

Representation Spectra of Lattices in Lie Groups and its applications to Locally Symmetric Spaces

विद्या वाचस्पति की

उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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Dedicated to
Maa, Baba and Mana

Certificate

Certified that the work incorporated in the thesis entitled “Representation Spectra of Lattices in Lie Groups and its applications to Locally Symmetric Spaces” submitted by Kaustabh Mondal was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institution.



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I declare that this written submission represents my idea in my own words and where others' ideas have been included; I have adequately cited and referenced the original sources.

I declare that I have acknowledged collaborative work and discussions wherever such work has been included. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/-source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

The work reported in this thesis is the original work done by me under the guidance of my advisor Dr. Chandrasheel Bhagwat and my co-advisor Dr. Gunja Sachdeva.



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Abstract

Let (τ, V_τ) be a finite dimensional representation of a maximal compact subgroup K of a connected non-compact semisimple Lie group G , and let Γ be a uniform torsion-free lattice in G . We obtain an infinitesimal version of the celebrated Matsushima-Murakami formula, which relates the dimension of the space of automorphic forms associated to τ and multiplicities of irreducible $\tau^\vee$ -spherical spectra in $L^2(\Gamma \backslash G)$. This result gives a promising tool to study the joint spectra of all central operators on the homogeneous bundle associated to the locally symmetric space and hence its infinitesimal τ -isospectrality. Along with this we prove that the almost equality of τ -spherical spectra of two lattices assures the equality of their τ -spherical spectra.

Let Γ_1 and Γ_2 be two lattices of finite covolume in a semisimple Lie group G . We prove a spectral rigidity result for the representation spectra of the right regular representations $L^2(\Gamma_1 \backslash G)$ and $L^2(\Gamma_2 \backslash G)$ of G . This can be thought of as an analogue of the strong multiplicity one theorem and it generalises [BR11, Thm 1.1] by Bhagwat and Rajan to the case of non-uniform lattices.

Let G be the real points of a connected semisimple simply connected algebraic group over $\mathbb{Q}$ such that G has a compact Cartan subgroup and let Γ be a uniform lattice in G . Let $\widehat{G}_d$ denote the set of equivalence classes of unitary discrete series representations of G . We prove that for any finite subset of $\widehat{G}_d$ satisfying a certain condition, the associated finite set of discrete series multiplicities in $L^2(\Gamma \backslash G)$ determines all discrete series multiplicities in $L^2(\Gamma \backslash G)$. This allows us to obtain a refinement of the strong multiplicity one result for discrete series representations. As an application, we deduce that for two given levels, the equality of the dimensions of the spaces of cusp forms over a suitable finite set of weights

implies the equality of the dimensions of the spaces of cusp forms for all weights.

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Chapter 1

Introduction

1.1 Isospectrality and Representation Equivalence

The mathematical description of classical spectral theory of light and sound is given by the *spectrum* of the Laplacian acting on certain space of smooth function on Riemannian manifold. An admissible source of such Riemannian manifold is locally symmetric spaces. These spaces are of the form $X_\Gamma = \Gamma \backslash G/K$; where, G is a reductive Lie group, K is a maximal compact subgroup and Γ is a lattice in G . The symmetric space G/K , equipped with G -invariant Riemannian metric, is the universal cover of X_Γ . In the recent past, one of the successful approaches to understand the spectrum of Laplace type operator on X_Γ is using its intimate relation with representation theory of G .

The question: *Can the space X_Γ (up to isometry) be determined by its spectrum?* has been of great interest over last few decades. The works of Milnor (see [Mil64]) and Vignéras (see [Vig80]) are in the frontline to answer this negatively. Later, the construction of non-isometric isospectral manifolds by Sunada brought significant arithmetic flavour in this

context (see [Sun85]). In [Raj07], C. S. Rajan studied the appropriate arithmetic properties which are determined by the spectrum.

The arithmetic properties of X_Γ have many similarities with the spectrum of X_Γ . The first such evidence appears in the work of Maass, who studied the arithmetic properties of eigenfunctions of Laplace operators on the complex upper half plane $\mathbb{H}$ (later well-known as Maass forms), that are invariant under congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$ (see [Gel75]). This provides a relation between the representation theory of $G := \mathrm{PSL}(2, \mathbb{R})$ on $L^2(\Gamma \backslash G)$ and the spectrum of Hyperbolic Laplacian acting on $\Gamma \backslash \mathbb{H}$. Later, Selberg came up with *Trace Formula* which happens to be a tremendous achievement linking the spectrum and arithmetic of X_Γ (see e.g. [Hej83]).

The purpose of this thesis is studying the representation theory of $L^2(\Gamma \backslash G)$ to address the spectrum of X_Γ . More explicitly, for given two lattices Γ_1 and Γ_2 in a Lie group G and for a fixed maximal compact subgroup K in G , we study a notion called *Representation Equivalence* of $L^2(\Gamma_1 \backslash G)$ and $L^2(\Gamma_2 \backslash G)$ to obtain some *Isospectrality* results for X_{Γ_1} and X_{Γ_2} . Before diving into the case for general reductive/semisimple Lie group, let us recall some classical results for finite group G .

Definition 1.1.1. A triplet (G, H_1, H_2) consisting of finite group G and two non-conjugate subgroups H_1 and H_2 of G is called a *Gassmann-Sunada system* if one of the following equivalent condition holds:

- (1) The right regular representation of G on $\mathbb{C}[H_1 \backslash G]$ and $\mathbb{C}[H_2 \backslash G]$ are isomorphic.
- (2) Any G -conjugacy class intersects H_1 and H_2 in the same number of elements.
- (3) For any finite dimensional representation V of G , $\dim V^{H_1} = \dim V^{H_2}$.

An example can be given as follows. Let H_1 and H_2 be two finite groups such that for any positive integer d , the number of elements of order d in H_1 and H_2 are same. In particular,

$|H_1| = |H_2| = k$. Then, (S_k, H_1, H_2) becomes a Gassmann-Sunada triplet. Here, S_k is the symmetric group in k symbols. The following is Sunada's result in [Sun85].

Theorem 1.1.2. *Suppose (G, H_1, H_2) is a Gassmann-Sunada system and assume that G acts freely and isometrically on a compact Riemannian manifold M . Then the quotient spaces $H_1 \backslash M$ and $H_2 \backslash M$ are isospectral.*

We will make a precise definition of Isospectrality in Chap. 4. For now, the reader can take an ad-hoc definition : Two locally symmetric spaces are isospectral if the spectrum of the corresponding Laplace operators acting on smooth functions on them coincides.

Sunada's Thm. 1.1.2 for Isospectrality had been extended by DeTurck and Gordon in [DG89]. To state their result, we need to define representation equivalence of lattices in Lie group. Let G be a Lie group. Let Γ be a cocompact lattice in G . In particular, G is unimodular. Consider the right regular representation R_Γ of G on the space $L^2(\Gamma \backslash G)$ of square integrable functions on G with respect to G -invariant measure on $\Gamma \backslash G$, defined by

$$(R_\Gamma(g)\varphi)(x) = \varphi(xg)$$

for all $\varphi \in L^2(\Gamma \backslash G)$ and $x, g \in G$. Then, we have

$$(R_\Gamma, L^2(\Gamma \backslash G)) = \widehat{\bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) \pi},$$

where $\widehat{G}$ is the set of isomorphism classes of irreducible representation of G and $m(\pi, \Gamma)$ is the number of π -isotypic component in $L^2(\Gamma \backslash G)$.

Definition 1.1.3. *Two co-compact lattices Γ_1 and Γ_2 in G are said to be representation equivalent if the right regular representations $(R_{\Gamma_1}, L^2(\Gamma_1 \backslash G))$ and $(R_{\Gamma_2}, L^2(\Gamma_2 \backslash G))$ are isomorphic*

i.e. $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all $\pi \in \widehat{G}$.

Theorem 1.1.4. (DeTurck-Gordon) *Let G be a Lie group acting on the left as isometries of a Riemannian manifold M . Suppose Γ_1 and Γ_2 are discrete, co-compact subgroups of G acting freely and properly discontinuously on M , such that the quotients $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ are compact Riemannian manifolds. If the lattices Γ_1 and Γ_2 are representation equivalent in G , then $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ are isospectral for the Laplacian acting on the space of smooth functions.*

So, we see that representation equivalent lattices give rise to spectrally indistinguishable spaces. It is natural to ask if the converse holds. This is quite ambitious inverse spectral problem. When $G = \mathrm{PSL}(2, \mathbb{R})$, the converse holds due to Pesce. Let X be a hyperbolic compact Riemann surface and Γ_X be its fundamental group. By uniformization of X , the fundamental group Γ_X can be realised as a lattice in $\mathrm{PSL}(2, \mathbb{R})$ via an inclusion $\rho_X : \Gamma_X \rightarrow \mathrm{PSL}(2, \mathbb{R})$.

Theorem 1.1.5. (Pesce; [Hub98]) *Suppose X and Y are compact Riemannian surfaces of genus at least 2. Then X and Y are isospectral with respect to the Hyperbolic Laplacian acting on the space of smooth functions if and only if the lattices Γ_X and Γ_Y are representation equivalent in $\mathrm{PSL}(2, \mathbb{R})$.*

Apart from the group $\mathrm{PSL}(2, \mathbb{R})$ this converse direction is not well-understood. See [Raj07] for a possible reformulation in generalities. While understanding the full representation spectra in $L^2(\Gamma \backslash G)$ at once seems difficult, we intend to focus on representation spectra with prescribed archimedean type. For an irreducible representation τ of K , an irreducible representation π in $\widehat{G}$ is called τ -spherical if $\mathrm{Hom}_K(\tau, \pi|_K) \neq 0$. Let us denote $\widehat{G}_\tau = \{\pi \in \widehat{G} : \mathrm{Hom}_K(\tau, \pi|_K) \neq 0\}$. Note that $\widehat{G} = \cup_{\tau \in \widehat{K}} \widehat{G}_\tau$. There is also the notion of τ -isospectrality for X_Γ defined in Sec. 4.1.1 for certain τ -spherical Laplace operator. Therefore, the τ -

equivariant version of inverse spectral problem reads : *Given two τ -isospectral spaces X_{Γ_1} and X_{Γ_2} , are Γ_1 and Γ_2 $\tau^\vee$ -representation equivalent?* Let us now define a weaker version of τ -representation equivalence and τ -isospectrality.

Definition 1.1.6. *Two lattices Γ_1 and Γ_2 are called almost- τ -representation equivalent if $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all but finitely many $\pi \in \widehat{G}_\tau$.*

Definition 1.1.7. *Two locally symmetric spaces X_{Γ_1} and X_{Γ_2} with the same universal cover G/K are said to be almost- τ -isospectral if the dimensions of the λ -eigenspaces for the corresponding τ -spherical Laplace operators are equal for all but finitely many $\lambda \in \mathbb{R}$.*

In the direction of “almost- τ -representation equivalence implies isospectrality”, several approaches have been made. In [LM20], E. Lauret and R. Miatello answered this question for the case where K is a closed subgroup of a compact group G , i.e., for the case of compact homogeneous space G/K for compact group G . They further proved that the multiplicity of an appropriate finite subset of $\widehat{G}_\tau$ determines all multiplicities (see [LM20] Thm. 1.1). Later, they studied the representation-spectral converse in the context of simply connected compact Riemannian symmetric space G/K of rank one and proved there are infinitely many $\tau \in \widehat{K}$ such that almost- τ -isospectrality implies $\tau^\vee$ -representation equivalence and hence τ -isospectral (see [LM21] Thm. 1.1).

For many non-compact Riemannian symmetric spaces, Pesce proved the validity of representation spectral converse where τ is trivial (see [Hub95]). When τ is trivial, the τ -spectra are called spherical spectra. Bhagwat and Rajan answered that almost-spherical representation equivalence implies spherical representation equivalence (see [BR11] Thm. 1.2). In fact, they described that for a spherical irreducible representation $\pi \in \widehat{G}_1$, there exists a character λ_π of the algebra of G -invariant differential operators on X_Γ such that

$\text{Mult}(\lambda_\pi) = m(\pi, \Gamma)$ and conversely. Later, Kelmer obtained several density results which relates the isospectrality and representation equivalence via the notion of length equivalence for the case of X_Γ with real rank one and of non-compact type (having non-compact universal cover), in particular for compact hyperbolic manifolds (see [Kel14]).

In the first half of Chap. 4, we give an affirmative answer of the question: *Does the almost- τ -representation equivalence implies τ -representation equivalence?* The following theorem can be regarded as τ -spherical spectral analogue of classical strong multiplicity one theorem.

Theorem 1.1.8. (Thm. 1.1.2; [BMS25]) *Let G be a semisimple, non-compact, connected Lie group with finite centre. Assume that τ is an irreducible representation of a maximal compact subgroup K . If two uniform torsion free lattices Γ_1 and Γ_2 are almost- τ -representation equivalent then Γ_1 and Γ_2 are τ -representation equivalent.*

Remark 1.1.9. *Consequently, the above hypothesis implies that X_{Γ_1} and X_{Γ_2} are $\tau^\vee$ -isospectral. When τ is trivial, Thm. 1.1.8 reduces to [BR11 Thm. 1.2]. A special case of Thm. 1.1.8 is [BS24 Thm. 4.1] for the group $\text{PSL}(2, \mathbb{R})$.*

Let us briefly outline our methods for obtaining Thm. 1.1.8. We employ the well-established and effective tool, namely the Selberg Trace Formula to study the representation spectra of $L^2(\Gamma \backslash G)$ with respect to a convolution operator. Since our focus is on the τ -spherical irreducible representations in $L^2(\Gamma \backslash G)$, we need to annihilate the non τ -spherical component using appropriate test functions. To achieve this, we instead consider the right regular representation $L^2(\Gamma \backslash G, V_\tau)$ (see 2.4.1) of G . We also utilize an algebra, denoted by $C_c^\infty(G, K, V_\tau)$ consisting of compactly supported smooth $\text{End}(V_\tau)$ -valued test functions on G which are τ -equivariant (see 2.4.3). One can suitably define the convolution operator

on $L^2(\Gamma \backslash G, V_\tau)$ for such test functions. It is straightforward to compute both the spectral and geometric expansions of the trace of these convolution operators, which leads to the required Selberg Trace Formula for $L^2(\Gamma \backslash G, V_\tau)$ (see [3.2.7](#)). The advantage of using such operator valued τ -equivariant test functions is the corresponding convolution operator annihilates the non τ -spherical representation spectra (see [4.3.4](#)). Moreover, due to our hypothesis, taking the difference between the trace formula for Γ_1 and Γ_2 yields a finite linear combination of Harish-Chandra character distribution associated with finitely many τ -spherical irreducible representations. In Prop. [4.3.6](#), we construct a left K -saturated open set in G that avoids all conjugacy classes $[\gamma]_G$ for $\gamma \in \Gamma_1 \cup \Gamma_2$. For the aforementioned test functions supported on this K -stable open set, the orbital integrals, and thus the entire geometric side vanish. Finally, the remainder of the argument relies on the analyticity of locally integrable character functions and the linear independence of character distributions for inequivalent irreducible representations.

1.2 Representation Equivalence for Arithmetic Lattices

In the previous section, we were dealing with lattices Γ in a Lie group G such that $\Gamma \backslash G$ is compact. For compact quotient $\Gamma \backslash G$ (equivalently, X_Γ is compact), one has the advantage that the right regular representation $(R_\Gamma, L^2(\Gamma \backslash G))$ is completely reducible. Now, we weaken the set up assuming merely $\Gamma \backslash G$ is of finite volume with respect to unique G -invariant measure. For example, principal congruence lattices in $\mathrm{SL}(2, \mathbb{R})$ are of finite covolume. In this case, R_Γ is no longer completely reducible. Let $L^2_{\mathrm{disc}}(\Gamma \backslash G)$ be the maxi-

mal semisimple subrepresentation of $L^2(\Gamma \backslash G)$. Then, we have

$$L^2(\Gamma \backslash G) = L^2_{\text{disc}}(\Gamma \backslash G) \oplus L^2_c(\Gamma \backslash G)$$

where, $L^2_c(\Gamma \backslash G)$ is the orthogonal complement of $L^2_{\text{disc}}(\Gamma \backslash G)$ in $L^2(\Gamma \backslash G)$. In the case of non co-compact lattice Γ , the presence of the continuous spectrum in $L^2(\Gamma \backslash G)$ always demands delicate analysis. For instance, for any compactly supported smooth function f , the convolution operator $R_\Gamma(f)$ is not trace class. Consequently, the classical Selberg trace formula does not work to achieve some result analogous to Thm. [1.1.8](#). The non-convergence of $\text{Trace}(R_\Gamma(f))$ is due to the presence of unipotent elements in the lattice Γ . However, Müller showed that under some *nice* conditions (satisfied by arithmetic lattices) on Γ , the restriction of $R_\Gamma(f)$ on $L^2_{\text{disc}}(\Gamma \backslash G)$ is trace class (see [\[Mul89\]](#)). The convergence of the trace of $R_\Gamma(f)|_{L^2_{\text{disc}}(\Gamma \backslash G)}$ is due to a control on the growth of the continuous spectrum by certain intertwining operators on principal series representations (see Sec. [5.3](#)).

To get rid of continuous spectrum one often tries to obtain an ample supply of test functions f such that the convolution operator $R_\Gamma(f)$ is supported on discrete spectrum. In [\[LV07\]](#), such a class of compactly supported test functions was constructed so that $R_\Gamma(f)$ is supported on the spherical irreducible representations (having nonzero K -fixed vectors); it has been refined recently in [\[Mai24\]](#). In the latest breakthrough by [\[BLZZ21\]](#), for a prescribed irreducible representation π occurring in the discrete spectrum of $L^2(\Gamma \backslash G)$ with congruence lattice Γ a source of test functions f (though merely Schwartz class functions) can be found so that the convolution operator $R_\Gamma(f)$ is supported on the *nearly* π -isotypic component inside the discrete spectrum. Consequently one can write the corresponding trace formula with quite simple spectral side. However, the cost against this simplicity of

spectral side is that there is no flexibility on the support of the test functions (as those are Schwartz class). For example, a key idea in co-compact case is choosing ample supply of compactly supported test functions supported away from Γ -conjugacy classes (see Sec. 4.3.7). This does not work for Schwartz class functions while comparing the geometric sides of trace formulas for two lattices.

