD$. For $\mathbb{C}SGD$, this may not be true. Theorem 3.0.2 is not measure dependent and thus holds in general.

3.1 Some Observable

Here, we will define some of the observables in classical and complex SGD. The main goal is to construct a measure of these observables. The following observables have a common property, i.e. they can be written as : "Given the observable is preserved till stage $n-1$, what is the transition amplitude that it is also preserved at stage n?". It can be written as following conditions

$$\mathcal{A}_n = \{ \text{ Observable is preserved till stage n} \} \tag{3.3}$$

From here, we directly see that $A_n \subset A_m$ if $n < m$. Thus, our observable will be

$$\mathcal{A} = \bigcap_{n=1}^{\infty} \mathcal{A}_n \tag{3.4}$$

We know that for positive measure, if $\mu(\mathcal{A}_1)$ is finite, then $\mu(\cap_{n=1}^{m} \mathcal{A}_n) \to \mu(\mathcal{A})$.[14] This is what we are going to do in the subsequent sections to calculate the measure of the observables in classical case. For complex or vector valued measure, this property will still hold given the measure extends to full σ-algebra. If not, we will have to see for individual observables if the measure can be extended for those or not.

3.1.1 Originary event

An orginary event means that a universe started from a point and all the subsequent elements are in the future of that point. It thus says that if we trace back to the past, we will get a unique starting point.

Definition *Originary event is an observable which contains an element e_0 such that for $\forall e$ in the event, $e_0 \prec e$.*

The measure of an originary event is given by

$$\prod_{n=1}^{\infty} (1 - q_n) \tag{3.5}$$

The proof of this is given after we find the measure of our next observable, break. The measure for complex transitive percolation[1] is given by the Euler function[8]

$$\phi(n) = \prod_{n=1}^{\infty} (1 - q^n) \tag{3.6}$$

---

[1] $q_n = q^n$ where $q \in \mathbb{C}$

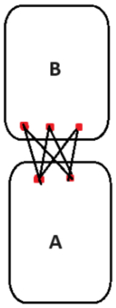


Figure 3.1: Break

3.1.2 Break

A break is an observable which represents the event of collapse and reformation of universe. The main idea is that universe collapses, not necessary to a point, and the new universe comes to the future of the existing universe. It this means that any element in the new universe is in the future of every element of the past universe. We will look at the definition of break occurring at stage n_1 in a labeled causet. We will later define it for

Definition *A **Break** at stage n_1 is an observable represented by $(A, B) \subset \tilde{\Omega}$, where $A \subset \tilde{\Omega}(n_1)$ and B are causets and $\forall y \in B$ and $\forall x \in A$, $x \prec y$.*

Definition *An unlabeled break is denoted by $(A, B) \in \Omega$ such that A is an unlabeled finite causet and any element in B lies in the future of A.*

The measure of a break can be computed first for labeled case and we can then generalize to the unlabeled case. The measure of an unlabeled break (A, B) with A containing n_1 elements is given by

$$\sum_{\mathcal{L}(A)} P(A) \times \prod_{n>n_1}^{\infty} \left[1 - (1 - \alpha^t_{n_1}) \frac{q_n}{q_{n_1}} \right], \tag{3.7}$$

where $\mathcal{L}(A)$ is the set of all the possible relabeling of an unlabeled set A with n elements. The sum of transition amplitudes that preserve the break is preserved at rank n, given it is preserved till rank $n-1$ is given as

$$\mathcal{R}_n(A) = 1 - \frac{q_{n+n_1}}{q_{n_1}} (1 - \alpha^t_{n_1}) \tag{3.8}$$

Now, we will prove the above results for breaks and use that result to prove equation 3.5 as well.

Proof of Equation 3.7

To begin, we first define clearly what we are willing to do. First, let's say that we have a break (A, B) in a causal set $c^i_{n+n_1}$. Suppose number of elements in A is denoted by $N(A) = n_1$. Let's denote the maximal element of causal set of n elements as $M(A_1)$. B can have finite or infinite elements depending whether $(A, B) \in \tilde{\Omega}_n$ or $(A, B) \in \tilde{\Omega}$. We will first show that labels of A can only be smaller than or equal to n_1 in any possible natural relabeling. We will then find the measure of a break in the labeled case. We will finally show that for unlabeled case, B is already label invariant and thus, we just need to add up over the measures of all the possible relabeling of A and then operate by the transition operator in $\mathcal{T}(\tilde{\Omega}(n_1) \to \tilde{\Omega})$.