In this scenario, one can take one of the two possible paths. If the test function f is moreover K -finite then Osborne-Warner trace formula provides a geometric expression for $\text{Trace}(R_\Gamma(f))|_{L^2_{\text{disc}}(\Gamma \backslash G)}$ in terms of Hyperbolic terms, scattering terms and unipotent terms (see Sec. 5.3). We use this to obtain Thm. 5.1.5 (cf. Thm. 1.5 in [BM23]) which is similar to Thm. 1.1 in [BR11], but under very strong assumptions. By those assumptions, in the comparison of the trace formulas for Γ_1 and Γ_2 , the continuous spectra cancel out. But on the geometric side the unipotent orbital integral still survives (see Sec. 5.4.3). Now the set of regular elements in G (denoted by G_{reg}) is an open and dense subset of G . Therefore we can take our test functions to be supported in G_{reg} which guarantees the vanishing of the unipotent orbital integral. Hence, we get a simple trace formula consisting of discrete spectral side and regular geometric side (like the cocompact case). Restricting the support of the test functions inside G_{reg} does not change the trace of $\pi(f)$ for any irreducible representation π because of Thm. 5.2.9 together with the fact that $G \setminus G_{\text{reg}}$ is of measure zero. The rest of the argument goes same as in the proof of ([BR11], Thm. 1.1).

The other path would be the Arthur-Selberg Trace formula. This trace formula rectifies the non-convergence issue of trace via adelic reformulation. So, it is natural to frame our question in terms of cuspidal automorphic representations for a linear reductive algebraic group $\mathcal{G}$ whose real points would be our prescribed Lie group G . It is well-known that

congruence lattices can be obtained from compact open subgroups in finite adeles.

Let us give an example. Let G be a semisimple simply connected algebraic group over $\mathbb{Q}$. $K \subset G(\mathbb{A}_F^\infty)$ be a compact open subgroup. Then we have the strong approximation :

$$\Gamma_K \backslash G(\mathbb{R}) \cong G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q}) / K$$

where, $\Gamma_K = G(\mathbb{Q}) \cap K$. When $G = GL(2)$ (which is neither semisimple nor simply connected), such isomorphism holds as well and the compact open subgroup $K = K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathbb{Z}}) : N|c \right\}$ gives rise to the congruence lattices $\Gamma_0(N)$.

We follow [Dei99] to write a simple trace formula for rank one semisimple Lie groups using the coarse version of Arthur-Selberg Trace formula (see [Art05] for higher rank case). Using the theory of Hecke twist, we prove the following representation equivalence result for cuspidal spectrum in Sec. 5.5.2:

Theorem 1.2.1. ¹ Let K_1 and K_2 be two compact open subgroups of $GL(n, \mathbb{A}_{F,f})$. Denote the congruence subgroup $GL(n, F) \cap K_i$ of $GL(n, F_\infty)$ by Γ_{K_i} for $i = 1, 2$. If $m(\pi, \Gamma_{K_1}) = m(\pi, \Gamma_{K_2})$ for all but finitely many $\pi \in \widehat{G(F_\infty)}$, then $L_{cusp}^2(\Gamma_{K_1} \backslash G(F_\infty)) \cong L_{cusp}^2(\Gamma_{K_2} \backslash G(F_\infty))$.

Remark 1.2.2. Indeed, for a co-compact lattice Γ we have, $L_{cusp}^2(\Gamma \backslash G) = L^2(\Gamma \backslash G)$. Therefore, Thm. 1.2.1 can be thought as an improvement of Thm. 1.1 in [BR11] for congruence lattices in $GL(n, \mathbb{R})$.

Remark 1.2.3. C. S. Rajan suggested us to frame a natural analogue of the above result in terms of character families. In other words, if for two congruent subgroups, the multiplicities

¹I thank Prof. A. Raghuram for useful discussion on this result.

are the same except for finitely many twisted by character families, then one can expect that the lattices are isospectral.

1.3 Linear Dependence of Discrete Series Multiplicities

Let us get back to the case for co-compact lattices. So far, we have been concerned with the cuspidal, discrete and τ -spherical spectra in $L^2(\Gamma \backslash G)$. We now shift our attention to a particular family of irreducible representations, namely the discrete series representations. Our goal is to study the multiplicities $m(\pi, \Gamma)$ for discrete series representations π occurring in $L^2(\Gamma \backslash G)$. The discrete series representations are analogous to the irreducible representations of compact Lie groups. For example, they *occur* in $L^2(G)$ by the Harish-Chandra Plancherel theorem. The precise definition and classification of discrete series representations are described in Sec. [6.2.1](#).

The arithmetic and geometric significance of the discrete series multiplicities is well-known. The appropriate cohomological meaning of these multiplicities can be attributed to Langlands, who conjectured it to be the dimension of twisted sheaf cohomology of a certain degree over $\Gamma \backslash G/T$ (see [\[Lan65\]](#)). This was proved partly by Schmid (see [\[Sch71\]](#)), and was completed by Williams (see [\[Wil85\]](#)). Williams used the L^2 -Dolbeault cohomology instead of sheaf cohomology, relaxing the compactness assumption on $\Gamma \backslash G$. The dimension of exactly one non-vanishing L^2 -cohomology turns out to be the multiplicity of a discrete series representation. In [\[BMS25\]](#) with Bhagwat and Sachdeva, the author developed a refinement of the classical Matsushima-Murakami formula and recovered the di-

mension of the L^2 -cohomology. To obtain a closed multiplicity formula for discrete series representation, Williams also proved the Thm. 7.47 of [Wil91] which is the best possible improvement of the Osborne-Warner formula for integrable discrete series multiplicities (see [OW78]). In [HP73], Hotta-Parthasarathy used Atiyah-Singer-Lefschetz fixed point formula to compute the index of a twisted Dirac operator explicitly. This along with Thm. 6.2.5 of Williams provides the expected geometric expression of discrete series multiplicities (e.g. see Thm. 6.2.7).

The multiplicities $m(\pi, \Gamma)$ of irreducible representations π of G occurring in $L^2(\Gamma \backslash G)$ have been of great interest also in the context of spectral geometry of locally symmetric spaces, arithmeticity of lattices in Lie groups, etc. As mentioned earlier, a spectral rigidity result of the multiplicities $m(\pi, \Gamma)$ was obtained by Bhagwat and Rajan (see [BR11]), and was refined later by Kelmer (see [Kel14]). In [LM20], Lauret and Miatello proved for compact group G (and therefore for finite Γ) the existence of a certain finite set of multiplicities $m(\pi, \Gamma)$ that determines all multiplicities. Since every irreducible representation of a compact semisimple group can be thought of as a discrete series representation (as they occur in $L^2(G)$), it is tempting to seek some suitable finite set of discrete series multiplicities for non-compact groups which determines all discrete series multiplicities. The main Thm. 6.1.2 in Chap. 6 answers this affirmatively. We state a simplest version of it in the context of representation equivalence of lattices in a rank one semisimple Lie group.

Theorem 1.3.1. (Cor. 1.1 in [MS25]) *Let G be a semisimple connected rank one Lie group whose complexification $G_{\mathbb{C}}$ is simply connected. Assume that G contains a compact Cartan subgroup. Consider two co-compact lattices Γ_1 and Γ_2 in G . There exists a finite subset S of $\hat{G}$ consisting of discrete series representations such that if $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all $\pi \in S$ then $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all discrete series representation π of G .*

Let Γ be a uniform lattice in $\mathrm{SL}(2, \mathbb{R})$ arising from a maximal order of a division quaternion algebra over a totally real number field. There are only finitely many conjugacy classes of elliptic elements in Γ . Denote the l.c.m. of their orders by N_Γ and call it the maximal order of elliptic elements in Γ . Let $\mathcal{S}_k(\Gamma)$ be the space of classical holomorphic cusp forms on the complex upper half plane of weight k and level Γ . It is well-known that $\mathcal{S}_k(\Gamma)$ provides an analytic realization of a discrete series representation of $\mathrm{SL}(2, \mathbb{R})$. In Sec. [6.5](#) we explicitly prove the equality of the dimension of $\mathcal{S}_k(\Gamma)$ and the multiplicities of discrete series representations occurring in $L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$ (see Prop. [6.5.2](#)), and obtain a similar finite linear generation result on the dimension of $\mathcal{S}_k(\Gamma)$ (see Cor. [6.5.1](#)). Combining this with Rem. [6.1.1](#), we have the following consequence of Thm. [1.3.1](#):

Corollary 1.3.1. *Assume that Γ_1 and Γ_2 are two uniform lattices in $\mathrm{SL}(2, \mathbb{R})$, also N_{Γ_1} and N_{Γ_2} are the maximal order of elliptic elements in Γ_1 and Γ_2 respectively. Let $N = \mathrm{l.c.m.}(N_{\Gamma_1}, N_{\Gamma_2})$. Then there exists a finite subset $\mathcal{F} \subset \mathbb{N}_{\geq 2}$ of cardinality $2N$ such that if $\dim \mathcal{S}_k(\Gamma_1) = \dim \mathcal{S}_k(\Gamma_2)$ for all $k \in \mathcal{F}$, then $\dim \mathcal{S}_k(\Gamma_1) = \dim \mathcal{S}_k(\Gamma_2)$ for all $k \in \mathbb{N}_{\geq 2}$.*

Remark 1.3.1. *In Cor. [1.3.1](#) the finite set $\mathcal{F}$ of size $2N$ is not unique. See Cor. [6.5.1](#) for the sufficient condition on the finite set $\mathcal{F}$ for which Cor. [1.3.1](#) holds.*

Let us sketch briefly the idea behind the proof of the general version of Thm. [1.3.1](#). Our method is based on a geometric expression for the multiplicities $m(\pi, \Gamma)$ of discrete series representations π of a semisimple Lie group G . For instance, if G is compact (and therefore any lattice Γ is finite) then for any irreducible representation π of G , the multiplicity $m(\pi, \Gamma)$ equals $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{Trace}(\pi(\gamma))$. In the case of non-compact G , the discrete series representations are parametrized by a set $\mathcal{D}(G)$ consisting of regular linear integral forms λ (see [6.2.1](#)). Let us denote the discrete series representation by π_λ that corresponds to

$\lambda \in \mathcal{D}(G)$. For any regular integral linear form $\lambda \in \mathcal{D}(G)$, we consider the Lefschetz number $\chi(\lambda, \Gamma)$ associated with a twisted Dirac operator acting on the space of smooth sections of a certain homogeneous bundle over G/K (see Subsec. [6.2.2](#)). We require that λ to be *sufficiently regular* which ensures that the multiplicity $m(\pi_\lambda, \Gamma)$ of the discrete series representation π_λ equals the Lefschetz number $\chi(\lambda, \Gamma)$ up to a sign (see Thm. [6.2.5](#)). Thanks to the work of Hotta and Parthasarathy (see Thm. [6.2.7](#)), this Lefschetz number and hence the multiplicity admits a geometric expression in terms of the elliptic elements in Γ . A key tool in our approach is the formulation of a “string” $\mathcal{S}(\lambda_1, \lambda_2)$ of regular integral forms in $\mathcal{D}(G)$, with fixed base λ_1 and direction λ_2 (see Def. [6.2.8](#)). For such a given string, we define a generating function $F_{\lambda_1, \lambda_2, \Gamma}$ as a formal power series in the complex variable z with coefficients as discrete series multiplicities associated to the string (see Sec. [6.3](#)). Using the geometric expression of the multiplicities, we prove that $F_{\lambda_1, \lambda_2, \Gamma}$ is a rational function with numerator polynomial $p(z) = \sum_i b_i z^i$ and denominator $(1 - z^{N_\Gamma})^{|\Phi^+|+1}$, where N_Γ is the maximal order of elliptic elements in Γ , and the degree of $p(z) < N_\Gamma(|\Phi^+| + 1)$ (see Prop. [6.3.1](#)). Finally, a technical condition on the finite set $\mathcal{S} \subset \mathbb{N} \cup \{0\}$ gives rise to a nonsingular system of $|\Phi^+| + 1$ many linear equations in the unknowns $\{b_i\}_i$ and constant terms $\{m(\pi_{\lambda_k}, \Gamma) : k \in \mathcal{S}\}$. Solving this system uniquely determines the desired linear expression for each discrete series multiplicity.

1.4 Organization

The rest of the thesis is organized as follows. In Chap. [2](#) we start with the basic structure theory of semisimple Lie groups e.g. Iwasawa decomposition of Lie groups and its Lie algebras, parabolic subgroups, root spaces decompositions etc. We also describe the

spectral decomposition of $L^2(\Gamma \backslash G)$ and $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}}))$ in Sec. 2.2 and Sec. 2.3 respectively. Lastly, we describe a τ -equivariant spectral decomposition i.e. the decomposition of $L^2(\Gamma \backslash G, V_{\tau})$ for a prescribed archimedean type τ (see 2.4).

In Chap. 3, we describe various trace formulas for the $L^2(\Gamma \backslash G)$. We first recall the classical version of Selberg Trace formula (see 3.1). Then, we prove a τ -equivariant version of Selberg trace formula (see 3.2). This will be used in proving Thm. 1.1.8. We moreover discuss a simple trace formula for congruence lattices in rank one semisimple Lie groups following [Dei99] (see 3.3).

The Chap. 4 concerns our results on τ -isospectrality (e.g Thm. 1.1.8). The section 4.1 states the main results on τ -isospectrality and infinitesimal τ -isospectrality. In Section 4.2 we set up the preliminaries and recall the Harish-Chandra character distributions, Isospectrality and Representation equivalence. In Section 4.3 we discuss the required lemmas and propositions to prove Thm. 4.1.3. In Section 4.4 we provide some observations and complete the proof of Thm. 4.1.7. In Section 4.5 we consider the case where the group G has discrete series representations and using the results from [Wil91] and Thm. 4.1.7, we show that the dimension of χ -eigenspace of the automorphic forms of type $\tau^{\vee}$ is equal to the q_{λ} -th L^2 -cohomology of the automorphic line bundle $\Gamma \backslash \mathcal{L}_{\lambda}$ associated to the discrete series representation π with minimal K -type τ and infinitesimal character χ .

In Chap. 5, we consider the representation equivalence of congruence lattices in rank one semisimple Lie groups. Sec. 5.1 describes the main Thm. 5.1.5 of Chap. 5, which can be regarded as a strong multiplicity one theorem for *isocuspidal* congruence lattices. In Sec. 5.2 we discuss some required lemmas on geometry of conjugacy classes in semisimple Lie groups. We derive a simple trace formula for arithmetic lattices from Osborne-Warner Trace formula in Sec. 5.3. Sec. 5.4 discusses the proof of the strong multiplicity one

theorem. Finally, in Sec. [5.5](#), we discuss some generalities of Hecke twist of cuspidal automorphic representations and consequently prove Thm. [1.2.1](#).

The Chap. [6](#) concerns the discrete series multiplicities. In Subsec. [6.2.1](#), we recall the Harish-Chandra parametrization of discrete series representations. The relation between discrete series representations and Lefschetz numbers, as well as the geometric expression of Lefschetz numbers, is described in Subsec. [6.2.2](#), and Subsec. [6.2.3](#) respectively. The Subsec. [6.2.4](#) describes the notion of string of regular linear integral forms. The Sec. [6.3](#) is devoted to introducing the generating function for a string and proving its rationality. The proof of the main results Thm. [6.1.2](#) and Cor. [6.1.2](#) are given in Subsec. [6.4.1](#) and Subsec. [6.4.2](#) respectively. In Subsec. [6.4.3](#), we introduce the notion of multi-string and provide a generalized version of Thm. [6.1.2](#). Subsequently we prove the infinitude of its remainder part in $\mathcal{D}^*(G)$. In Sec. [6.5](#), as an application of our results, we showcase a finite linear dependence result for the dimension of cusp forms.

Chapter 2

General Preliminaries

2.1 Structure Theory of Semisimple Lie Groups

2.1.1 Cartan Decomposition

Let G be a linear connected real semisimple Lie group e.g. $SL(n, \mathbb{R})$, $SO(n, 1)$ etc. Let $\mathfrak{g}$ be the real Lie algebra of G . We denote by $\mathfrak{g}_{\mathbb{C}}$ its complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Being linear G always has a Cartan involution Θ ; namely, the inverse conjugate transpose. $\Theta^2 = 1$. Let $K = \{g \in G : \Theta g = g\}$. This is a maximal compact subgroup in G . The derivative of Θ at 1 gives rise to an automorphism θ of $\mathfrak{g}$. Again, we have $\theta^2 = 1$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the eigenspaces corresponding to the eigenvalues $+1$ and -1 . The eigenspace decomposition for θ

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is called the Cartan decomposition of $\mathfrak{g}$. It can be seen easily that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{k}$ is the Lie algebra of K .

2.1.2 Root Space Decomposition

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra consisting of semisimple elements in $\mathfrak{g}$. Such a subalgebra exists because $\mathfrak{g}$ is finite dimensional. A Cartan subgroup T of G is a maximal connected abelian subgroup whose Lie algebra is a Cartan subalgebra $\mathfrak{h}$.

The complexified Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ acts on the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ via adjoint action. We can regard $\{Ad(\mathfrak{h}_{\mathbb{C}})\}$ to be a commutative family of linear operators on $\mathfrak{g}_{\mathbb{C}}$. For a linear functional $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$, let $\mathfrak{g}_{\alpha} := \{v \in \mathfrak{g}_{\mathbb{C}} : [h, v] = \alpha(h)v \text{ for all } h \in \mathfrak{h}_{\mathbb{C}}\}$. The joint eigenspace $\mathfrak{g}_{\alpha}$ is called the root space corresponding to the root $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$. Moreover, we have the decomposition of $\mathfrak{g}_{\mathbb{C}}$ as :

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \mathfrak{h}_{\mathbb{C}}^*} \mathfrak{g}_{\alpha}.$$

The above decomposition is called root space decomposition. By finite dimensionality, there are only finitely many $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$ such that $\mathfrak{g}_{\alpha} \neq 0$. Also, note that corresponding to the zero functional the eigenspace $\mathfrak{g}_0$ is a nonzero space as it contains $\mathfrak{h}_{\mathbb{C}}$. Therefore, the above direct sum essentially runs over a finite subset $\Phi \subset \mathfrak{h}_{\mathbb{C}}^*$. The non-zero linear functionals in the finite set Φ are called the roots for the root system $\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

2.1.3 Iwasawa Decomposition

We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ (which exists by finite dimensionality). The collection $\{AdH : H \in \mathfrak{a}\}$ is a commuting family of linear transformations on $\mathfrak{g}$. They can be simultaneously

diagonalised with real eigenvalues and for each linear functional λ on $\mathfrak{a}$,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

The set $\{\lambda \neq 0 : \mathfrak{g}_\lambda \neq 0\}$ is called the set of restricted roots and denoted by $\Sigma(\mathfrak{g}, \mathfrak{a})$. Then, we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\lambda$$

Indeed, $\mathfrak{a} \subset \mathfrak{g}_0$. $\Sigma(\mathfrak{g}, \mathfrak{a})$ turns out to be an abstract root system (see Prop. 5.9 in [Kna01]). We choose a positive system Σ^+ in $\Sigma(\mathfrak{g}, \mathfrak{a})$. Define

$$\mathfrak{n} = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

Then, by Prop. 5.9 in [Kna01], $\mathfrak{n}$ is a Lie subalgebra in $\mathfrak{g}$. Moreover, we have

Proposition 2.1.4. (Prop. 5.10 of [Kna01]) *For a linear connected semisimple Lie group G , the Lie algebra $\mathfrak{g}$ has the decomposition (as a vector space sum)*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

Here, $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent and $\mathfrak{a} \oplus \mathfrak{n}$ is solvable.

One can lift this decomposition from Lie algebra to Lie group. Let A and N be the analytic subgroups with Lie algebra $\mathfrak{a}$ and $\mathfrak{n}$. Then, A , N and AN are simply connected closed

subgroups of G . Finally, by Thm. 5.12 in [Kna01], we have the multiplication map

$$K \times A \times N \rightarrow G \quad \text{given by } (k, a, n) \mapsto kan$$

is a diffeomorphism.

2.1.5 Parabolic subgroups

Keeping the notation as above, we have the Iwasawa decomposition $G = KAN$; where, K is a maximal compact subgroup in G . Recall that K acts on A via conjugation. Let $M = Z_K(A)$ be the centralizer of A in K and $\mathfrak{m}$ be the Lie algebra of M . A conjugate of the closed subgroup MAN is called a minimal parabolic subgroup. A parabolic subgroup is a closed subgroup which contains a minimal parabolic subgroup.

Recall the restricted roots $\Sigma(\mathfrak{g}, \mathfrak{a})$ and choose a positive system $\Sigma^+ \subset \Sigma(\mathfrak{g}, \mathfrak{a})$. Then, Σ^+ determines the nilpotent subalgebra $\mathfrak{n}$. There exists a subset Π of Σ^+ such that no $\lambda \in \Pi$ can be written as sum of two elements λ_1 and λ_2 in Σ^+ . The set Π is called the set of simple roots. The positive system Σ^+ , hence $\mathfrak{n}$ determines a minimal parabolic subgroup MAN . The parabolic subgroups containing MAN are classified by the following :

Proposition 2.1.6. (Prop. 5.23 in [Kna01]) *The parabolic subgroups $S' = M'A'N'$ containing MAN are in one-to-one correspondence with the subsets Π' of Π . No two of these parabolic subgroups are conjugate within G .*

2.1.7 An example : $SO(n, 1)$

We illustrate an example for the group $G = SO(n, 1)$. $SO(n, 1) = \{A \in SL(n+1, \mathbb{R}) : {}^T A I_{n,1} A = I_{n,1}\}$; where, ${}^T A$ is the transpose of A and $I_{n,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$. There is a bilinear form $\varphi_{n,1}(x_1, \dots, x_n, x_{n+1}) = \sum_{i=1}^n x_i^2 - x_{n+1}^2$ on $\mathbb{R}^{n+1}$. $A \in SO(n, 1)$ if and only if the columns of A form an orthonormal basis with respect to $\varphi_{n,1}$. $SO(n, 1)$ is not connected for $n \geq 2$. In fact, it has two connected component. We are interested in the connected component $SO(n, 1)^\circ$. It is homeomorphic to $SO(n) \times \mathbb{R}^n$.

The Lie algebra $\mathfrak{g}$ of $G = SO(n, 1)$ can be expressed as $\mathfrak{g} = \mathfrak{so}(n, 1) = \{X \in M_{n+1}(\mathbb{R}) : {}^T X I_{n,1} + I_{n,1} X = 0; \text{Trace}(X) = 0\}$. Now, if $X = (a_{ij})_{i,j} \in \mathfrak{so}(n, 1)$, then $a_{ii} = 0$ for $1 \leq i \leq n$; $a_{ij} + a_{ji} = 0$ for all $1 \leq i, j \leq n$ and $a_{i,n+1} = a_{n+1,i}$ for $1 \leq i \leq n$. Therefore, we have

$$X = \begin{pmatrix} Q & v \\ {}^T v & 0 \end{pmatrix} \text{ such that } Q \in \mathfrak{so}(n); {}^T Q + Q = 0.$$

We have the Cartan involution $\theta(X) = -{}^T X$ on $\mathfrak{so}(n, 1)$. Then, the $+1$ and -1 -eigenspaces are $\mathfrak{so}(n)$ and $\left\{ \begin{pmatrix} 0 & v \\ {}^T v & 0 \end{pmatrix} : v \in \mathbb{R}^n \right\}$, respectively. So, we denote $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p} = \left\{ \begin{pmatrix} 0 & v \\ {}^T v & 0 \end{pmatrix} : v \in \mathbb{R}^n \right\}$ and get the Cartan decomposition $\mathfrak{so}(n, 1) = \mathfrak{so}(n) \oplus \left\{ \begin{pmatrix} 0 & v \\ {}^T v & 0 \end{pmatrix} : v \in \mathbb{R}^n \right\}$. In particular, $SO(n)$ is a maximal compact subgroup in $SO(n, 1)$.

For $1 \leq i < j \leq n$ and $1 \leq k \leq n$, we write $W_{ij} = e_{ij} - e_{ji}$ and $V_k = e_{k,n+1} + e_{n+1,k}$; where, e_{ij} is the elementary matrix with 1 at the ij -th place and 0 everywhere else. Let us choose a maximal abelian subspace $\mathfrak{a} = \mathbb{C}.V_1$ in $\mathfrak{p}$. Then, $Ad_{\mathfrak{a}}$ acts on $\mathfrak{g}$. One can check that the 0-th joint eigenspace $\mathfrak{g}_0$ equals the $\mathbb{C}$ -linear span of $\langle V_1, W_{ij} : 2 \leq i < j \leq n \rangle$. The nonzero

weights are λ and $-\lambda$ such that $\lambda(V_1) = 1$. The corresponding weight spaces are given by $\mathfrak{g}_\lambda = \mathbb{C}.\langle V_j + W_{1,j} : 2 \leq j \leq n \rangle$ and $\mathfrak{g}_{-\lambda} = \mathbb{C}.\langle -V_j + W_{1,j} : 2 \leq j \leq n \rangle$. Therefore, the restricted root space decomposition is

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}.$$

Regarding $\{\lambda\}$ as a positive system, we have the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{g}_\lambda.$$

Let A and N be the closed analytic subgroup for the Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then the group A can be written as :

$$A = \text{Exp}(tV_1) = \left\{ \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Similarly, exponentiating the nilpotent subalgebra $\mathfrak{g}_\lambda$, we have the unipotent group

$$N = \left\{ \begin{pmatrix} 1 - \frac{1}{2} \sum_{j=2}^n t_j^2 & t_2 & t_3 & \cdots & t_n & \frac{1}{2} \sum_{j=2}^n t_j^2 \\ -t_2 & 1 & 0 & \cdots & 0 & t_2 \\ -t_3 & 0 & 1 & \cdots & 0 & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -t_n & 0 & 0 & \cdots & 1 & t_n \\ -\frac{1}{2} \sum_{j=2}^n t_j^2 & t_2 & t_3 & \cdots & t_n & 1 + \frac{1}{2} \sum_{j=2}^n t_j^2 \end{pmatrix} : t_2, t_3, \dots, t_n \in \mathbb{R} \right\}.$$

Finally, by [ER06], we have the Iwasawa decomposition (at the group level) $G = K \times A \times N$;

where, $K = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = SO(n)$ and A, N are as above.

Consider $\mathfrak{a}_r = \{H \in \mathfrak{a} : \lambda(H) \neq 0 \text{ for all } \lambda \in \Sigma\} = \left\{ \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} : c \neq 0 \right\}$. Then, $\mathfrak{a}_r$ has two components; called Weyl chambers. It is well-known that the number of Weyl chambers is same as the number of positive systems in restricted root system. In this case, those are $\{\lambda\}$ and $\{-\lambda\}$. They determine the nilpotent subalgebras $\mathfrak{n}, \bar{\mathfrak{n}}$ and the corresponding analytic subgroups $N, \bar{N}$, respectively. Indeed, $\bar{N}$ is the opposite unipotent subgroup

$$\bar{N} = \left\{ \begin{pmatrix} 1 - \frac{1}{2} \sum_{j=2}^n t_j^2 & -t_2 & -t_3 & \cdots & -t_n & \frac{1}{2} \sum_{j=2}^n t_j^2 \\ t_2 & 1 & 0 & \cdots & 0 & -t_2 \\ t_3 & 0 & 1 & \cdots & 0 & -t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_n & 0 & 0 & \cdots & 1 & -t_n \\ -\frac{1}{2} \sum_{j=2}^n t_j^2 & -t_2 & -t_3 & \cdots & -t_n & 1 + \frac{1}{2} \sum_{j=2}^n t_j^2 \end{pmatrix} : t_2, t_3, \dots, t_n \in \mathbb{R} \right\}.$$

It is easy to check that $M := Z_K(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for some $A \in SO(n-1)$. Moreover, by

Prop. 2.1.6, there are only two proper parabolic subgroups $P = MAN$ and $\bar{P} = M\bar{A}\bar{N}$ in $SO(n, 1)$.

2.2 Spectral decomposition of L^2 spaces : Real

Let G be a connected reductive real Lie group. We fix a Haar measure dg on G . Let Γ be a discrete subgroup in G . The Haar measure dg induces a G -invariant measure (again denoted by) dg on the quotient $\Gamma \backslash G$. We moreover assume that the total volume with respect to dg on $\Gamma \backslash G$ is finite. At first, we consider the space $C^\infty(\Gamma \backslash G)$ consisting of all smooth complex-valued functions f on G satisfying $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$ and $g \in G$ and $\int_{\Gamma \backslash G} |f(g)|^2 dg < \infty$. We can define the L^2 -inner product on $C^\infty(\Gamma \backslash G)$ by $\langle f_1, f_2 \rangle = \int_{\Gamma \backslash G} f_1(g) \overline{f_2(g)} dg$. We moreover take the Hilbert completion of $C^\infty(\Gamma \backslash G)$ with respect to this L^2 -inner product. The following proposition describes the completion.

Proposition 2.2.1. (Prop. 2.1.6 in [Bum97]) *Let $L^2(\Gamma \backslash G) = \{\phi : G \rightarrow \mathbb{C} : \phi \text{ is measurable function ; } \phi(\gamma x) = \phi(x) \text{ for all } \gamma \in \Gamma, x \in G; \int_{\Gamma \backslash G} |\phi(x)|^2 dx < \infty\}$. Then, $C^\infty(\Gamma \backslash G)$ is dense in $L^2(\Gamma \backslash G)$.*

G acts on the quotient $\Gamma \backslash G$ from the right. This induces an action of G on the space $L^2(\Gamma \backslash G)$ defined by

$$\text{for } g \in G, g \cdot \phi(x) = \phi(xg) \text{ for all } x \in G.$$

This induces a continuous representation $R_\Gamma : G \longrightarrow \text{GL}(L^2(\Gamma \backslash G))$, called the right regular representation R_Γ (see Prop. 2.1.7 in [Bum97]). Moreover, the action R_Γ preserves the L^2 -inner product; so it is a unitary representation. The decomposition of this representation R_Γ into irreducible representations is known as the *spectral problem*.

The representation $(R_\Gamma, L^2(\Gamma \backslash G))$ is not completely reducible (also known as semisimple) in general. Let us denote the maximal semisimple subrepresentation of $L^2(\Gamma \backslash G)$ by $L^2_{\text{disc}}(\Gamma \backslash G)$ (known as discrete spectrum). The orthogonal complement of $L^2_{\text{disc}}(\Gamma \backslash G)$ in $L^2(\Gamma \backslash G)$ is denoted by $L^2_c(\Gamma \backslash G)$ (known as continuous spectrum). The following proposi-

tion says that if $\Gamma \backslash G$ is compact then $L_c^2(\Gamma \backslash G) = 0$.

Proposition 2.2.2. *Assume that $\Gamma \backslash G$ is compact. The space $L^2(\Gamma \backslash G)$ decomposes into Hilbert direct sum of subspaces that are invariant and irreducible under the right regular representation R_Γ .*

In other words, we can write

$$L^2(\Gamma \backslash G) \cong \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) \pi;$$

where, $m(\pi, \Gamma)$ is the number of π -isotypic component occurring in $L^2(\Gamma \backslash G)$. Alternatively, $m(\pi, \Gamma) = \dim_{\mathbb{C}} \text{Hom}_G(\pi, L^2(\Gamma \backslash G))$. A central goal of this thesis is to understand the behaviour of these multiplicities $m(\pi, \Gamma)$.

Let us now consider the case when $\Gamma \backslash G$ is non-compact. The continuous spectrum $L_c^2(\Gamma \backslash G)$ possesses a Hilbert direct integral decomposition by the abstract decomposition as in Thm. 14.10.5 in [Wal92]. Explicitly, there exists a Borel measure μ_Γ on $\hat{G}$ such that

$$L_c^2(\Gamma \backslash G) \cong \int_{\hat{G}}^{\oplus} \pi \, d\mu_\Gamma(\pi).$$

For arbitrary Γ , the measure μ_Γ is not known explicitly. However, in Chap. 5, we will see that when Γ is an arithmetic lattice (e.g. congruence lattice) the principal series representations of G occur in the Hilbert direct integral decomposition of the continuous spectrum and the measure μ_Γ is quite simple. In Chap. 4 and Chap. 6, we will focus on the lattices Γ in G with compact quotient $\Gamma \backslash G$ and therefore no continuous spectrum would appear.

If Γ is an arithmetic lattice in G , the discrete spectrum $L_{\text{disc}}^2(\Gamma \backslash G)$ consists of two parts:

Cuspidal Spectrum and Residual spectrum. In other words,

$$L^2(\Gamma \backslash G) = L^2_{\text{cusp}}(\Gamma \backslash G) \oplus L^2_{\text{res}}(\Gamma \backslash G) \oplus L^2_{\text{c}}(\Gamma \backslash G).$$

For instance, the trivial representation of G occurs in the residual spectrum $L^2_{\text{res}}(\Gamma \backslash G)$.

2.3 Spectral decomposition of L^2 spaces : Adelic

Let G be an affine algebraic group over a number field F . Let $\mathbb{A}_F$ be the ring of adèles of F . We consider the locally compact group $G(\mathbb{A}_F)$. Let us denote the adelic quotient $G(F) \backslash G(\mathbb{A}_F)$ by $[G]$. Similar to the real case, there is the right regular action of $G(\mathbb{A}_F)$ on the space of all complex valued smooth functions on $[G]$. As $[G]$ possesses the $G(\mathbb{A}_F)$ -invariant Haar measure, we consider the Hilbert space $L^2([G])$ with respect to the L^2 -inner product defined as in the real case. Therefore, we get the unitary right regular representation $R : G(\mathbb{A}_F) \rightarrow GL(L^2([G]))$. This representation space decomposes (see Thm. 10.5.1 in [GH24]):

$$L^2([G]) = L^2_{\text{cusp}}([G]) \oplus L^2_{\text{res}}([G]) \oplus L^2_{\text{cont}}([G])$$

The explicit description of $L^2_{\text{res}}([G])$ and $L^2_{\text{cusp}}([G])$ are quite involved and we refer the reader to Chap. 10 of [GH24] for details. However, we restrict ourselves to the cuspidal spectrum $L^2_{\text{cusp}}([G])$.

The cuspidal subspace

$$L^2_{\text{cusp}}([G]) \subseteq L^2([G])$$

is the space of $\phi \in L^2([G])$ such that for every proper parabolic subgroup $P < G$ with

unipotent radical N , one has that

$$\int_{N(F) \backslash N(\mathbb{A}_F)} \phi(n g) \, dn = 0$$

for almost every $g \in G(\mathbb{A}_F)$ with respect to the fixed Haar measure on $G(\mathbb{A}_F)$.

Proposition 2.3.1. (Lem. 9.2.1 and Cor. 9.1.3 in [GH24]) *The subspace $L^2_{\text{cusp}}([G])$ is closed and $G(\mathbb{A}_F)$ -subrepresentation of $L^2([G])$. It has a Hilbert direct sum decomposition*

$$L^2_{\text{cusp}}([G]) \cong \bigoplus_{\Pi \in \widehat{G(\mathbb{A}_F)}} L^2_{\text{cusp}}(\Pi) = \bigoplus_{\Pi \in \widehat{G(\mathbb{A}_F)}} m(\Pi) \Pi$$

where, $L^2_{\text{cusp}}(\Pi)$ is the Π -isotypic subspace of $L^2_{\text{cusp}}([G])$.