Lemma 3.1.1. *In a break $(A, B) \subset \tilde{\Omega}_{n+n_1}$, any relabeling*[2] *will have $ej \in A, \forall j \leq n_1$.*

Proof. Assume that $N(A) = n_1, N(B) = n$. We have to show that for any element $e_{L(i)} \in A$,we must have $L(i) \leq n_1$. To prove this, let's assume otherwise. Say that for some i, $L(i) > n_1$ in a break $(A, B) \subset \tilde{\Omega}_{n+n_1}$. But $e_{L(i)} \prec e_{L(j)} \forall e_{L(j)} \in B$ and as $N(B) = n$, There are n elements greater than $L(i)$. Thus, total elements in the set (A,B) will be atleast $L(i)+n > n_1+n$. This is a contradiction and thus $L(i) \leq n_1$. Thus, first n_1 elements belong to A regardless of labeling. ■

We have to define a few things before we do the next proof. Before that, we talk about a notation. $c^i_{n+n_1}$ denotes a set with a break at level n_1, and $i \in \mathcal{J}(n + n_1)$.

Definition *Given a break (A_1,B) and $N(A_1) = n_1$, where , the children of $c^i_{n+n_1}$ which are not allowed for the set to have a break at n_1 can be formally written as*

$$I^i_n(A_1) = \{j(i) : c^{j(i)}_{n+n_1+1} \text{ is child of } c^i_{n+n_1} \exists m \in M(A_1) \text{ such that } m \not\prec e_{n+n_1+1}\} \tag{3.9}$$

Definition *A transition amplitude in $\mathcal{T}(\tilde{\Omega}(n + n_1) \to \tilde{\Omega}(n_1 + n + 1))$ from $c^i_{n+n_1}$ is said to preserve break at stage n_1, if $c^{j(i)}_{n+1}$ has a break at stage n_1 given c^i_n had a break at n_1.*

For simplicity, we will write j(i) as j for now considering just one of the possible sets that have a break at n_1 i.e we are fixing i for now. We will find the measure in the case of labeled causet and then find the measure for unlabeled case.

Theorem 3.1.2. *Suppose that we have a break (A_1, B), with $N(A_1) = n_1, N(B) = n$ with labeled causet A_1, B. The sum of transition amplitudes corresponding to the transitions*

---

[2] Remember relabeling is a bijection $L : \tilde{\Omega} \to \tilde{\Omega}$ such that natural labeling is preserved i.e. $e_i \prec e_j \Leftrightarrow e_{L(i)} \prec e_{L(j)}$.

$c^i_{n+n_1} \to c^{j(i)}_{n_1+n+1}$ *preserves the break is*

$$\mathcal{R}_n(A_1) = 1 - \frac{q_{n_1+n}}{q_{n_1}}\left(1 - \alpha_{n_1}^{t_{A_1}}\right) \tag{3.10}$$

Proof. We need to remove all such transitions for which $\exists m \in M(A_1)$ such that $m \not\prec e_{n+n_1+1}$. Thus, we need to remove all $c^{j(i)}_{n+n_1+1}$ for $j \in I^i_n(A_1)$ for all possible i. Let's first fix an i i.e. we are looking at a particular causet $c^i_{n+n_1}$. Thus for simplicity, we write $j(i) \equiv j$. We write, using Markov property

$$\text{sum of possible transitions } = 1 - \sum_{j \in I^i_n(A_1)} \alpha^i_{n+n_1} \tag{3.11}$$

By Bell causality, we can look at the transitions of a different, smaller causet and compare the ratios. We know that the transitions that we are removing have precursor sets as the subset of A_1, so any element of B is always a spectator for such transitions. We will compare the gregarious transitions with these transitions and we can do that at stage n_1, where there is no element of B i.e. spectators are removed.

$$\frac{\alpha^j_{n+n_1}}{q_{n+n_1}} = \frac{\alpha^{f(j)}_{n_1}}{q_{n_1}} \tag{3.12}$$

We need to show that $f : I^i_n(A_1) \to I_0(A_1)$ is one-one.[3] Let's assume that this is not the case i.e. f is multivalued.[4]. Then, e_{n+n_1+1} is now the e_{n_1+1} with the same precursor set. Thus, in all transitions $\alpha^{f(i)}_{n_1}$, e_{n_1+1} will have the same 'labeled' precursor set and thus the same spectator set. But then, these transitions are actually the same and thus $f(j)$ is one-one. Thus, we have

$$\sum_{j \in I^i_n(A_1)} \frac{\alpha^i_{n+n_1}}{q_{n+n_1}} = \sum_{j \in I^i_n(A_1)} \frac{\alpha^{f(j)}_{n_1}}{q_{n_1}} \tag{3.13}$$