The isomorphism classes of Π for which a non-zero isotypic component occurs in the above decomposition are called cuspidal automorphic representations.

Let K^∞ be a compact open subgroup of finite adèles $G(\mathbb{A}_F^\infty)$. It gives rise to the congruence subgroup $\Gamma_{K^\infty} = G(F) \cap K^\infty$ inside $G(F_\infty)$. Here, F_∞ is the product of the completion at the archimedean places of F . Assume that G/F has class number one. Then by the strong approximation

$$\Gamma_{K^\infty} \backslash G(F_\infty) \cong G(F) \backslash G(\mathbb{A}_F) / K^\infty.$$

Theorem 2.3.2. (see Thm. 6.6.1 in [GH24]) *For a cuspidal automorphic representation (Π, V) occurring in $L^2_{\text{cusp}}([G])$, there exist unitary irreducible representations (Π_∞, V_∞) and (Π^∞, V^∞) of $G(F_\infty)$ and $G(\mathbb{A}_F^\infty)$ respectively, such that*

$$(\Pi, V) \cong (\Pi_\infty \otimes \Pi^\infty, V_\infty \widehat{\otimes} V^\infty).$$

In particular, (Π^∞, V^∞) is an admissible representation. So, $(V^\infty)^{K^\infty}$ is finite dimensional. Therefore, taking K^∞ -fixed vectors in $L^2_{\text{cusp}}([G])$ we get

$$L^2_{\text{cusp}}(\Gamma_{K^\infty} \backslash G(F_\infty)) \cong \bigoplus_{\Pi \in \widehat{G(\mathbb{A}_F)}} m(\Pi) \dim(\Pi^{K^\infty}) \Pi_\infty.$$

We will use this decomposition in Chap. 5.

2.4 τ -spherical Representation Spectrum

Let G be a connected non-compact semisimple Lie group and let K be a maximal compact subgroup of G with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Then the homogeneous space G/K is a symmetric space with a G -invariant metric. Let (τ, V_τ) be a finite dimensional representation of K . Let Γ be a uniform torsion-free lattice in G .

We consider the right regular representation ρ of G on the space

$$\begin{aligned} L^2(\Gamma \backslash G, V_\tau) := \{ \phi : G \rightarrow V_\tau \mid \phi \text{ is measurable,} \\ \phi(\gamma g) = \phi(g) \text{ for all } \gamma \in \Gamma, \int_{\Gamma \backslash G} |\phi(g)|^2 dg < \infty \}. \end{aligned} \quad (2.4.1)$$

Since $\Gamma \backslash G$ is compact, the right regular representation $L^2(\Gamma \backslash G)$ of G is completely reducible and each irreducible subrepresentation occurs with finite multiplicity. In other words,

$$L^2(\Gamma \backslash G) \cong \widehat{\bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) W_\pi}$$

with $m(\pi, \Gamma) < \infty$ for all $\pi \in \widehat{G}$.

We choose an orthonormal basis for each copy of π . Taking the union of those we get an orthonormal basis $\mathcal{W}$ of $L^2(\Gamma \backslash G)$. Let $\{v_i\}_{i=1}^n$ be an orthonormal basis of V_τ . For any $\psi \in \mathcal{W}$, we let $\psi_i(x) = \psi(x)v_i$. Then $\{\psi_i \mid \psi \in \mathcal{W}, 1 \leq i \leq n\}$ forms an orthonormal basis of $L^2(\Gamma \backslash G, V_\tau)$. Note that $L^2(\Gamma \backslash G, V_\tau) = L^2(\Gamma \backslash G) \otimes V_\tau$.

Therefore, we get

$$L^2(\Gamma \backslash G, V_\tau) \cong \widehat{\bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) (W_\pi \otimes V_\tau)}. \quad (2.4.2)$$

For each copy of $W_\pi \otimes V_\tau$, we identify a subspace V_π in $L^2(\Gamma \backslash G, V_\tau)$. The subspace V_π is not an irreducible G -subspace, rather it is a direct sum of $\dim V_\tau$ many copies of W_π .

We introduce a space $C_c^\infty(G, K, V_\tau)$ defined by

$$\begin{aligned} \{f : G \rightarrow \text{End}(V_\tau) \mid f \text{ is compactly supported and smooth such that} \\ f(kx) = f(x)\tau(k^{-1}) \text{ for all } x \in G, k \in K\}. \end{aligned} \quad (2.4.3)$$

For $f \in C_c^\infty(G, K, V_\tau)$, let $h_f(x) = \text{Trace}(f(x))$ for all $x \in G$. Now for any $f \in C_c^\infty(G, K, V_\tau)$, we define the convolution operator $\rho(f) : L^2(\Gamma \backslash G, V_\tau) \rightarrow L^2(\Gamma \backslash G, V_\tau)$ by

$$\rho(f)\phi(x) = \int_G f(y)(\phi(xy)) dy \quad \text{for all } \phi \in L^2(\Gamma \backslash G, V_\tau). \quad (2.4.4)$$

Proposition 2.4.1. *Each V_π is a $\rho(f)$ -stable subspace for all $f \in C_c^\infty(G, K, V_\tau)$.*

Proof. Let $\mathcal{U} \subset \mathcal{W}$ be the subset such that $\mathcal{U}$ is an orthonormal basis of W_π . Then $\{\phi_i : \phi \in \mathcal{U}, 1 \leq i \leq n\}$ is an orthonormal basis of V_π . Thus, it suffices to show that

$$\langle \rho(f)\phi_i, \psi_j \rangle = 0 \quad \text{for any } \phi \in \mathcal{U}, \psi \in \mathcal{W} \setminus \mathcal{U} \text{ and any } 1 \leq i, j \leq n.$$

We compute,

$$\begin{aligned}
\langle \rho(f)\phi_i, \psi_j \rangle &= \int_{\Gamma \backslash G} \langle \rho(f)\phi_i(g), \psi_j(g) \rangle dg \\
&= \int_{\Gamma \backslash G} \left\langle \int_G \phi(gy) f(y) v_i, \psi(g) v_j \right\rangle dy dg \\
&= \int_{\Gamma \backslash G} \int_G \phi(gy) \overline{\psi(g)} \langle f(y) v_i, v_j \rangle dy dg \tag{2.4.5} \\
&= \int_G \langle f(y) v_i, v_j \rangle \int_{\Gamma \backslash G} \phi(gy) \overline{\psi(g)} dg dy \\
&= 0.
\end{aligned}$$

The last equality is due to the fact that ψ is orthogonal to any R_Γ -translates of ϕ in $L^2(\Gamma \backslash G)$. This completes the proof. $\square$

Chapter 3

Trace Formula

3.1 Selberg Trace Formula for uniform lattice

Let Γ be a uniform lattice in a semisimple Lie group G . We have the right regular representation R_Γ on the space $L^2(\Gamma \backslash G)$ defined by $(R_\Gamma(g)(\varphi))(x) = \varphi(xg)$ for all $\varphi \in L^2(\Gamma \backslash G)$ and $x, g \in G$. In Sec [2.2](#), we have seen that the representation space $L^2(\Gamma \backslash G)$ possesses a discrete direct sum decomposition into irreducible representations of G with finite multiplicity. Let us fix a Haar measure dg on G . This induces a right G -invariant measure on the compact quotient $\Gamma \backslash G$.

Let us denote $C_c^\infty(G)$ to be the algebra of compactly supported smooth functions on G . Here, the algebra structure is with respect to the convolution operation : for any $f, g \in C_c^\infty(G)$, $(f \star g)(x) = \int_G f(xy^{-1})g(y)dy$. For any $f \in C_c^\infty(G)$, we have the convolution operator $R_\Gamma(f)$ on $L^2(\Gamma \backslash G)$, defined by

$$R_\Gamma(f)(\varphi)(x) = \int_G f(y)\varphi(xy)dy \quad \text{for all } \varphi \in L^2(\Gamma \backslash G) \text{ and } x \in G.$$

Indeed, for any $\varphi \in L^2(\Gamma \backslash G)$, $R_\Gamma(f)(\varphi)$ is left Γ -invariant and belongs to $L^2(\Gamma \backslash G)$.

We can rewrite this as :

$$R_\Gamma(f)(\varphi)(x) = \int_{\Gamma \backslash G} \left[\sum_{\gamma \in \Gamma} f(x^{-1}\gamma y) \right] \varphi(y) dy$$

Note that the sum $K_f(x, y) := \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$ is a finite sum because the test function f is compactly supported. Therefore, the operator $R_\Gamma(f)$ is a Hilbert-Schmidt operator with kernel $K_f(x, y) := \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$. Moreover, it is a trace class operator and its trace is given by

$$\text{Trace}(R_\Gamma(f)) = \int_{\Gamma \backslash G} \left[\sum_{\gamma \in \Gamma} f(x^{-1}\gamma x) \right] dx.$$

Now the right-hand side of the above equation can be expressed in terms of *orbital integrals*. To do this, we introduce a few notations. Let $[\gamma]_G$ and $[\gamma]_\Gamma$ be the conjugacy classes of γ in G and Γ , respectively. Let $[\Gamma]$ (resp. $[\Gamma]_G$) be the set of conjugacy classes in Γ (resp. G -conjugacy classes of elements in Γ). We denote by G_γ the centralizer of γ in G . We write $\Gamma_\gamma := \Gamma \cap G_\gamma$. It can be seen easily that Γ_γ is a uniform lattice in G_γ (see [Wal76]). Also, since any $\gamma \in \Gamma$ is a semisimple element the centralizer subgroups G_γ are unimodular. So, we have a G -invariant measure $d_\gamma x$ on $G_\gamma \backslash G$.

We write the right-hand side as

$$\sum_{[\gamma] \in [\Gamma]} \int_{\Gamma \backslash G} \left(\sum_{\gamma' \in [\gamma]} f(x^{-1}\gamma' x) \right) dx.$$

Let us select a conjugacy class $[\gamma]$ in $[\Gamma]$. The conjugacy class $[\gamma]$ consists of the elements $\gamma' = \gamma''^{-1}\gamma\gamma''$; where, γ'' runs over Γ . Ranging γ'' over Γ , each such γ' might be obtained

several times. In fact, two elements γ_1'' and γ_2'' determine the same γ' if and only if $\gamma_1''\gamma_2''^{-1}$ belongs to Γ_γ . So we have,

$$\sum_{\gamma' \in [\gamma]} f(x^{-1}\gamma'x) = \sum_{\gamma'' \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\gamma''^{-1}\gamma\gamma''x).$$

Consequently,

$$\int_{\Gamma \backslash G} \sum_{\gamma' \in [\gamma]} f(x^{-1}\gamma'x)dx = \int_{\Gamma \backslash G} \sum_{\gamma'' \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\gamma''^{-1}\gamma\gamma''x)dx = \int_{\Gamma_\gamma \backslash G} f(x^{-1}\gamma x)dx.$$

Furthermore,

$$\begin{aligned} \int_{\Gamma_\gamma \backslash G} f(x^{-1}\gamma x)dx &= \int_{G_\gamma \backslash G} \int_{\Gamma_\gamma \backslash G_\gamma} f(x^{-1}x'^{-1}\gamma x'x)dx'dx, \\ &= \int_{G_\gamma \backslash G} \int_{\Gamma_\gamma \backslash G_\gamma} f(x^{-1}\gamma x)dx'dx, \\ &= \text{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx. \end{aligned}$$

Therefore, we obtain the trace as

$$\text{Trace}(R_\Gamma(f)) = \sum_{[\gamma] \in [\Gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx.$$

For our purpose, we want the outer sum ranging over the G -conjugacy classes of elements

of Γ . So we write,

$$\text{Trace} (R_\Gamma(f)) = \sum_{[\gamma] \in [\Gamma]_G} a(\gamma, \Gamma) \mathcal{O}_\gamma(f),$$

where,

$$a(\gamma, \Gamma) := \sum_{[\gamma']_\Gamma \subset [\gamma]_G} \text{vol} (\Gamma_{\gamma'} \backslash G_{\gamma'}), \text{ and } \mathcal{O}_\gamma(f) := \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx.$$

Recall from Sec. [2.2](#) that R_Γ possesses a spectral decomposition

$$R_\Gamma \cong \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) \pi.$$

So, by convoluting with test functions $f \in C_c^\infty(G)$ we have,

$$\text{Trace} (R_\Gamma(f)) = \sum_{\pi \in \widehat{G}} m(\pi, \Gamma) \text{Trace} (\pi(f)).$$

Hence, the final form of the Selberg Trace formula for compact quotients takes the following form :

$$\sum_{\pi \in \widehat{G}} m(\pi, \Gamma) \text{Trace} (\pi(f)) = \sum_{[\gamma] \in [\Gamma]_G} a(\gamma, \Gamma) \mathcal{O}_\gamma(f) \quad \text{for all } f \in C_c^\infty(G).$$

3.2 Selberg trace formula for $L^2(\Gamma \backslash G, V_\tau)$

Recall $\{\psi_j : \psi \in \mathcal{W}, j \in 1, \dots, n\}$ is an orthonormal basis of $L^2(\Gamma \backslash G, V_\tau)$ (recall [2.4.1](#)).

For any $f \in C_c^\infty(G, K, V_\tau)$ (recall [2.4.3](#)), let $[f_{ij}]_{n \times n}$ be the matrix representation of f with respect to the basis $\{v_i\}_{i=1}^n$ of V_τ . For $f \in C_c^\infty(G, K, V_\tau)$, observe that,

$$\begin{aligned}
\rho(f)\psi_j(x) &= \int_G f(x^{-1}y)(\psi_j(y)) dy \\
&= \int_G f(x^{-1}y)(\psi(y)v_j) dy \\
&= \sum_i \int_G \psi(y)f_{ij}(x^{-1}y)v_i dy.
\end{aligned} \tag{3.2.1}$$

Then, we have

$$\begin{aligned}
\text{Trace } (\rho(f)) &= \sum_{\psi \in \mathcal{W}} \sum_j \langle \rho(f)\psi_j, \psi_j \rangle \\
&= \sum_{\psi \in \mathcal{W}} \sum_j \int_{\Gamma \backslash G} \langle \sum_i \int_G \psi(y)f_{ij}(x^{-1}y)v_i dy, \psi(x)v_j \rangle dx \\
&= \sum_{\psi \in \mathcal{W}} \sum_j \int_{\Gamma \backslash G} \int_{\Gamma \backslash G} \psi(y) \psi(x) \sum_{\gamma \in \Gamma} \langle \sum_i f_{ij}(x^{-1}\gamma y)v_i, v_j \rangle dy dx \\
&= \sum_{\psi \in \mathcal{W}} \int_{\Gamma \backslash G \times \Gamma \backslash G} \sum_{\gamma \in \Gamma} \text{Trace } (f(x^{-1}\gamma y)) \overline{\psi(x)} \psi(y) dy dx \\
&= \sum_{\psi \in \mathcal{W}} \int_{\Gamma \backslash G \times \Gamma \backslash G} \sum_{\gamma \in \Gamma} h_f(x^{-1}\gamma y) \overline{\psi(x)} \psi(y) dy dx.
\end{aligned} \tag{3.2.2}$$

Put $K_f(x, y) = \sum_{\gamma \in \Gamma} h_f(x^{-1}\gamma y)$. Above equals

$$\sum_{\psi \in \mathcal{W}} \int_{\Gamma \backslash G \times \Gamma \backslash G} K_f(x, y) \overline{\psi(x)} \psi(y) dy dx. \tag{3.2.3}$$

Consider the integral operator T on $L^2(\Gamma \backslash G)$ defined by

$$T\psi(x) = \int_{\Gamma \backslash G} K_f(x, y) \psi(y) dy. \quad (3.2.4)$$

Then T is a trace class operator and its trace is given by

$$\text{Trace}(T) = \sum_{\psi \in \mathcal{W}} \int_{\Gamma \backslash G \times \Gamma \backslash G} K_f(x, y) \overline{\psi(x)} \psi(y) dy dx. \quad (3.2.5)$$

On the other hand the trace of such an operator T equals $\int_{\Gamma \backslash G} K_f(x, x) dx$. Therefore,

$$\text{Trace}(\rho(f)) = \int_{\Gamma \backslash G} K_f(x, x) dx = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} h_f(x^{-1} \gamma x) dx.$$

From Prop. [2.4.1](#), we have

$$\text{Trace}(\rho(f)) = \sum_{\pi \in \widehat{G}} m(\pi, \Gamma) \text{Trace}(\rho(f)|_{V_\pi}).$$

Now,

$$\begin{aligned} \text{Trace}(\rho(f)|_{V_\pi}) &= \sum_j \sum_{\psi \in \mathcal{W} \cap W_\pi} \langle \rho(f) \psi_j, \psi_j \rangle \\ &= \sum_{\psi \in \mathcal{W} \cap W_\pi} \int_{\Gamma \backslash G} \int_G \psi(xy) \overline{\psi(x)} \text{Trace}(f(y)) dy dx \\ &= \text{Trace}(\pi(h_f)). \end{aligned} \quad (3.2.6)$$

Therefore, we have the following *Selberg Trace formula*:

$$\begin{aligned}
\sum_{\pi \in \widehat{G}} m(\pi, \Gamma) \operatorname{Trace} (\pi(h_f)) &= \int \sum_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} h_f(x^{-1}\gamma x) dx \\
&= \sum_{[\gamma] \in [\Gamma]_G} \operatorname{vol} (\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} h_f(x^{-1}\gamma x) dx \\
&= \sum_{[\gamma] \in [\Gamma]_G} a(\gamma, \Gamma) O_\gamma(h_f),
\end{aligned} \tag{3.2.7}$$

where $a(\gamma, \Gamma)$ and $O_\gamma(h_f)$ denote $\operatorname{vol} (\Gamma_\gamma \backslash G_\gamma)$ and $\int_{G_\gamma \backslash G} h_f(x^{-1}\gamma x) dx$ respectively.

Remark 3.2.1. *This trace formula is a generalisation of the well-known Selberg trace formula. The explicit description of the ‘geometric side’ is given in Subsec. [3.1](#).*

3.3 Simple trace formula for congruence lattices in rank one groups

In this subsection, we discuss a simple trace formula for congruence lattices derived from the Arthur-Selberg Trace formula.

Let G be a semisimple simply connected linear algebraic group over a number field $\mathbb{Q}$. We keep the example of $\mathrm{SL}(2)$ in mind, for simplicity. Let $\mathbb{A}$ be the ring of adeles over $\mathbb{Q}$. Then $\mathbb{A} = \mathbb{A}_{fin} \times \mathbb{R}$; where, $\mathbb{A}_{fin}$ is the ring of finite adeles. Moreover, $G(\mathbb{A}) \cong G(\mathbb{A}_{fin}) \otimes G(\mathbb{R})$. Let $\Gamma \subset G(\mathbb{R})$ be a congruence subgroup, i.e., there is a compact open subgroup K_Γ of $G(\mathbb{A}_{fin})$ such that $\Gamma = G(\mathbb{Q}) \cap K_\Gamma$. Using strong approximation we have the following identification

$$\Gamma \backslash G(\mathbb{R}) \longrightarrow G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\Gamma,$$

given by $\Gamma g \mapsto G(\mathbb{Q})gK_\Gamma$. We further assume that Γ is weakly neat i.e. Γ is torsion free and for any $\gamma \in \Gamma$ the adjoint action $Ad(\gamma)$ on the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$ has no roots of unity other than 1 as eigenvalue. For a given congruence subgroup Γ_0 , it is always possible to find a weakly neat subgroup Γ of Γ_0 of finite index.

Recall from Subsec. [2.3](#) that we have the right regular representation R of $G(\mathbb{A})$ on the Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Let $C_c^\infty(G(\mathbb{A}))$ denote the algebra consisting of compactly supported smooth complex functions on $G(\mathbb{A})$. For any $f \in C_c^\infty(G(\mathbb{A}))$, we have the convolution operator $R(f)$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ defined as:

$$\text{for any } \varphi \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) \quad R(f)(\varphi)(x) = \int_{G(\mathbb{A})} f(y)\varphi(yx)dy \quad \text{for all } x \in G(\mathbb{Q})\backslash G(\mathbb{A}).$$

When $G(\mathbb{Q})\backslash G(\mathbb{A})$ is noncompact, this convolution operator is not trace class operator. Therefore, the kernel function $K_f(x)$ of the operator $R(f)$ defined by

$$x \mapsto \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma x);$$

is not absolutely integrable over $G(\mathbb{Q})\backslash G(\mathbb{A})$. The main reason behind this nonconvergence is the presence of unipotent elements in $G(\mathbb{Q})$ i.e. nontrivial parabolic subgroups intersect with conjugacy classes in $G(\mathbb{Q})$. In fact, for a general group G over $\mathbb{Q}$, the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ is noncompact if and only if G contains a proper parabolic subgroup defined over $\mathbb{Q}$. The reader is referred to ([GH24](#), Sec. 1.9) for the details of the parabolic subgroups.

Our idea is to put some conditions on our test function $f \in C_c^\infty(G(\mathbb{A}))$ to make the kernel $K(x, x)$ converge. Consider the function $f = f_\infty \otimes 1_{K_\Gamma}$ in $C_c^\infty(G(\mathbb{A}))$; where, $f_\infty \in C_c^\infty(G(\mathbb{R}))$ and $1_{K_\Gamma} = \frac{1}{\text{vol}(K_\Gamma)} \mathbf{1}$, the characteristic function of the compact open subgroup

K_Γ divided by its volume.

Now we assume that for any $\mathbb{Q}$ -parabolic subgroup $P = MAN \neq G$, $f_\infty(x^{-1}qx) = 0$ for all $q \in MN$ and $x \in G$. The elements in MN for a parabolic subgroup P are called P -singular. Therefore, we assume that f_∞ vanishes on all $\mathbb{Q}$ -parabolically singular elements. Recall that an element is called regular if its centralizer is a torus. Let G_{reg} be the set of all regular elements in G . If an element is regular then it is not P -singular for all $P \neq G$. Hence, our assumption on f_∞ tells that f_∞ is supported in G_{reg} .

In [Art05], Arthur replaces $k_f(x)$ by a modified kernel function to make the geometric side of the trace formula converge. We do not recall the Trace formula by Arthur in full generality here and refer the reader to [Art05]. However, from [Dei99] one can see that for the test functions $f \in C_c^\infty(G(\mathbb{A}))$ satisfying the above assumptions, the geometric side of Arthur Trace formula reduces to a simpler version.

To state the result in [Dei99], we recall a few notations from sec. 3.1. For any $\gamma \in \Gamma$, the centralizer of γ in Γ and $G(\mathbb{R})$ are denoted by Γ_γ and $G(\mathbb{R})_\gamma$, respectively. Also, for any γ , $[\gamma]$ denotes the conjugacy class of γ in Γ . For any $f_\infty \in C_c^\infty(G(\mathbb{R}))$, we have the orbital integral

$$\mathcal{O}_\gamma(f_\infty) := \int_{G(\mathbb{R})_\gamma \backslash G(\mathbb{R})} f_\infty(x^{-1}\gamma x) d_\gamma x.$$

Now we state Thm. 1.12 from [Dei99] as follows.

Theorem 3.3.1. *Let $f \in C_c^\infty(G(\mathbb{A}))$ be of the form $f := f_\infty \otimes \frac{1}{\text{vol } K_\Gamma} \mathbf{1}$; where, $\mathbf{1}$ is the characteristic function of K_Γ and $f_\infty \in C_c^\infty(G(\mathbb{R}))$ is supported in $G(\mathbb{R})_{\text{reg}}$. Then we have,*

$$\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} K_f(x) dx = \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G(\mathbb{R})_\gamma) \mathcal{O}_\gamma(f_\infty);$$

where, the sum in the right hand side runs over all conjugacy classes $[\gamma]$ in Γ .

Now we move to the spectral side of the trace formula. To describe it we need to recall the principal series representations of $G(\mathbb{A})$ and certain intertwining operators acting on it.

Since G is a rank one group, there is, up to conjugation, only one parabolic subgroup P over $\mathbb{Q}$ in G . The adelic parabolic subgroup $P(\mathbb{A})$ possesses the Langlands decomposition $N(\mathbb{A})L(\mathbb{A})^1A(\mathbb{A})^0$. Moreover, we have the Iwasawa decomposition $G(\mathbb{A}) = N(\mathbb{A})L(\mathbb{A})^1A(\mathbb{A})^0K$ for a maximal compact open subgroup K in $G(\mathbb{A})$. Let $\mathfrak{a}_{L,\mathbb{C}} := X^*(L) \otimes_{\mathbb{Z}} \mathbb{C}$. Since G is of rank one, the quotient $L(\mathbb{Q}) \backslash L(\mathbb{A})^1$ is compact. We have the right regular representation $L^2(L(\mathbb{Q}) \backslash L(\mathbb{A})^1)$ of $L(\mathbb{A})$. Now for any $\lambda \in \mathfrak{a}_{L,\mathbb{C}}$ we consider the induced representation $Ind_{P(\mathbb{A})}^{G(\mathbb{A})}(L^2(L(\mathbb{Q}) \backslash L(\mathbb{A})^1) \otimes \lambda \otimes 1)$ of $G(\mathbb{A})$. We denote it by I_λ . Indeed, the principal series representation I_λ is unitary if and only if $\Re(\lambda) = 0$. Let us denote the Hilbert space for the representation I_λ of $G(\mathbb{A})$ by H_P . There exists an intertwining operator $M(\lambda) : I_\lambda \rightarrow I_{-\lambda}$ whose precise description is omitted in this thesis; the reader is referred to [Dei99].

Let $Pr_{K_\Gamma} : H_P \longrightarrow H_P^{K_\Gamma}$ be the projection onto the space of K_Γ -fixed vectors. We denote by $M(\lambda)^\Gamma$ the composition $Pr_{K_\Gamma} \circ M(\lambda)^\Gamma \circ Pr_{K_\Gamma}$. Let $L^2_{\text{disc}}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ be the maximal semisimple subrepresentation of $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. Then $G(\mathbb{R})$ naturally acts on the subspace $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))^{K_\Gamma}$ of K_Γ -fixed vectors in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. For a given $\pi \in \widehat{G(\mathbb{R})}$ let us denote the set $\{\Pi \in L^2_{\text{disc}}(G(\mathbb{Q}) \backslash G(\mathbb{A})) : \Pi_\infty = \pi\}$ by $\mathcal{A}(\pi)$. Then the representation space $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))^{K_\Gamma}$ of $G(\mathbb{R})$ decomposes as

$$\bigoplus_{\pi \in \widehat{G(\mathbb{R})}} m(\pi, \Gamma) \pi;$$

where, $m(\pi, \Gamma) = \sum_{\Pi \in \mathcal{A}(\pi)} m(\Pi) \dim(\Pi^{K_\Gamma})$. Here, Π^{K_Γ} denotes the space of K_Γ -fixed vectors in the representation space of Π , which is indeed finite dimensional. If $\mathcal{A}(\pi)$ is empty then we conventionally set $m(\pi, \Gamma) = 0$. Now we can state the simple trace formula following Thm. 1.19 of [Dei99].

Theorem 3.3.2. *Let $f \in C_c^\infty(G(\mathbb{A}))$ is of the form $f := f_\infty \otimes \frac{1}{\text{vol } K_\Gamma} \mathbf{1}$; where, $\mathbf{1}$ is the characteristic function of K_Γ and $f_\infty \in C_c^\infty(G(\mathbb{R}))$ is supported in $G(\mathbb{R})_{\text{reg}}$. Then we have,*

$$\begin{aligned} \sum_{\pi \in \widehat{G(\mathbb{R})}} m(\pi, \Gamma) \pi - \frac{1}{4} \text{Tr}(M^\Gamma(0) I_0(f)) + \frac{1}{4\pi i} \int_{\Re(\lambda)=0} \text{Tr} \left((M(\lambda)^\Gamma)^{-1} \frac{d}{d\lambda} (M(\lambda)^\Gamma) I_\lambda(f) \right) d\lambda \\ = \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G(\mathbb{R})_\gamma) \mathcal{O}_\gamma(f_\infty), \end{aligned}$$

where, the sum in the right-hand side runs over all conjugacy classes $[\gamma]$ in Γ .

Remark 3.3.1. *We actually need merely the infinity part $M(\lambda)^\Gamma$ of the adelic intertwining operator $M(\lambda)$. In Chap. 5 we will see how the Warner-Osborn trace formula is reminiscent of the above simple trace formula. This will be used in proving the main Thm. 5.1.5 in Chap.*

5

Chapter 4

Isospectrality of Locally Symmetric Spaces

4.1 Statements of Main Results

4.1.1 τ -Isospectrality and Representation Equivalence

Let G be a connected non-compact semisimple Lie group with finite center. Let K be a maximal compact subgroup in G . Then G/K is a symmetric space which carries a G -invariant Riemannian metric induced by the $\text{Ad}(G)$ -invariant inner product on the Lie algebra $\mathfrak{g}$ of G . The subgroup K acts on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. Let $\mathfrak{U}(\mathfrak{g})^K$ denote the subalgebra of its K -invariants. For any finite dimensional complex representation (τ, V_τ) of K , one has the homogeneous vector bundle E_τ on G/K (see Subsection [4.2.4](#) for the details) whose smooth sections are given by the space

$$\mathcal{A}^\infty(G/K, \tau) := \{\phi : G \rightarrow V_\tau \mid \text{smooth, } \phi(gk) = \tau(k^{-1})(\phi(g)) \text{ for all } k \in K, g \in G\}.$$

This space $\mathcal{A}^\infty(G/K, \tau)$ is $\mathfrak{U}(\mathfrak{g})^K$ -stable. In particular, the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra acts on $\mathcal{A}^\infty(G/K, \tau)$, and the Casimir element in the center induces a second order elliptic differential operator Δ_τ on $\mathcal{A}^\infty(G/K, \tau)$. When τ is the trivial representation of K , Δ_τ coincides with the Laplace-Beltrami operator acting on the smooth functions on G/K .

Let Γ be a uniform lattice in G . Then $X_\Gamma := \Gamma \backslash G/K$ is a compact locally symmetric space which is a manifold if Γ acts freely on G/K . The space X_Γ has a Riemannian metric induced from G/K . We denote $V_{\Gamma, \tau}$ to be the space of all Γ -invariant smooth sections in $\mathcal{A}^\infty(G/K, \tau)$. Then $V_{\Gamma, \tau}$ is stable under the action of $\mathfrak{U}(\mathfrak{g})^K$, in particular of the center $\mathfrak{Z}(\mathfrak{g})$. The Casimir element C induces the second order self-adjoint elliptic operator $\Delta_{\tau, \Gamma}$ on $V_{\Gamma, \tau}$, which has non-negative discrete spectrum of eigenvalues with finite multiplicities. The multiset of eigenvalues with multiplicity is denoted by $\text{Spec}(\Delta_{\tau, \Gamma})$. Such spectrum of a locally symmetric space is closely related to the multiplicity of irreducible representations occurring in the right regular representation $L^2(\Gamma \backslash G)$ of G . Here we mention the famous Matsushima-Murakami formula (see [Mat68], see also [LMR15] Prop. 2.4):

$$\text{Mult}_{\Delta_{\tau, \Gamma}}(\lambda) = \sum_{\pi \in \widehat{G}, \pi(C) = \lambda} m(\pi, \Gamma) \dim(\text{Hom}_K(\tau^\vee, \pi|_K)), \quad (4.1.1)$$

where, $\tau^\vee$ is the dual representation of τ and $m(\pi, \Gamma)$ is the multiplicity of π in the right regular representation $L^2(\Gamma \backslash G)$.

Let $\widehat{G}_\tau = \{\pi \in \widehat{G} : \text{Hom}_K(\tau, \pi|_K) \neq 0\}$. Two uniform lattices Γ_1 and Γ_2 are called τ -representation equivalent if $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all $\pi \in \widehat{G}_\tau$. From Eq. 4.1.1 it is easily observed that if $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all $\pi \in \widehat{G}_{\tau^\vee}$, then $\text{Spec}(\Delta_{\tau, \Gamma_1}) = \text{Spec}(\Delta_{\tau, \Gamma_2})$ for X_{Γ_1} and X_{Γ_2} , respectively. Therefore, if two lattices Γ_1 and Γ_2 are $\tau^\vee$ -representation equiv-

alent, then the corresponding elliptic operators have the same spectrum for the locally symmetric spaces X_{Γ_1} and X_{Γ_2} , respectively, i.e. they are τ -isospectral (see [LMR15, Prop. 2.5]).

Definition 4.1.2. *Two co-compact lattices Γ_1 and Γ_2 are called almost- τ -representation equivalent if $m(\pi, \Gamma_1)$ is equal to $m(\pi, \Gamma_2)$ for all but finitely many $\pi \in \widehat{G}_\tau$.*

Now we state one of the main results in Chap. 4.

Theorem 4.1.3. *Let G be a non-compact connected semisimple Lie group with finite center. If for two uniform torsion free lattices Γ_1 and Γ_2 ,*

$$m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$$

for all but finitely many $\pi \in \widehat{G}_\tau$, then Γ_1 and Γ_2 are τ -representation equivalent lattices.

Remark 4.1.4. *Consequently, the above hypothesis implies that X_{Γ_1} and X_{Γ_2} are $\tau^\vee$ -isospectral.*

4.1.5 Infinitesimal τ -Isospectrality

For an infinitesimal character χ of $\mathfrak{Z}(\mathfrak{g})$, let $V_{\chi, \Gamma, \tau} = \{\phi \in V_{\Gamma, \tau} \mid z \cdot \phi = \chi(z)\phi \text{ for all } z \in \mathfrak{Z}(\mathfrak{g})\}$.

We define a refinement of the notion of τ -isospectrality as follows:

Definition 4.1.6. *Two locally symmetric spaces X_{Γ_1} and X_{Γ_2} of non-compact type are infinitesimally τ -isospectral if*

$$\dim V_{\chi, \Gamma_1, \tau} = \dim V_{\chi, \Gamma_2, \tau} \text{ for all } \chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}.$$

Let $[\chi]$ be the set of all irreducible representations of G which have infinitesimal character χ . It is known that $[\chi]$ is a finite subset of $\widehat{G}$ (see [Kna01] Cor. 10.37). We have obtained the following variant of Matsushima-Murakami formula:

Theorem 4.1.7. *Let G be a connected non-compact semisimple Lie group. Assume that Γ is a uniform lattice in G . Then for any $\chi \in \widehat{\mathfrak{z}(\mathfrak{g})}$ and for any finite dimensional representation τ of K ,*

$$\dim V_{\chi, \Gamma, \tau} = \sum_{\pi \in [\chi]} m(\pi, \Gamma) \dim(\text{Hom}_K(\tau^\vee, \pi|_K)).$$

Remark 4.1.8. *For a rank one semisimple Lie group G , an infinitesimal character $\chi \in \widehat{\mathfrak{z}(\mathfrak{g})}$ is completely determined by its value on the Casimir element. Therefore, the above result becomes the earlier mentioned Matsushima-Murakami formula Eq. 4.1.1*

Remark 4.1.9. *Let $\tau = \tau_p$ be the p -th exterior power of the adjoint representation of K on $\mathfrak{p}_{\mathbb{C}}^*$. The associated homogeneous vector bundle is identified with the p -th exterior product of the co-tangent bundle on G/K . In [Mat67], Matsushima proved the relationship between the dimension of harmonic p -forms on X_Γ and the multiplicity of irreducible representations in $L^2(\Gamma \backslash G)$ that occur with a non-zero τ_p -isotypic component. Therefore Thm. 4.1.7 can be seen as an infinitesimal version of Eq. 4.1.1*

Corollary 4.1.1. *If Γ_1 and Γ_2 are $\tau^\vee$ -representation equivalent then the spaces X_{Γ_1} and X_{Γ_2} are infinitesimally τ -isospectral.*

Remark 4.1.10. *If τ is trivial, the above formula in Thm. 4.1.7 reduces to [BR11] Thm. 1.3].*

Remark 4.1.11. *The finite set $[\chi]$ of all irreducible representations of a real reductive group (informally a χ -packet) is quite close to Langlands L -packet that consists of irreducible admissible representations having the same L -parameter. It is known that the irreducible rep-*

representations from a fixed L -packet have the same infinitesimal character i.e., they cannot be distinguished by the spectral data. To our knowledge, the converse of this is not known in general. It is interesting to formulate a suitable variant of Matsushima-Murakami formula in terms of L -packet instead, and that might reflect some analogous relationship between the multiplicities of irreducible representations in a given L -packet and isospectrality.