The above sum includes all transitions except the timid transition, as that will have all the maximal elements in it's past. Thus, for transition at stage n_1, we get that

$$\sum_{j \in I^i_n(A_1)} \alpha^{f(j)}_{n_1} = 1 - \alpha^{t_{A_1}}_{n_1} \tag{3.14}$$

This means that

$$\begin{aligned} \sum_{j \in I^i_n(A_1)} \alpha^i_{n+n_1} &= \frac{q_{n_1+n}}{q_{n_1}}\left(1 - \alpha^{t_{A_1}}_{n_1}\right) && (3.15)\\ \Rightarrow \text{sum of possible transitions} &= 1 - \frac{q_{n_1+n}}{q_{n_1}}\left(1 - \alpha^{t_{A_1}}_{n_1}\right) && (3.16)\end{aligned}$$

---

[3] Note that $I_0(A_1)$ has no labels as there is only one such set, as A_1 is already given.

[4] In that sense, f is not a function but a relation.

The result is not dependent of $c^i_{n_1+n}$ and thus, we get that this is true for all $i \in \mathcal{J}(n+n_1)$

$\blacksquare$

Now, suppose that we are given the measure of A_1 as $P(A_1)$. We now have to calculate the measure of a break at n_1 where $N(A_1) = n_1$.

Theorem 3.1.3. *The measure of a labeled break* (A_1, B) *is given as*

$$P(A_1) \times \prod_{n>n_1}^{\infty} \left[1 - (1-\alpha^t_{n_1})\frac{q_{n+n_1}}{q_{n_1}}\right] \tag{3.17}$$

where $\mathcal{L}(A_1)$ *is the set of all relabeling of* A_1.

Proof. As we have already seen that the transition amplitude for the break to be preserved for transitions at stage $n+n_1$ is $\mathcal{R}_n(A_1)$. We will prove this by principle of mathematical induction. First, at stage n_1+1, we know that sum of transitions that preserve break is $\times \left(1 - \frac{q_{n_1+n}}{q_{n_1}}\left(1-\alpha^{t_{A_1}}_{n_1}\right)\right)$. Now, assume this to be true for stage $n+n_1-1$ i.e. The transitions to A_1 that preserve the break till stage n

$$\prod_{i>n_1}^{n+n_1-1} \left[1 - (1-\alpha^t_{n_1})\frac{q_i}{q_{n_1}}\right] \tag{3.18}$$

Then at stage $n+n_1$, we have transitions from sets $c^i_{n+n_1}$ such that break is preserved. Thus, total probability that a break is preserved at stage $n+n_1$ is

$$\begin{aligned} P(c^i_{n+n_1}) &\times \left[1 - (1-\alpha^t_{n_1})\frac{q_{n+n_1}}{q_{n_1}}\right] \\ \Rightarrow P(A_1) \times \prod_{i>n_1}^{n+n_1-1} \left[1 - (1-\alpha^t_{n_1})\frac{q_i}{q_{n_1}}\right] &\times \left[1 - (1-\alpha^t_{n_1})\frac{q_{n+n_1}}{q_{n_1}}\right] \\ \Rightarrow P(A_1) &\times \prod_{i>n_1}^{n+n_1} \left[1 - (1-\alpha^t_{n_1})\frac{q_i}{q_{n_1}}\right] \end{aligned} \tag{3.19}$$

Thus, by induction, we know that at any stage $n+n_1$, we have

$$P(A_1) \times \prod_{n>n_1}^{n+n_1} \left[1 - (1-\alpha^t_{n_1})\frac{q_n}{q_{n_1}}\right] \tag{3.20}$$

As we discussed earlier, for a positive measure, the measure of a Break can be found as limit of the above measure which preserves break till stage $n+n_1$. We thus get

$$\mu(\text{Break at } n_1) = P(A_1) \times \prod_{n>n_1}^{\infty} \left[1 - (1-\alpha^t_{n_1})\frac{q_n}{q_{n_1}}\right] \tag{3.21}$$

$\blacksquare$

Theorem 3.1.4. *The measure of an unlabeled break (A,B) is*

$$\sum_{\mathcal{L}(A)} P(A) \times \prod_{n>n_1}^{\infty} \left[1 - (1 - \alpha^t_{n_1}) \frac{q_i}{q_{n_1}}\right] \tag{3.22}$$