4.2 Preliminaries

4.2.1 Results on Harish-Chandra characters

We will be using the following two important results on *Harish-Chandra characters*:

Harish-Chandra character distribution: Let (π, W_π) be an irreducible unitary representation of G . Let $C_c^\infty(G)$ be the space of all compactly supported smooth functions on G . For any $f \in C_c^\infty(G)$, the convolution operator $\pi(f)$ on W_π is defined by $\pi(f)v = \int_G f(g)\pi(g)v \, dg$ for all $v \in W_\pi$. It is a trace class operator (see [Kna01, Thm. 10.2]) and

$$\chi_\pi(f) = \text{Trace}(\pi(f)) \text{ for all } f \in C_c^\infty(G).$$

Theorem 4.2.2. *Let $\{\pi_i\}$ be a finite collection of mutually inequivalent unitary irreducible representations of G . Then their characters $\{\chi_{\pi_i}\}$ are linearly independent distributions on G .*

Proof. Reader is referred to [Kna01, Thm. 10.6]. □

Theorem 4.2.3. *Let π be an irreducible unitary representation of G . Then the distribution*

character χ_π is given by a locally integrable function ϕ_π on G i.e. for $f \in C_c^\infty(G)$,

$$\chi_\pi(f) = \int_G f(g) \phi_\pi(g) dg.$$

Moreover, the restriction of ϕ_π to the regular set of G is a real analytic function that is invariant under conjugation.

Proof. Reader is referred to [Kna01, Thm. 10.25]. □

4.2.4 Isospectrality and representation equivalence

There is a natural homogeneous vector bundle E_τ on G/K that is associated with the representation (τ, V_τ) of K . The space of all smooth global sections of E_τ can be realized as $\mathcal{A}^\infty(G/K, \tau) = \{\phi : G \rightarrow V_\tau \mid \phi \text{ is smooth, } \phi(xk) = \tau(k^{-1})(\phi(x)) \text{ for all } x \in G, k \in K\}$. Note that $\mathcal{A}^\infty(G/K, \tau)$ is a $\mathfrak{U}(\mathfrak{g})^K$ module. In particular, $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})^K$ acts on $\mathcal{A}^\infty(G/K, \tau)$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$. We choose a basis $\{X_i\}$ and $\{Y_j\}$ of $\mathfrak{k}$ and $\mathfrak{p}$, respectively with respect to a bilinear form B on $\mathfrak{g}$ induced from the Killing form such that $B(X_k, X_l) = -\delta_{kl}$ and $B(Y_m, Y_n) = \delta_{mn}$. Let C be the Casimir element given by $C = -\sum X_i^2 + \sum Y_j^2$. The Casimir element C induces a second order symmetric elliptic operator Δ_τ on $\mathcal{A}^\infty(G/K, \tau)$.

Let Γ be a uniform torsion-free lattice in G , and let $X_\Gamma = \Gamma \backslash G/K$ be the associated compact locally symmetric space of non-compact type. We consider the vector bundle $E_{\tau, \Gamma}$ on X_Γ defined by the relation $[\gamma g, w] \sim [g, w]$ for all $\gamma \in \Gamma$ and $[g, w] \in E_\tau$. The space of all smooth global sections of $E_{\tau, \Gamma}$ can be realized as the space $V_{\Gamma, \tau} = \{\phi \in \mathcal{A}^\infty(G/K, \tau) \mid \phi(\gamma x) = \phi(x) \text{ for all } \gamma \in \Gamma\}$. Hence, the center $\mathfrak{Z}(\mathfrak{g})$ acts on $V_{\Gamma, \tau}$.

Let $\Delta_{\tau,\Gamma} = \Delta_{\tau}|_{V_{\Gamma,\tau}}$. This is again a second order symmetric elliptic operator on X_{Γ} . Its spectrum $\text{Spec}(\Delta_{\tau,\Gamma})$ is a discrete subset of the non-negative real numbers. Let $\text{Mult}_{\Delta_{\tau,\Gamma}}(\lambda) =$ the multiplicity of $\lambda \in \text{Spec}(\Delta_{\tau,\Gamma})$. Recall Eq. [4.1.1](#)

$$\text{Mult}_{\Delta_{\tau,\Gamma}}(\lambda) = \sum_{\pi \in \widehat{G}, \pi(C)=\lambda} m(\pi, \Gamma) \dim(\text{Hom}_K(\tau^{\vee}, \pi|_K)),$$

where $\pi(C)$ is the scalar by which the Casimir element C acts on W_{π} .

Definition 4.2.5. X_{Γ_1} and X_{Γ_2} are called τ -isospectral if the operators Δ_{τ,Γ_1} and Δ_{τ,Γ_2} have the same spectrum (with multiplicity).

We denote $\widehat{G}_{\tau} = \{\pi \in \widehat{G} : \text{Hom}_K(\tau, \pi|_K) \neq 0\}$.

Definition 4.2.6. Γ_1 and Γ_2 are called τ -representation equivalent if $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all $\pi \in \widehat{G}_{\tau}$.

It is clear from Eq. [4.1.1](#) that, if Γ_1 and Γ_2 are $\tau^{\vee}$ -representation equivalent, then X_{Γ_1} and X_{Γ_2} are τ -isospectral.

4.2.7 A further refinement

We have seen that $\mathfrak{Z}(\mathfrak{g})$ acts on $V_{\Gamma,\tau}$ (see Subsection [4.1.1](#)). For any character $\chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}$, we have the χ -eigenspace defined as

$$V_{\chi,\Gamma,\tau} = \{\phi \in V_{\Gamma,\tau} \mid z \cdot \phi = \chi(z)\phi \text{ for all } z \in \mathfrak{Z}(\mathfrak{g})\}.$$

In fact, $V_{\Gamma,\tau}$ decomposes as $V_{\Gamma,\tau} = \bigoplus_{\chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}} V_{\chi,\Gamma,\tau}$.

We here introduce a notion of two locally symmetric spaces being *infinitesimally τ -isospectral*, defined as follows:

Definition 4.2.8. *Two locally symmetric spaces X_{Γ_1} and X_{Γ_2} are infinitesimally τ -isospectral if for all characters $\chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}$,*

$$\dim V_{\chi, \Gamma_1, \tau} = \dim V_{\chi, \Gamma_2, \tau}.$$

Remark 4.2.9. *If G is a real rank one Lie group, the center $\mathfrak{Z}(\mathfrak{g})$ is the polynomial algebra over $\mathbb{C}$ in the Casimir element C . Therefore, the χ -eigenspace is simply the λ -eigenspace, where λ is an eigenvalue of the Laplace-Beltrami operator $\Delta_{\tau, \Gamma}$ on $E_{\tau, \Gamma}$. In this case, infinitesimal τ -isospectrality reduces to τ -isospectrality as defined above.*

As in [BR11] and [BS24], we can expect that the $\dim V_{\chi, \Gamma, \tau}$ is related to the multiplicities $m(\pi, \Gamma)$ of $\pi \in \widehat{G}_{\tau^\vee}$ occurring in $L^2(\Gamma \backslash G)$ with infinitesimal character χ . This is the content of Thm. 4.1.7.

4.3 Proof of the first main result Thm. 4.1.3

4.3.1 Some preliminary results

We will describe the lemmas and propositions required to prove Thm. 4.1.3.

Lemma 4.3.2. *Let Kx be a left coset of K in G for some $x \in G$. Let U be an open set containing the left coset Kx . Then there exists $f \in C_c^\infty(G)$ with $\text{Supp}(f) \subset U$ such that $\int_K f(kx) \chi_\tau(k) dk \neq 0$. (Here χ_τ is the character function of τ .)*

Proof. For any $f \in C_c^\infty(G)$ with $\text{Supp}(f) \subset U$, we define $\psi \in C^\infty(K)$ by $\psi(k) = f(kx)$. Conversely, for any $\psi \in C^\infty(K)$, define $f(kx) = \psi(k)$, and extend f smoothly to U .

Now, if $\int_K f(kx)\chi_\tau(k) dk = 0$ for all $f \in C_c^\infty(G)$ such that $\text{Supp}(f) \subset U$, then

$$\int_K \psi(k) \chi_\tau(k) dk = 0 \text{ for all } \psi \in C^\infty(K).$$

But that implies χ_τ is identically zero leading to a contradiction. $\square$

Lemma 4.3.3. *Let $Kx \neq Ky$ be two distinct left cosets of K in G . Then there exists $F \in C_c^\infty(G, K, V_\tau)$ such that $h_F(x) \neq 0$ and $h_F|_{Ky} = 0$.*

Proof. $Kx \neq Ky$ are disjoint compact sets in G . Choose disjoint open sets U_1, U_2 with $Kx \subset U_1, Ky \subset U_2$, and $U_1 \cap U_2 = \emptyset$. From Lem. 4.3.2 there exists $f \in C_c^\infty(G)$ with $\text{Supp}(f) \subset U_1$ such that $\int_K f(kx)\chi_\tau(k) dk \neq 0$.

Let $F(g) = \int_K f(kg)\tau(k) dk$. Then, $F \in C_c^\infty(G, K, V_\tau)$.

Furthermore, $h_F(g) = \int_K f(kg)\chi_\tau(k) dk$. Hence $h_F(x) \neq 0$ and $h_F(y) = 0$ and hence $h_F|_{Ky} = 0$. $\square$

Proposition 4.3.4. *Let π be an irreducible representation of G occurring as a subrepresentation of ρ . Assume that π is not τ -spherical i.e. $\text{Hom}_K(V_\tau, V_\pi|_K) = 0$. Then $\rho(f)$ is zero on V_π .*

Proof. It is enough to show that $\rho(f)$ is zero at each element of the orthonormal basis $\{\psi_j : \psi \in \mathcal{U}, 1 \leq j \leq n\}$. Here, $\mathcal{U} = \mathcal{W} \cap W_\pi$. Recall that (see Eq. 2.4.4)

$$\rho(f)\psi_j(g) = \int_G \psi(gy)f(y)(v_j) dy.$$

For a fixed $\psi \in \mathcal{U}$, consider the subspace $U_\psi = \text{span} \{\psi_j : j \in \{1, \dots, n\}\}$. If $v = \sum_{j=1}^n a_j v_j$, we write $\psi_v = \sum_{j=1}^n a_j \psi_j$. Then the subspace U_ψ is same as $\{\psi_v : v \in V_\tau\}$. There is an action

of K on U_ψ by $\tau(k)\psi_v = \psi_{\tau(k)v}$.

Clearly, U_ψ is a representation of K isomorphic to τ . Note that $\widehat{\bigoplus_{\psi \in \mathcal{W}} U_\psi} = V_\pi$. We show that $\rho(f)|_{U_\psi} = 0$. In fact, we show that $\rho(f) \in \text{Hom}_K(U_\psi, V_\pi|_K)$.

Let $\psi_v \in U_\psi$ and $k_0 \in K$. Then for every $g \in G$, we have

$$\begin{aligned}
 (\rho(k_0)\rho(f)\psi_v)(g) &= (\rho(f)\psi_v)(gk_0) \\
 &= \int_G f(y) (\psi(gk_0y)v) dy \\
 &= \int_G \psi(gy) f(k_0^{-1}y)v dy \\
 &= \int_G \psi(gy) f(y)\tau(k_0)v dy \\
 &= \rho(f) \tau(k_0) \psi_v(g).
 \end{aligned} \tag{4.3.1}$$

Therefore, $\rho(f)$ is zero on U_ψ for all $\psi \in \mathcal{W}$. Consequently, $\rho(f)$ is zero on V_π .

□

Lemma 4.3.5. *Let Γ be a torsion-free uniform lattice in G . For a non-trivial element $\gamma \in \Gamma$, the conjugacy class $[\gamma]_G$ of γ in G is disjoint from K .*

Proof. Reader is referred to [BR11, Lem. 4.2].

□

Proposition 4.3.6. *Let Γ_1 and Γ_2 be two uniform lattices in G . Then there exists an open set B in G such that $[\gamma]_G \cap B$ is empty for all $\gamma \in \Gamma_1 \cup \Gamma_2$, and B is stable under left K action on G .*

Proof. Let U' be a relatively compact open set containing the identity element e in G . Let $U = KU'$. Then U is relatively compact and therefore it intersects at the most finitely

many conjugacy classes $[\gamma]_G$. Since, the natural map $G \rightarrow K \backslash G$ is proper, $K[\gamma]_G$ is closed in G . Since U is K -stable, $K[\gamma]_G \cap U \neq \emptyset$ if and only if $[\gamma]_G \cap U \neq \emptyset$. Let $E = \bigcup_{\gamma \neq e} (K[\gamma]_G \cap U)$. Since E is a finite union of closed sets, it is closed and K -stable subset of U . If there is $k \in K$, $\gamma \neq e$ and $x \in G$ such that $kx^{-1}\gamma x = e$, then $x^{-1}\gamma x \in K$; which contradicts the previous lemma. Therefore $e \notin K[\gamma]_G$ for any $\gamma \neq e$ and hence $e \notin E$. Choose an open set V containing e such that $E \cap V = \emptyset$. Now, let $B = KV \cap K^c$. Then B is the desired open set in G . $\square$

4.3.7 Proof of Theorem 4.1.3

We have two uniform torsion free lattices Γ_1 and Γ_2 in G . So by Eq. 2.4.2,

$$(\rho_{\Gamma_i}, L^2(\Gamma_i \backslash G, V_\tau)) = \widehat{\bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma_i) V_\pi}$$

for $i = 1, 2$. Let $t_\pi = m(\pi, \Gamma_1) - m(\pi, \Gamma_2)$. By hypothesis there exists a finite set $\mathcal{S} \subset \widehat{G}_\tau$ such that $t_\pi = 0$ for all $\pi \in \widehat{G}_\tau \setminus \mathcal{S}$. For $f \in C_c^\infty(G, K, V_\tau)$, $\rho_{\Gamma_i}(f)$ is zero on V_π if $\pi \notin \widehat{G}_\tau$ by Prop. 4.3.4. Recall that $h_f(y) = \text{Trace}(f(y))$. Therefore from the Selberg trace formula 3.2.7, we have

$$\begin{aligned}
\sum_{[\gamma] \in [\Gamma_1]_G \cup [\Gamma_2]_G} (a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2)) O_\gamma(h_f) &= \sum_{\pi \in \mathcal{S}} t_\pi \text{Trace}(\pi(h_f)) \\
&= \sum_{\pi \in \mathcal{S}} t_\pi \int_G h_f(y) \phi_\pi(y) dy \\
&= \int_G h_f(y) \phi(y) dy,
\end{aligned}$$

where $\phi = \sum_{\pi \in \mathcal{S}} t_\pi \phi_\pi$.

Let B be the open set from Prop. 4.3.6. For any $f \in C_c^\infty(G, K, V_\tau)$ supported in B , the orbital integrals on the right-hand side are zero. For such functions f , we have

$$\int_B h_f(y) \phi(y) dy = 0.$$

From Lem. 4.3.3, the functions h_f separates points in B . Hence ϕ must vanish on the open subset B of G . Since ϕ is real analytic (see Thm. 4.2.3), it vanishes on all of G . By the linear independence of functions ϕ_π (see Thm. 4.2.2), we conclude that $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all $\pi \in \widehat{G}_\tau$.

4.4 Proof of the second main result Thm. 4.1.7

4.4.1 Some observations

In this subsection, we make some observations that will be useful in the proof of Thm. 4.1.7.

Recall from Subsection 4.2.4 that $\mathcal{A}^\infty(G/K, \tau)$ is the space of smooth sections of the

vector bundle E_τ and $V_{\Gamma,\tau}$ is the subspace defined by

$$V_{\Gamma,\tau} = \{\phi \in \mathcal{A}^\infty(G/K, \tau) \mid \phi(\gamma x) = \phi(x) \text{ for all } \gamma \in \Gamma, x \in G\}.$$

Let $n = \dim(V_\tau)$. Recall the choice of an orthonormal basis $\{v_i\}_{i=1}^n$ of V_τ from Subsection 2.4. For a fixed $\phi \in V_{\chi,\Gamma,\tau}$, write $\phi(x) = \sum_{i=1}^n \phi_i(x) v_i$ for all $x \in G$, where each ϕ_i is smooth complex-valued function on $\Gamma \backslash G$. Recall from Subsection 2.4 that $V_{\chi,\Gamma,\tau}$ is the χ -eigenspace of $V_{\Gamma,\tau}$ with respect to the action of $\mathfrak{Z}(\mathfrak{g})$. For any $X \in \mathfrak{Z}(\mathfrak{g})$ and for all $x \in G$, we can see that

$$X \cdot \phi(x) = \sum_{i=1}^n X \cdot \phi_i(x) v_i. \quad (4.4.1)$$

Therefore, we conclude that $X \cdot \phi_i = \chi(X) \phi_i$ for all i .

Each $\tau(k)$ has a matrix representation with respect to the chosen orthonormal basis of V_τ . Let the (i, j) -th entry of $\tau(k)$ be denoted by $a_{ij}(k)$.

Note that $\phi(xk) = \tau(k^{-1})(\phi(x))$ for all $x \in G$ and $k \in K$. Thus we have

$$\begin{aligned} \sum_i \phi_i(xk) v_i &= \tau(k^{-1}) \left(\sum_j \phi_j(x) v_j \right) \\ &= \sum_l \left(\sum_j a_{lj}(k^{-1}) \phi_j(x) \right) v_l. \end{aligned} \quad (4.4.2)$$

Therefore, $\phi_i(xk) = \sum_{j=1}^n a_{ij}(k^{-1}) \phi_j(x)$ for all $x \in G$, $k \in K$.