Proof. The main idea here is to find the measure $\mu = \tilde{\mu} \circ p^{-1}$ where $p : \tilde{\Omega} \to \Omega$ is the quotient map. We will assume that our labeled break is $(\tilde{A}, \tilde{B})$ and that $p(\tilde{A}, \tilde{B}) = (A, B)$. By lemma 3.1.1, we can write $\tilde{A}$ and $\tilde{B}$ as separate causets with the timid child of $\tilde{A}$ to be considered as gregarious child in $\tilde{B}$[5] If we assume that $c_n \in \tilde{B}$ and there is a relabeling $\tilde{c}_n$ of c_n. By the definition of break, $\tilde{c}_n \in \tilde{B}$. This tells us that $\tilde{B}$ is already label invariant or that $p^{-1}(\tilde{B}) = \tilde{B}$ and $p(\tilde{B}) = \tilde{B}$. We also note that for some unlabeled causet A with n elements, $p^{-1}(A) = \tilde{A}$ are the set of all possible natural relabeling of A.This set is denoted by $\mathcal{L}(\tilde{A})$, which is a finite set as $\tilde{A}$ is finite, we get that $\mathcal{L}(\tilde{A}), \tilde{B} = p^{-1}((A, \tilde{B}))$. We also see that the relabelings of A lie in distinct cylinder sets and that we can operate over their measure by an operator of the type $\mathcal{T}(\tilde{\Omega}(n_1) \to \tilde{\Omega})$. We thus have to sum over all relabeling of A and multiply by $\prod_{n>n_1}^{n+n_1} \left[1 - (1 - \alpha^t_{n_1}) \frac{q_n}{q_{n_1}}\right]$. We thus get,

$$\sum_{\mathcal{L}(\tilde{A})} P(A) \times \prod_{n>n_1}^{n+n_1} \left[1 - (1 - \alpha^t_{n_1}) \frac{q_n}{q_{n_1}}\right] \tag{3.23}$$

■

Corollary 3.1.4.1. *For the case of an* ***originary event****, the measure is*

$$\sum_{\mathcal{L}(A)} P(A) \times \prod_{n=1}^{\infty} [1 - q_n] \tag{3.24}$$

Proof. We know that originary event is a break with $n(A) = 1$. Rest follows directly. ■

3.1.3 Post

A post is a special case of break where A has only one maximal element. It is analogous to the case where the universe collapses to a point and re-emerges from that point i.e. that point lies in the past of all elements of B. In other words, we can say that a post is a break where A has only one maximal element.

---

[5] Note that we should be careful that we are not writing the measure of $\tilde{A}, \tilde{B}$ separately. What we are doing is to just look at the label dependence of causets and this separation provides an easier way to do so. The main idea is to see these in terms of quotient map of all relabeling so that we can analyze which set is label invariant and which is not.

Definition *A* ***post*** *is a break (A,B) where A has only one maximal element.*

The measure of a post can thus be given by the equation 3.7. We can actually simplify that equation putting $m = 1$ to get

$$\sum_{\mathcal{L}(A)} P(A) \times \prod_{n>n_1}^{\infty} \left[1 - \frac{q_n}{q_{n_1-1}}\right] \tag{3.25}$$

where $\mathcal{L}(A)$ are all the possible relabeling of an unlabeled set A with n elements. The probability that the post is preserved at rank n, given it is preserved till rank $n-1$ is given as

$$\mathcal{R}_n(A) = 1 - \frac{q_{n+n_1}}{q_{n_1-1}} \tag{3.26}$$

All these results are actually the subcases of theorems 3.1.2 and 3.1.4. With this, we have constructed the measures of observables like originary event, posts and breaks in classical SGD.

Chapter 4

Renormalization in Causets

In this chapter, we wish to answer some questions related to the dynamics of universe in which several breaks or posts occur. We can ask that if we have a break (A,B), how will the dynamics in B be affected by the causal structure of A? What if we have more than one break? If the number of elements in A become very large or if there are a lot of breaks before set B(which represents the epoch we want to study), how are the dynamics affected? Our main goal here is to see where these dynamics tend to in the presence of a large number of breaks i.e. what are the effective dynamics.

4.1 Modeling this new dynamics

We said that we want to look at the form of dynamics after a break(or post) occurs i.e. we want to see the dynamics of a universe after the universe has collapsed and reformed. For that, we assume that a break has occurred at stage n_1. To talk about the dynamics, we have to know what is the probability that a break is preserved at stage $n + n_1$ given that it is preserved till stage $n + n_1 - 1$. We have already proven this result as in theorem 3.1.2. For classical SGD, we need that the probability sums up to 1. We can put a similar restriction for the complex SGD, that the sum of all possible transitions should be 1. We do that so that the quantum measure of full poscau is 1. With that, we can write the renormalized probabilities at stage n after the break (A,B) where A has n_1 elements and m maximal elements

$$\alpha_n^i(A) = \frac{\alpha_{n+n_1}^i}{\mathcal{R}_n(A)} \tag{4.1}$$

Here, $\mathcal{R}_n(A) = 1 - \frac{q_{n_1+n}}{q_{n_1}}\left(1 - \alpha_{n_1}^{t_A}\right)$. Here, $\alpha_{n_1}^{t_A}$ is the timid child to A.

$$\alpha_{n_1}^{t_A} = q_{n_1} \sum_{k=m}^{n_1} \binom{n-m}{k-m} t_k \tag{4.2}$$

We can make some simplifications to this if we write the t_n in a way that it acts as new parameters for dynamics that starts after n_1. For posts, this is actually much more natural to do as for them, $\mathcal{R}_n(A) = 1 - \frac{q_{n_1+n}}{q_{n_1-1}}$. The numerator for them can be written in a way that it looks like the dynamics of a universe that started after n_1. The nth transition after a post is given by

$$\begin{aligned}
\alpha_n^i(A) &= \frac{\alpha_{n+n_1}^i}{1 - \frac{q_{n_1+n}}{q_{n_1-1}}} \qquad (4.3)\\
&= \frac{\sum_{k=m}^{n_1+\varpi} \binom{\varpi + n_1 - m}{k-m} t_k}{\frac{1}{q_{n+n_1}} + \frac{1}{q_{n_1-1}}} \qquad (4.4)
\end{aligned}$$

where $\varpi + n_1$ represents the number of elements in the precursor set and m denotes the number of maximal elements. We can use the formula

$$\sum_{i+j=k} \binom{m}{i}\binom{n}{i} = \binom{n+m}{k} \tag{4.5}$$

Using this property and the fact that $1/q_n = \sum_{k=0}^{n} \binom{n}{k}$, we can write the above equation as

$$\begin{aligned}
\alpha_n^i(A) &= \frac{\sum_{k=m}^{n_1+\varpi} \binom{\varpi + n_1 - m}{k-m} t_k}{\sum_{k=0}^{n+n_1} \binom{n+n_1}{k} t_k - \sum_{k=0}^{n_1-1} \binom{n_1-1}{k} t_k} \\
&= \frac{\sum_{k=m}^{n+n_1} \sum_{\ell+j=k} \binom{n_1-1}{j} \binom{\varpi+1-m}{\ell-m} t_k}{\sum_{k=0}^{n+n_1} \binom{n+n_1}{k} t_k - \sum_{k=0}^{n_1-1} \binom{n_1-1}{k} t_k} \\
&= \frac{\sum_{\ell} \sum_{j} \binom{n_1-1}{j} \binom{\varpi+1-m}{\ell-m} t_{\ell+j}}{\sum_{k=0}^{n+n_1} \binom{n+n_1}{k} t_k - \sum_{k=0}^{n_1-1} \binom{n_1-1}{k} t_k} \qquad (4.6)
\end{aligned}$$

(4.7)

Here, we will call the quantity $\sum_j \binom{n_1-1}{j} t_{\ell+j} = \tilde{t}_\ell^{n_1-1}$. With this, we can write the dynamics as

$$\alpha_n^i(A) = \frac{\sum_{\ell=m}^{\varpi+1-m} \binom{\varpi+1-m}{\ell-m} \tilde{t}_\ell^{n_1-1}}{\sum_{\ell=0}^{n+1} \binom{n+1}{\ell} \tilde{t}_\ell^{n_1-1} - \tilde{t}_\ell^{n_1-1}} \tag{4.8}$$

From above, we see why we split things in terms of $t_n^{(n_1-1)}$. These dressed parameters act as the new parameters of the dynamics. We can thus write the dynamics as above in terms of these new parameters. This allows us to see the dynamics after a post as a new universe starting from a point with parameters $\tilde{t}_n^{(n_1-1)}$.

With that we can now talk about the limit points of these renormalizations and also talk about the cases where the dynamics tend to the limit points.