4.4.2 Proof of Thm. 4.1.7

We denote $[\chi] = \{\pi \in \widehat{G} : \text{the infinitesimal character of } \pi \text{ is } \chi\}$. Recall that

$$L^2(\Gamma \backslash G) = \widehat{\bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) W_\pi}.$$

There are $m(\pi, \Gamma)$ copies of W_π inside $L^2(\Gamma \backslash G)$ say, $\{W_{\pi_t} : 1 \leq t \leq m(\pi, \Gamma)\}$. Let P_{π_t} be the projection onto W_{π_t} .

Clearly, $\phi_i \in L^2(\Gamma \backslash G)$ for all $\phi \in V_{\chi, \Gamma, \tau}$. We have

$$\phi_i = \sum_{\pi \in [\chi]} \sum_{t=1}^{m(\pi, \Gamma)} P_{\pi_t} \phi_i.$$

For any $1 \leq t \leq m(\pi, \Gamma)$, clearly $P_{\pi_t} \in \text{Hom}_G(L^2(\Gamma \backslash G), W_{\pi_t})$. Using Eq. 4.4.2 we get

$$\begin{aligned} \pi(k) P_{\pi_t} \phi_i &= P_{\pi_t} \rho(k) \phi_i \\ &= P_{\pi_t} \sum_{j=1}^n a_{ij}(k^{-1}) \phi_j \\ &= \sum_{j=1}^n a_{ij}(k^{-1}) P_{\pi_t} \phi_j. \end{aligned} \tag{4.4.3}$$

Let $F_{\pi_t} = \text{Span of } \{P_{\pi_t} \phi_j : 1 \leq j \leq n\}$. From Eq. 4.4.3, it follows that F_{π_t} is a finite dimensional representation of K . Let $\tau^\vee$ be the dual representation of τ of K on the dual space V_τ^* . Let $\{v_i^*\}_{i=1}^n$ be the dual basis of V_τ^* . Then there is a K -isomorphism between F_{π_t} and V_τ^* which maps each $P_{\pi_t} \phi_i$ to v_i^* .

Now we assume that τ is irreducible (and hence $\tau^\vee$ as well). For a fixed π_t , let $W_{\pi_t}(\tau^\vee)$

be the isotypic component of $\tau^\vee$ in W_{π_t} . Then $F_{\pi_t} \subset W_{\pi_t}(\tau^\vee)$. We have a decomposition

$$W_{\pi_t}(\tau^\vee) = \bigoplus_{p=1}^{M_{\tau^\vee}} H_p,$$

where, each H_p is an irreducible representation of K isomorphic to $\tau^\vee$. Note that $M_{\tau^\vee} = \dim(\text{Hom}_K(\tau^\vee, \pi|_K))$.

For each p , let $\{\phi_{t,p,s}\}_{s=1}^n$ be a basis of H_p satisfying $\phi_{p,t,s}(xk) = \sum_{u=1}^n a_{us}(k^{-1}) \phi_{t,p,u}(x)$. Note that this is possible because each H_p is isomorphic to F_{π_t} .

Now, $P_{\pi_t} \phi_i = \sum_p \sum_s \alpha_{p,s,i} \phi_{t,p,s}$. For each p , the matrix $(\alpha_{p,s,i})_{s,i}$ represents a K -homomorphism between F_{π_t} and H_p . Therefore, $\alpha_{p,s,i} = \alpha_p \delta_{s,i}$ for some scalar α_p by Schur's lemma.

Therefore, $P_{\pi_t} \phi_i = \sum_p \alpha_p \phi_{t,p,i}$. We define $\phi_{t,p}(x) = \sum_{i=1}^n \phi_{t,p,i}(x) v_i$ for all t and p . Hence, we get

$$\begin{aligned} \phi &= \sum_{i=1}^n \phi_i v_i \\ &= \sum_{i=1}^n \left(\sum_{\pi \in [\chi]} \sum_{t=1}^{m(\pi, \Gamma)} P_{\pi_t} \phi_i \right) v_i \\ &= \sum_{i=1}^n \sum_{\pi \in [\chi]} \sum_{t=1}^{m(\pi, \Gamma)} \sum_{p=1}^{M_{\tau^\vee}} \alpha_p \phi_{t,p,i} v_i \\ &= \sum_{\pi \in [\chi]} \sum_{t=1}^{m(\pi, \Gamma)} \sum_{p=1}^{M_{\tau^\vee}} \alpha_p \phi_{t,p}. \end{aligned} \tag{4.4.4}$$

We conclude that $\dim V_{\chi, \Gamma, \tau} \leq \sum_{\pi \in [\chi]} m(\pi, \Gamma) \dim(\text{Hom}_K(\tau^\vee, \pi|_K))$.

Conversely, for every t and for every p , choose a basis $\{\phi_{t,p,i}\}_{i=1}^n$ of H_p such that

$$\phi_{t,p,i}(xk) = \sum_{j=1}^n a_{ji}(k^{-1}) \phi_{t,p,j}(x).$$

Let $\phi_{t,p} = \sum_{i=1}^n \phi_{t,p,i} v_i$. Then $\phi_{t,p}(xk) = \tau(k^{-1})(\phi_{t,p}(x))$. Since $X \cdot \phi_{t,p,i} = \chi(X) \phi_{t,p,i}$ for all $X \in \mathfrak{Z}(\mathfrak{g})$, it follows that $\phi_{t,p} \in V_{\chi,\Gamma,\tau}$.

We conclude that $\sum_{\pi \in [\chi]} m(\pi, \Gamma) \dim(\text{Hom}_K(\tau^\vee, \pi|_K)) \leq \dim V_{\chi,\Gamma,\tau}$. Therefore, we have the equality

$$\dim V_{\chi,\Gamma,\tau} = \sum_{\pi \in [\chi]} m(\pi, \Gamma) \dim(\text{Hom}_K(\tau^\vee, \pi|_K)) \text{ for all } \tau \in \widehat{K}. \quad (4.4.5)$$

Now, let us consider the general case that τ is a finite dimensional representation of K (possibly reducible). Let $\tau \cong \bigoplus_{i=1}^q m_i \tau_i$ be a decomposition of τ into irreducible representations of K . Hence $\tau^\vee \cong \bigoplus_{i=1}^q m_i (\tau_i)^\vee$.

Let $V_{\Gamma,\tau}$ be as in Subsection 4.2.4. The center $\mathfrak{Z}(\mathfrak{g})$ acts on this space. Therefore for any character $\chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}$, we can consider the χ -eigenspace $V_{\chi,\Gamma,\tau} \subset V_{\Gamma,\tau}$. We observe that

$$\dim V_{\chi,\Gamma,\tau} = \sum_{i=1}^q m_i \dim V_{\chi,\Gamma,\tau_i}.$$

We can use the argument culminating into Eq. 4.4.5 for each τ_i and the additivity properties of Hom spaces with respect to decomposition of τ to complete the proof of Thm. 4.1.7.

4.5 Discrete series representations and cohomology

In this section, we comment about the case when G is non-compact connected semisimple group that admits a discrete series representation. (This is same as saying rank $G = \text{rank } T$, where T is a ‘compact’ Cartan subgroup.) It is well-known that for a given $\chi \in \widehat{\mathfrak{z}(\mathfrak{g})}$, and a K -type τ there exists at most one (up to infinitesimal equivalence) discrete series representation $\pi_{\lambda+\rho}$ with the infinitesimal character χ and minimal K -type τ where $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ such that $\lambda + \rho$ is regular and integral linear form. Also, $\dim (\text{Hom}_K(\tau, \pi_{\lambda+\rho}|_K)) = 1$.

Therefore, Thm. 4.1.7 implies the dimension of $V_{\chi, \Gamma, \tau^\vee}$ is equal to the multiplicity of the above discrete series $\pi_{\lambda+\rho}$ in $L^2(\Gamma \backslash G)$.

For $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, there exists a G -equivariant holomorphic line bundle $\mathcal{L}_\lambda$ on G/T . Under a mild condition on λ (see [Wil91] Eq. 7.66), [Wil91] Thm. 7.65] implies the L^2 -cohomology $H_2^*(\Gamma \backslash \mathcal{L}_\lambda)$ is non-vanishing in exactly one degree q_λ (which depends on λ) and

$$\dim H_2^{q_\lambda}(\Gamma \backslash \mathcal{L}_\lambda) = m(\pi_{\lambda+\rho}, \Gamma).$$

Hence, the dimension of the χ -eigenspace of the automorphic forms of type $\tau^\vee$ is equal to the dimension of q_λ -th L^2 -cohomology of the automorphic line bundle $\Gamma \backslash \mathcal{L}_\lambda$.

Chapter 5

Representation Equivalence for Arithmetic Lattices : Rank one Case

5.1 Statement of Main Results

Let G be a linear semisimple Lie group with finite center and of real rank one (e.g. $\mathrm{SL}(2, \mathbb{R})$).

Let Γ be a lattice in G such that the covolume $\mathrm{vol}(\Gamma \backslash G)$ (with respect to the unique G -invariant measure on $\Gamma \backslash G$) is finite. Consider the Hilbert space

$$L^2(\Gamma \backslash G) = \{\phi : G \longrightarrow \mathbb{C} : \phi(\gamma x) = \phi(x) \text{ for all } \gamma \in \Gamma, x \in G; \text{ and } \int_G |\phi(x)|^2 dx < \infty\}.$$

The group G acts on $L^2(\Gamma \backslash G)$ via $g.\phi(x) = \phi(xg)$ for all $x, g \in G$. This gives rise to a right regular representation R_Γ of G on $L^2(\Gamma \backslash G)$. Let $L^2_{\mathrm{disc}}(\Gamma \backslash G)$ be the maximal semisimple

subrepresentation of $L^2(\Gamma \backslash G)$ (called *discrete spectrum*). Then

$$L^2_{\text{disc}}(\Gamma \backslash G) \cong \widehat{\bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) \pi},$$

where the finite multiplicities $m(\pi, \Gamma)$ can be interpreted as $\dim_{\mathbb{C}} \text{Hom}_G(\pi, L^2(\Gamma \backslash G))$.

The orthogonal complement of $L^2_{\text{disc}}(\Gamma \backslash G)$ in $L^2(\Gamma \backslash G)$ is called *continuous spectrum* and is denoted by $L^2_c(\Gamma \backslash G)$. It admits a Hilbert direct integral decomposition with respect to a certain Radon measure on the unitary dual of G (see Subsec. 5.2.6).

Definition 5.1.1. Two lattices Γ_1 and Γ_2 are called *representation equivalent* if the right regular representations $(R_{\Gamma_1}, L^2(\Gamma_1 \backslash G))$ and $(R_{\Gamma_2}, L^2(\Gamma_2 \backslash G))$ are unitarily equivalent as representations of G , i.e. there is a unitary isomorphism (surjective isometry) between them.

The purpose of this chapter is to provide a sufficient condition for the representation equivalence of two lattices Γ_1 and Γ_2 . This can be thought of as an analogue of the classical strong multiplicity one theorem for the group $\text{GL}(n)$. Before we describe our main result, let us define some notations.

Since G is of real rank one, every proper parabolic subgroup is minimal. A parabolic subgroup P of G has a Levi decomposition LN , where L is a Levi part of P and N is the unipotent radical of P . Moreover, P possesses a decomposition (called Langlands decomposition) as $P = MAN$, where A is the split torus in L and M is a compact subgroup contained in a maximal compact subgroup K ; in fact, $M = Z_K(A)$. Let $\mathfrak{a}$ be the Lie algebra of A (see Subsec. 5.2.1 for details).

Definition 5.1.2. For a lattice Γ in G , a proper parabolic subgroup P is called Γ -*cuspidal* if $P \cap \Gamma \subset MN$; $\frac{N}{N \cap \Gamma}$ and $\frac{MN}{MN \cap \Gamma}$ are compact.

The set of all Γ -cuspidal parabolic subgroups is denoted by $E(G, \Gamma)$. Then Γ acts on $E(G, \Gamma)$ by conjugation, and the set of cusps for Γ is parametrised by the orbits of this action. Note that if Γ is a cocompact lattice then $E(G, \Gamma)$ is empty i.e., there is no cusp for Γ .

Definition 5.1.3. *Two lattices Γ_1 and Γ_2 of finite covolume in G are said to be isocuspidal if there exists a finite set $\{P_1, P_2, \dots, P_n\}$ of cuspidal parabolic subgroups which serves as a common set of representatives of cusps for both lattices Γ_1 and Γ_2 .*

Remark 5.1.4. *If Γ_1 and Γ_2 both have only one cusp then they are isocuspidal.*

Let $\widehat{M}$ denote the set of equivalence classes of irreducible representations of M . For every $\sigma \in \widehat{M}$ and $\nu \in i\mathfrak{a}^*$, we have a representation space $H_\Gamma(\sigma, \nu)$ consisting of finitely many principal series representations of G . These representation spaces possess certain unitary intertwining operators (see Sec. [5.3](#) for details)

$$C_{\Gamma\sigma}(\nu) : H_\Gamma(\sigma, \nu) \longrightarrow H_\Gamma(\sigma, -\nu).$$

The following *Spectral Rigidity Theorem* is the main result of this chapter.

Theorem 5.1.5. *Let Γ_1 and Γ_2 be two isocuspidal lattices with finite covolume in G such that the following holds:*

1. *There exists a parabolic subgroup $P = MAN$ in the common set of representatives of cusps for Γ_1 and Γ_2 such that $L^2((\Gamma_1 \cap M) \backslash M) \cong L^2((\Gamma_2 \cap M) \backslash M)$ as representations of M and for every $\sigma \in \widehat{M}$, $C_{\Gamma_1\sigma}(\nu) = C_{\Gamma_2\sigma}(\nu)$ for all $\nu \in i\mathfrak{a}^*$.*

2. The multiplicities $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all but finitely many $\pi \in \widehat{G}$.

Then the lattices Γ_1 and Γ_2 are representation equivalent in G .

Remark 5.1.6. If two lattices Γ_1 and Γ_2 are isocuspidal, then by the assumption (1) it turns out that the representation spaces $H_{\Gamma_1}(\sigma, \nu)$ and $H_{\Gamma_2}(\sigma, \nu)$ are equal; (see Subsec. 5.4.1). Moreover, assumption (1) says that $C_{\Gamma_1\sigma}(\nu)(v) = C_{\Gamma_2\sigma}(\nu)(v)$ for all $v \in H_{\Gamma_1}(\sigma, \nu) = H_{\Gamma_2}(\sigma, \nu)$.

Remark 5.1.7. Note that if Γ_1 and Γ_2 are cocompact lattices then the condition (1) in Thm. 5.1.5 is vacuously true and hence it essentially recovers ([BR11], Thm. 1.1). However, in the case of non-cocompact lattices, condition (1) assures the equality of the continuous spectra (see Subsec. 5.4.1 for details).

Remark 5.1.8. If G is reductive but not semisimple, then for congruence lattices Γ_1 and Γ_2 in G , it is expected that only condition (2) as in Thm. 5.1.5 is sufficient to conclude $L_{disc}^2(\Gamma_1 \backslash G) \cong L_{disc}^2(\Gamma_2 \backslash G)$. This is proved in Sec. 5.5 for $GL(n)$.

5.2 Preliminaries

5.2.1 Associate parabolics in G

Let G be a linear semisimple Lie group with finite center and of real rank one and K be a maximal compact subgroup in G which we fix henceforth. In this situation, any proper parabolic subgroup P of G is a minimal parabolic. Moreover, any two parabolic subgroups are pairwise associate, i.e., G -conjugate.

Let $P = LN$ be the Levi decomposition of a given proper parabolic P in G , where L is the Levi part of P and N is the unipotent radical of P . We also have the Langlands decomposition $P = MAN$ where A is the split torus inside L and $M := Z_K(A)$ is the centraliser

of A in K . In particular, M is a semisimple (compact) subgroup of K . Let $\Sigma = \Sigma(G, A)$ be the set of roots for the pair (G, A) . If Σ^+ is a positive system relative to the unipotent radical N then Σ^+ consists of at most two roots $\{\alpha, 2\alpha\}$. The root spaces of α and 2α are denoted by $\mathfrak{n}_\alpha$ and $\mathfrak{n}_{2\alpha}$ respectively and the corresponding Lie groups are denoted by $N_\alpha, N_{2\alpha}$ respectively. So $\mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha}$ is the Lie algebra of N . Let $m_1 = \dim \mathfrak{n}_\alpha$ and $m_2 = \dim \mathfrak{n}_{2\alpha}$.

5.2.2 Γ -cuspidal parabolics

Let Γ be a lattice in G such that $\text{vol}(\Gamma \backslash G) < \infty$ (i.e. of finite covolume). A proper parabolic subgroup $P = MAN$ is called a Γ -cuspidal parabolic subgroup of G if $P \cap \Gamma \subset MN$; $N/N \cap \Gamma$ is compact and $MN/MN \cap \Gamma$ is compact. Let $E(G, \Gamma)$ be the set of all Γ -cuspidal parabolic subgroups in G . The set $E(G, \Gamma)$ decomposes as a disjoint union of finitely many orbits under the conjugation action of Γ . Let us fix the representatives $P_1, P_2, \dots, P_{n_\Gamma}$ of these orbits. They parametrise the set of all cusps of Γ . Let $P_1 = P$. Then there exists $k_i \in K$ with $k_1 = 1$ such that for each $i \in \{1, 2, \dots, n_\Gamma\}$ and $P_i = M_i A_i N_i$, $M_i = k_i^{-1} M k_i$, $A_i = k_i^{-1} A k_i$ and $N_i = k_i^{-1} N k_i$. Note that if Γ is a co-compact lattice in G , then $E(G, \Gamma)$ is empty; equivalently there is no cusp.