4.2 What do these dynamics tend to ?

We have written the given dynamics in terms of new parameters[11][9]

$$\tilde{t_n}^{(p)} = \sum_k \binom{p}{k} t_{k+n} \tag{4.9}$$

We can find out the limit to which $\tilde{t}_n^{(p)}$ tend to as $p \to \infty$. What we want to achieve from this is to get a general idea of what the dynamics of universe will look like if there are several posts/ breaks in the universe. We first define the notion of fixed points of these dynamics which means that the transition amplitudes have the same probability for similar type of transitions.

Definition *Fixed point under renormalization after posts or breaks with m_1 maximal elements are the dynamics $(t_1, t_2, \cdots, t_n, \cdots)$ such that for (A_1, B) and (A_1, A_2, B), if the transitions have same precursor set in B, they have the same transition amplitude.*

It was shown in [11] that the **fixed points** are $t_n = t^n$ and these are the only cycles as well.

4.2.1 Flows of dynamics

In Classical sequential growth dynamics, the dynamics tend to the fixed point in the following special cases,

1. If $t_n^{1/n} \to t$ as $n \to \infty$, where $t > 0$, we know that $\frac{\tilde{t}_{n+1}^{(p)}}{\tilde{t}_n^{(p)}} \to t$ as $p \to \infty$.
2. If $t_n^{1/n} \to \infty$, we don't reach any limit.
3. If $t_n^{1/n} \to 0$ as $n \to \infty$, we get that $\frac{\tilde{t}_{n+1}^{(p)}}{\tilde{t}_n^{(p)}} \to 0$. In this case, the terms $\tilde{t}_n^{(p)}$ might become zero after $n > n_L$, where t_{n_L} is the last T_n that is non zero. Using these, we can find the effective dynamics.

Limiting dynamics for Posts

Let's assume that we have a post (A,B), with n_1 elements in A.

$$\alpha_n^i(A) = \frac{\sum_{k=m}^{\varpi} \binom{\varpi - m}{k - m} \tilde{t}_k^{(n_1-1)}}{\sum_{k=1}^{n} \binom{n}{k} \tilde{t}_k^{(n_1-1)}} \tag{4.10}$$

Divide numerator and denominator by $\tilde{t}_0^{(n_1-1)}$,(if $t > 0$) we get

$$\alpha_n^i(A) \to \frac{\sum_{k=m}^{\varpi} \binom{\varpi - m}{k - m} t^k}{\sum_{k=1}^{n} \binom{n}{k} t^k} \tag{4.11}$$

4.2.2 The problem with break

We know that for a general break, we can have some arbitrary number of maximal elements. We do not have this problem for posts as there, we can have only one maximal element. Due to this, it does not seem possible to find a limiting dynamics in case of multiple breaks as the number of maximal elements may be arbitrary. But we can ask for limiting dynamics in a special case where there are only m_1 maximal elements i.e. number of maximal elements in the set A is fixed. We can see that this is similar to the case of posts, where there was only 1 maximal element. In this case too, we will get similar answer to posts with

1 replaced by m_1. In such a scenario, as we will see, the renormalized parameters are $\tilde{t}^{(n_1-m_1)}$.

First we find the fixed points of dynamics with breaks (A,B) having m_1 maximal elements in A. We assume that even if multiple break occurs $(A_1, A_2, \cdots, A_n, B)$, the set A_n has m_1 maximal elements. So, for a break (A,B), a transition that preserves break at stage $n + n_1$ for a break at n_1 and has m maximal elements and in its precursor set which has $\varpi + n_1$ elements is

$$\alpha_n^i(A) = \frac{\sum_{k=m}^{n_1+\varpi} \binom{\varpi + n_1 - m}{k - m} t_k}{\sum_{k=0}^{n+n_1} \binom{n + n_1}{k} t_k - \sum_{k=0}^{n_1} \binom{n_1}{k} t_k + \sum_{k=m_1}^{n_1} \binom{n_1 - m}{k - m_1} t_k} \tag{4.12}$$

Here, we can see that the natural way to write this in terms of dressed coordinates is to use the coordinates $\tilde{t}_n^{n_1-m_1}$. Note that with these coordinates, it may not seem exactly like a dynamics starting after the collapse of universe. But this is the best way we can dress the parameters and look for dynamics as $n_1 \to \infty$. This is because we want to find what does the dynamics look like after several collapses and reformations of universe. We will write the $\alpha_{n_1}^{t(A)}$ in a more useful way so that t_k can be dresses for this term too and this will also explain why we dress it as $\tilde{t}_n^{n_1-m_1}$

$$\begin{aligned} \alpha_{n_1}^{t(A)} &= q_{n_1} \sum_{k=m_1}^{n_1} \binom{n_1 - m_1}{k - m_1} t_k \\ &= q_{n_1} \sum_{k=0}^{n_1-m_1} \binom{n_1 - m_1}{k} t_{k+m_1} \\ \alpha_{n_1}^{t(A)} &= q_{n_1} \tilde{t}_{m_1}^{(n_1-m_1)} \end{aligned} \tag{4.13}$$

With this, we can write $\alpha_n^i(A)$ as

$$\alpha_n^i(A) = \frac{\sum_{k=m}^{m_1+\varpi} \binom{\varpi + m_1 - m}{k - m} \tilde{t}_k^{(n_1-m_1)}}{\sum_{k=0}^{n+m_1} \binom{n + m_1}{k} \tilde{t}_k^{(n_1-m_1)} - \sum_{k=0}^{m_1} \binom{m_1}{k} \tilde{t}_k^{(n_1-m_1)} + \tilde{t}_{m_1}^{(n_1-m_1)}} \tag{4.14}$$

Thus, we can talk about the limiting dynamics in this case as we take $n_1 \to \infty$ in the same way we did for posts.

Conclusion

In this thesis, we first reviewed the idea of causal set theory, a discrete framework of quantum gravity. We then studied the fundamental dynamics of this theory, known as Sequential Growth Dynamics(SGD) and found new results in the case of the Quantum SGD. We studied the dynamics under two possible generalizations of the local causality condition called the Bell causality and found that in either case, if the transition amplitudes are invertible, the dynamics is commutative. This is an interesting result as it states that Bell causality is actually a very strong condition. We note that this was possible because there are transitions that are unordered i.e. they can occur in any order. Such transitions eventually provided us with an extra condition that allowed us to prove commutativity and as such transitions will always be there in SGD, it seems likely that any possible generalization of Bell causality will eventually lead to commutative algebra if the transition amplitudes are invertible.

In chapter 3, we studied some observable in classical SGD. We constructed the measure of originary event(a universe evolving from one point), post (a universe collapsing to a point and re-forming) and break(a universe collapsing and a new universe forming to the future of it). But the measures are harder to construct in the quantum SGD as the measures may not even extend beyond the Boolean algebra generated by cylinder sets for these observables.

In chapter 4, we studied about the renormalization in casual sets. The main question that we wanted to answer was: "What do the dynamics tend to as the number of posts or breaks tend to infinity?" We found out that it may not be possible to answer this question in case of arbitrary breaks, but we can answer this if we assume that the number of maximal elements in the set A in a break (A,B) is a fixed number m. The post is a special case of this, where $m = 1$.

Chapter I

Appendix : Measure Theory

What is measure theory? In essence, it is a generalized theory of Riemann intergrals. We know that when it comes to very rough functions, Riemann integral may not exist. To overcome this problem, we look at the way Riemann integrals were defined. Suppose we had to integrate a function on the interval (a,b) , we take partitions of this interval and calculate upper and lower sums. We then make the step functions finer and finer and see if lower and upper sums converge to the same number. If yes, then integral exist. To deal with rough functions(they can be functions that are nowhere continuous either), we generalize this approach. We note that step functions were in some sense approximating the functions that are integrated. To approximate sufficiently rough functions, it is better to deal with a generalization of step functions called the simple functions. We take characteristic functions on arbitrary sets and construct simple function from these. A characteristic function χ_A is defined as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{I.1}$$

we thus take such disjoint sets A_n such that $\cup A_n = [a, b]$. On each of these, we take our function approximating $f(x)$ on A_n to be $c_n = \inf_A\{f(x)\}$. The simple function is then written as

$$s = \sum c_n \chi_{A_n} \tag{I.2}$$

We can always find a sequence of simple functions $s_n \to f$[14]. Now, we have to assign some length to these sets A_n. In case of step functions, it was a simple thing to do, as the length of an interval (c, d) is $d - c$. It is not so simple for arbitrary sets. We can assign lengths to arbitrary sets A_n by covering them with intervals, and making the covering finer and finer and calculate the sum of lengths of those intervals. We can then take the infima of these lengths to find the length of the set A_n. This length is called the outer measure. But if we provide length to all arbitrary sets using outer measure, we face inconsistency which tells us that we have to restrict the class of sets that we can assign lengths to [1]. We thus restrict the class of sets on which lengths/measures are assigned. We call this a σ-algebra or a measure space. What should these sets be? We know that we should be able to integrate on open sets/ intervals atleast. So, we want to assign length to those. The total set should have a length.What else do we need if we want to generalize this even further and integrate on some arbitrary set X. Suppose $f : X \to \mathbb{R}$ and we want to integrate this. It is easier to form simple functions if $f^{-1}((a, b))$ is a set that can be assigned length, for then we can simply make the intervals smaller and simple functions over these sets will tend to $f(x)$. If we also say that $f^{-1}(\mathbb{R}/(a, b))$ is also measurable, then, we know that we can talk about integrals on separate regions and can even actually talk about splitting the integral over disjoint regions. All these motivate us to define,

Definition *A **Boolean algebra** $\mathcal{B}$ on X is a collection of subsets of X such that*

1. *If $\{A_i \in \mathcal{B}\}$ be a collection of $n \in \mathbb{N}$ sets in the Boolean algebra. We have $\cup_{n=1}^{n} A_n \in \mathcal{B}$. Thus it is closed under finite union*
2. *If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$, where A^c refers to compliment of A i.e. X/A.*
3. *$X \in \mathcal{B}$ and $\emptyset \in \mathcal{B}$*

Definition *A set $\mathcal{M}$ is called σ-algebra over some set X is some collection of subsets of X such that*

1. *If $A_n \in \mathcal{M}$ $\forall n \in \mathbb{N}$, we have $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$. Thus it is closed under countable union*
2. *If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$, where A^c refers to compliment of A i.e. X/A.*
3. *$X \in \mathcal{M}$ and $\emptyset \in \mathcal{M}$*

Definition *A positive **measure** is a function $\mu : \mathcal{M} \to \mathbb{R}^+ \cup \{0\}$ that is countably additive i.e. if $A_i \cap A_j = \emptyset$ $\forall i \neq j \in \mathbb{N}$,*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n}^{\infty} \mu(A_n) \tag{I.3}$$

Definition *A **vector valued measure** is a function $\mu : \mathcal{M} \to \mathcal{H}$ where $\mathcal{H}$ is some Hilbert space. For this, convergence of the following sum is a required property*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n}^{\infty} \mu(A_n) \tag{I.4}$$

For vector valued measures, it is often useful to first construct the measure on the Boolean algebra $\mathcal{B}$ and then go on to extend it to the σ-algebra. This is what is done in the text so far. We give another useful definition that will be useful in construction of measures

Definition *Given a collection of sets* $\{A_\alpha : \alpha \in I\}$ *where* I *is some index set, the* σ*-algebra generated by this collection of sets is the smallest* σ*-algebra that contains these sets.*

Chapter II

Rings and Algebras

In this thesis, we recognised the set of transition amplitudes as rings. We will just state the definition of a ring and an algebra in this appendix.
What is a ring? We can begin answering this by considering an example that is simple and used by most people, the set of integers. What are the properties of a set of integers? We know that we can add them, they commute, they have additive inverse and additive identity. Besides, we have multiplication defined on them. But not all integers have a multiplicative inverse. Thus, it does not have all the properties of the real numbers, but we still have various interesting properties like, given two integers x, y, we can write $y = ax + b$ where a, b are also integers. We can prove other interesting properties of integers given this algebraic structure. Thus, we can take abstract sets with addition and multiplication defined in a similar way to integers and have similar properties in those sets. We call these sets, **rings**. We need a ring to have identity element for multiplication. We define a ring with identity as follows

Definition *A* ***ring*** *with identity is a set* R *such that*

1. *If* $a, b \in \mathsf{R}$*, then* $a + b \in \mathsf{R}$
2. $a + b = b + a$ *(commutativity)*
3. $a + (b + c) = (a + b) + c$ *(associativity)*
4. $\exists 0 \in \mathsf{R}$ *such that* $a + 0 = a$ *(existence of additive identity)*
5. $\forall a \in \mathsf{R}\ \exists(-a) \in \mathsf{R}$ *such that* $a + (-a) = 0$
6. *If* $a, b \in \mathsf{R}$*, then* $ab \in \mathsf{R}$

7. *$a(bc) = (ab)c$ (associativity)*

8. *$\exists 1 \in \mathsf{R}$ such that $(a)1 = a$. This is the property due to which we call it a ring with identity. Not all rings are required to have this property.*

9. *$a(b + c) = ab + ac$. (distributivity)*

We thus defined a ring. We did not use the concept of an algebra much in the thesis as we were only concerned with the properties of a ring. Informally speaking, an **algebra** is a ring with scalar multiplication defined on it. Thus, we have a notion of scaling in some sense.

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