5.2.3 Conjugacy classes of hyperbolic/elliptic elements

In this subsection, we mention some results about the geometry of conjugacy classes of hyperbolic/elliptic elements in a lattice in a semisimple Lie group.

Lemma 5.2.4. *For any hyperbolic or elliptic element $\gamma \in \Gamma$ the G -conjugacy class $[\gamma]_G$ is a closed set of measure zero.*

Proof. Any hyperbolic or elliptic element is semisimple and therefore its conjugacy class

must be closed. To prove that $[\gamma]_G$ is a set of measure zero, it is enough to show that the conjugacy class $[\gamma]_G$ endows a submanifold structure in G with lower dimension. Via the pullback of the projection map, $[\gamma]_G$ is homeomorphic to G/G_γ and G_γ contains a Cartan subgroup, where G_γ is a centraliser of γ in G . $\square$

Lemma 5.2.5. *Let E be the union of all conjugacy classes $[\gamma]_G$ of all hyperbolic and elliptic elements $\gamma \in \Gamma$. Then E is a closed set of measure zero.*

Proof. Let C be a compact subset of G . Since E is a countable union of closed sets $E \cap C$ is a finite union of closed sets. Therefore $E \cap C$ is closed. This holds for all compact subsets C . So E is closed. $\square$

5.2.6 Spectral Decomposition

For the parabolic subgroup $P = MAN$ let $\Gamma_M = \Gamma \cap M$. So, Γ_M is a finite subgroup of the compact group M . Let $\widehat{M}$ be the set of irreducible representations of M . Consider the right regular representation of M in the Hilbert space $L^2(\Gamma_M \backslash M)$. We have the Hilbert space decomposition

$$L^2(\Gamma_M \backslash M) \cong \widehat{\bigoplus_{\sigma \in \widehat{M}} m(\sigma, \Gamma_M) \sigma}.$$

Let $W = N_K(A)/M$ be the Weyl group. Since G is of real rank one (i.e. $\dim A = 1$) we have $|W| = 2$. Let w be the nontrivial element of W . Let $\mathfrak{a}$ be the Lie algebra of A . An irreducible representation $\sigma \in \widehat{M}$ is called ramified if $w\sigma = \sigma$. Otherwise, σ is called unramified.

For any $\sigma \in \widehat{M}$, $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let $\pi_{\sigma, \nu}$ denote the principal series representation of G . We write,

$$\pi(\sigma, \nu) = \begin{cases} m(\sigma, \Gamma_M) \pi_{\sigma, \nu} & \text{if } \sigma \text{ is ramified,} \\ m(\sigma, \Gamma_M) \pi_{\sigma, \nu} \oplus m(w\sigma, \Gamma_M) \pi_{w\sigma, \nu} & \text{if } \sigma \text{ is unramified.} \end{cases}$$

Let $H(\sigma, \nu)$ be the representation space of $\pi(\sigma, \nu)$. Let $\mathfrak{a}_i$ be the Lie algebra of A_i . Via conjugation by appropriate k_i , each $\sigma \in \widehat{M}$ gives rise to a representation $\sigma_i \in \widehat{M}_i$ and each $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ gives rise to $\nu_i \in \mathfrak{a}_{i, \mathbb{C}}^*$. For each $\sigma \in \widehat{M}$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ we put

$$\pi_{\Gamma}(\sigma, \nu) = \bigoplus_{i=1}^{n_{\Gamma}} \pi(\sigma_i, \nu_i).$$

We denote the corresponding representation space by $H_{\Gamma}(\sigma, \nu)$.

The right regular representation $L^2(\Gamma \backslash G)$ of G is not semisimple if Γ is not co-compact (See [OW81], Chapter 1, Page 22). Let $L_{\text{disc}}^2(\Gamma \backslash G)$ be the maximal semisimple subrepresentation of $L^2(\Gamma \backslash G)$. The orthogonal complement of $L_{\text{disc}}^2(\Gamma \backslash G)$ in $L^2(\Gamma \backslash G)$ is the continuous spectrum, denoted by $L_c^2(\Gamma \backslash G)$. Following ([OW81], Chapter 4, Page 112), the Hilbert direct integral decomposition of $L_c^2(\Gamma \backslash G)$ can be written as

$$L_c^2(\Gamma \backslash G) \cong \bigoplus_{\sigma \in \widehat{M}} \frac{1}{2\pi} \int_{i\mathbb{R}^+}^{\oplus} \pi_{\Gamma}(\sigma, \nu) d\nu. \quad (5.2.1)$$

5.2.7 Results on Harish-Chandra characters

We will be using the following two important results about Harish-Chandra characters of irreducible unitary representations of semisimple Lie groups (see [Kna01]).

Harish-Chandra character distribution: Let (π, W_{π}) be an irreducible unitary representation of G . For $f \in C_c^{\infty}(G)$, the convolution operator $\pi(f)$ on W_{π} is trace class and its

trace is denoted by $\chi_\pi(f) := \text{Trace}(\pi(f))$.

Theorem 5.2.8. *Let $\{\pi_i\}$ be a finite collection of mutually inequivalent unitary irreducible representations of G . Then their distribution characters $\{\chi_{\pi_i}\}$ are linearly independent distributions on G .*

Proof. The reader is referred to ([Kna01], Thm. 10.6) □

Theorem 5.2.9. *Let π be an irreducible unitary representation of G . Then the distribution character χ_π is given by a locally integrable function ϕ_π on G , i.e. for $f \in C_c^\infty(G)$,*

$$\chi_\pi(f) = \int_G f(g) \phi_\pi(g) dg.$$

Moreover, the restriction of ϕ_π to the set G_{reg} of regular elements is a real analytic function and is invariant under conjugation.

Proof. The reader is referred to ([Kna01], Thm. 10.25). □

5.3 Simple Trace Formula for Arithmetic Lattices

We describe the simple trace formula for the representation $L^2(\Gamma \backslash G)$ following ([SW99]; Sec. 2.1). Let $C_c^\infty(G)$ be the algebra of all compactly supported smooth functions on G . Because of the presence of continuous spectrum in the right regular representation R_Γ on $L^2(\Gamma \backslash G)$, the convolution operator $R_\Gamma(f)$ for $f \in C_c^\infty(G)$ is not a trace class operator in general. However, the restriction of $R_\Gamma(f)$ on the discrete part $L_{\text{disc}}^2(\Gamma \backslash G)$ is a trace class operator. To describe the trace, we need to set up a few notations. For $\gamma \in \Gamma$, let Γ_γ and G_γ denote the centraliser of γ in Γ and G , respectively. For any $\sigma \in \widehat{M}$ and $\nu \in i\mathfrak{a}^*$ there

exists a unitary intertwining operator

$$C_{\Gamma\sigma}(\nu) : H_{\Gamma}(\sigma, \nu) \longrightarrow H_{\Gamma}(\sigma, -\nu).$$

Moreover, for any K -finite compactly supported smooth function f on G one can use the functional equation of Eisenstein series (see [HC68], [Lan76]) to derive

$$C_{\Gamma\sigma}(\nu) \pi_{\Gamma}(\sigma, \nu)(f) = \pi_{\Gamma}(\sigma, -\nu)(f) C_{\Gamma\sigma}(\nu).$$

Let us mention that there is an ample supply of non-zero K -finite compactly supported smooth functions. For instance, such test functions are primary setup in ([Art83]; Page 1). In literature $C_{\Gamma\sigma}(\nu)$ is known as scattering matrix for Γ associated with holonomy class $\sigma \in \widehat{M}$.

Now we are ready to display the required trace formula following ([SW99]; Sec. 2.1).

$$\text{Trace } R_{\Gamma}(f)|_{L^2_{\text{disc}}(\Gamma \backslash G)} = I_{\Gamma}(f) + H_{\Gamma}(f) + S_{\Gamma}(f) + T_{\Gamma}(f) + U_{\Gamma}(f); \quad (5.3.1)$$

where the terms in the sum are given, respectively, by

$$\begin{aligned}
(\text{identity}) \quad I_\Gamma(f) &= \text{Vol}(\Gamma \backslash G) f(1), \\
(\text{regular}) \quad H_\Gamma(f) &= \sum_{\{\gamma\}: \text{hyperbolic or elliptic}} \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1} \gamma x) d(G_\gamma x), \\
(\text{scattering}) \quad S_\Gamma(f) &= \frac{1}{4\pi} \sum_{\sigma \in \widehat{M}} \int_{\text{Re}(\nu)=0} \text{tr} \left(C_{\Gamma\sigma}(\nu)^{-1} \frac{d}{d\nu} C_{\Gamma\sigma}(\nu) \pi_\Gamma(\sigma, \nu)(f) \right) |d\nu|, \\
(\text{threshold}) \quad T_\Gamma(f) &= -\frac{1}{4} \sum_{\sigma \in \widehat{M}} \text{tr} (C_{\Gamma\sigma}(0) \pi_\Gamma(\sigma, 0)(f)), \\
(\text{unipotent}) \quad U_\Gamma(f) &= \frac{1}{|\alpha_1|} (C_1(\Gamma)T_1(f) + C_2(\Gamma)T_2(f) + C'_1(\Gamma)T'_1(f)).
\end{aligned}$$

Here, the definition of $T_i(f), T'_1(f)$ can be recalled from ([SW99]; Sec. 3, Page 21) as follows:

$$\begin{aligned}
T_1(f) &= \frac{1}{A(\mathfrak{n}_\alpha)} \int_K \int_N f(k^{-1}nk) dn dk, \\
T_2(f) &= \frac{|\alpha_1|}{2} \left\{ \int_{G/G_{n_0}} f(xn_0x^{-1}) d(G/G_{n_0})(x) + \int_{G/G_{n_0^{-1}}} f(xn_0^{-1}x^{-1}) d(G/G_{n_0^{-1}})(x) \right\}, \\
T'_1(f) &= \frac{(m_1 + 2m_2)}{A(\mathfrak{n}_\alpha)} \int_{A(n_1)} \int_{N_1} \int_{N_2} f(k^{-1}n_1n_2k) \cdot \log |\log(n_1)| dk dn_1 dn_2,
\end{aligned}$$

where n_0 is a representative of the nontrivial unipotent orbit in $\mathfrak{n}_{2\alpha}$, $A(\mathfrak{n}_\alpha)$ denotes the area of the unit sphere in $\mathfrak{n}_\alpha$, and G_n represents the centraliser of an element n in G .

Remark 5.3.1. *If Γ is torsion free, then there is no contribution of elliptic orbital integral in the regular term. The constants $C_i(\Gamma), C'_1(\Gamma)$ that appear in the unipotent contribution were*

explicitly computed by DeGeorge (see [\[DG89\]](#)).

For our purpose, we put a certain condition on the K -finite function $f \in C_c^\infty(G)$ which make the trace formula much simpler. An element $x \in G$ is called regular if the centraliser $C_G(x)$ of x in G is a torus. Consider the set G_{reg} to be the set of all regular elements in G . Then G_{reg} is an open and dense subset of G . Let f be a K -finite function in $C_c^\infty(G_{\text{reg}})$. Since any element n of a unipotent radical N is not regular, the identity term and unipotent term in the above trace formula vanish for $f \in C_c^\infty(G_{\text{reg}})$. Consequently, we have,

$$\text{Trace } R_\Gamma(f)|_{L_{\text{disc}}^2(\Gamma \backslash G)} = H_\Gamma(f) + S_\Gamma(f) + T_\Gamma(f).$$

Moreover, if,

$$L_d^2(\Gamma \backslash G) \cong \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) \pi,$$

then we have the simple trace formula

$$\sum_{\pi \in \widehat{G}} m(\pi, \Gamma) \text{Trace } (\pi(f)) = H_\Gamma(f) + S_\Gamma(f) + T_\Gamma(f).$$

Remark 5.3.2. 1. *The scattering and threshold terms are essentially coming from the spectral side (more specifically given by the continuous spectrum).*

2. *If G has square-integrable representations then one can take its matrix coefficient of some square-integrable representation as the test function. In this case the scattering and threshold terms vanish. But then the sum in the spectral side is supported only at a finite set of square-integrable irreducible representations (for example, discrete series representations) and so the information about the multiplicity of principal series representations is lost.*

5.4 Proof of the Thm. 5.1.5

In this section, we compare the representation spectra for two lattices Γ_1 and Γ_2 in G and prove Thm. 5.1.5

5.4.1 Equality of Continuous Spectrum

Let us recall the Def. 5.1.3

Definition 5.4.2. *Two lattices Γ_1 and Γ_2 of finite covolume in G are said to be isocuspidal if there exists a finite set $\{P_1, P_2, \dots, P_n\}$ of cuspidal parabolic subgroups which serves as a common set of representatives of cusps for both lattices Γ_1 and Γ_2 .*

Note that this is automatic if both lattices are co-compact. Moreover, it also holds if the lattices Γ_1 and Γ_2 have only one cusp.

Let us denote $P := P_1$. Then P admits a decomposition say, MAN . Let us denote $\Gamma_{i,M} = \Gamma_i \cap M$ for $i = 1, 2$. Then $\Gamma_{i,M}$ are finite subgroups of M . Consider the representation spaces $L^2(\Gamma_{i,M} \backslash M)$ of M . We further assume that they are isomorphic as representations of M i.e., $L^2(\Gamma_{1,M} \backslash M) \cong L^2(\Gamma_{2,M} \backslash M)$. Note that if the lattices Γ_i are torsion-free, then both of $\Gamma_{1,M}$ and $\Gamma_{2,M}$ are trivial; so this assumption holds automatically. Furthermore, if

$$L^2(\Gamma_{i,M} \backslash M) \cong \bigoplus_{\sigma \in \widehat{M}} m(\sigma, \Gamma_{i,M}) \sigma \quad \text{for } i = 1, 2$$

and $m(\sigma, \Gamma_{1,M}) = m(\sigma, \Gamma_{2,M})$ for all but finitely many $\sigma \in \widehat{M}$ then by ([LM20], Thm. 1.1), $m(\sigma, \Gamma_{1,M}) = m(\sigma, \Gamma_{2,M})$ for all $\sigma \in \widehat{M}$ i.e. $L^2(\Gamma_{1,M} \backslash M) \cong L^2(\Gamma_{2,M} \backslash M)$.

By the definition of $H_\Gamma(\sigma, \nu)$ given in Sec. 5.2.6 it is clear that if two lattices Γ_1 and Γ_2 are isocuspidal and $m(\sigma, \Gamma_{1,M}) = m(\sigma, \Gamma_{2,M})$ for all $\sigma \in \widehat{M}$ then the representation spaces

$H_{\Gamma_1}(\sigma, \nu)$ and $H_{\Gamma_2}(\sigma, \nu)$ are equal for all $\sigma \in \widehat{M}, \nu \in i\mathfrak{a}^*$. Hence, by Eq. 5.2.1, $L_c^2(\Gamma_1 \backslash G) \cong L_c^2(\Gamma_2 \backslash G)$.

5.4.3 Equality of Discrete Spectrum

For all $\sigma \in \widehat{M}$ and $\nu \in i\mathfrak{a}^*$, we have that $H_{\Gamma_1}(\sigma, \nu) = H_{\Gamma_2}(\sigma, \nu)$ and by assumption (1) the intertwining operators $C_{\Gamma_1 \sigma}(\nu)$ and $C_{\Gamma_2 \sigma}(\nu)$ are equal on it. Now the scattering term $S_{\Gamma}(f)$ in the trace formula Eq. 5.3.1 depends on the representation spaces $H_{\Gamma}(\sigma, \nu)$ and the intertwining operators $C_{\Gamma \sigma}(\nu)$. For two lattices Γ_1 and Γ_2 , they are equal for all $\sigma \in \widehat{M}$ and $\nu \in i\mathfrak{a}^*$. Therefore, $S_{\Gamma_1}(f) = S_{\Gamma_2}(f)$ for all K -finite f in $C_c^\infty(G)$. Similarly, $T_{\Gamma_1}(f) = T_{\Gamma_2}(f)$ for all K -finite f in $C_c^\infty(G)$.

We argue that the unipotent terms $U_{\Gamma_1}(f)$ and $U_{\Gamma_2}(f)$ of the trace formula Eq. 5.3.1 vanish for all $f \in C_c^\infty(G_{\text{reg}})$. The integrands in $T_1(f), T_2(f)$ and $T'_1(f)$ involve f evaluated at the unipotent elements which are not regular in G . Since f is supported in the set of regular elements in G , all the integrals $T_1(f), T_2(f)$ and $T'_1(f)$ must be zero.

Now by the assumption on the lattices Γ_1 and Γ_2 , comparison of the trace formulas for the lattices Γ_1 and Γ_2 gives

$$\text{Trace } R_{\Gamma_1}(f)|_{L_{\text{disc}}^2(\Gamma_1 \backslash G)} - \text{Trace } R_{\Gamma_2}(f)|_{L_{\text{disc}}^2(\Gamma_2 \backslash G)} = H_{\Gamma_1}(f) - H_{\Gamma_2}(f) \text{ for all } f \in C_c^\infty(G_{\text{reg}}).$$

Let us denote the set of all G -conjugacy classes of hyperbolic and elliptic elements of Γ_1 and Γ_2 by $[\Gamma_1]_G$ and $[\Gamma_2]_G$, respectively. For a G -conjugacy class $[\gamma]_G$ of a hyperbolic or elliptic element $\gamma \in \Gamma_1 \cup \Gamma_2$ we denote $\text{vol}(\Gamma_\gamma \backslash G_\gamma)$ by $a(\gamma, \Gamma)$. So we have,

$$\sum_{\pi \in \mathcal{S}} t_\pi \chi_\pi(f) = \sum_{[\gamma] \in [\Gamma_1]_G \cup [\Gamma_2]_G} (a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2)) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) d(G_\gamma x),$$

where $t_\pi = m(\pi, \Gamma_1) - m(\pi, \Gamma_2)$, $\mathcal{S}$ is a finite set such that $t_\pi = 0$ for all $\pi \in \widehat{G} \setminus \mathcal{S}$ and $\chi_\pi(f) = \text{Trace}(\pi(f))$ for all $f \in C_c^\infty(G_{\text{reg}})$.

Since $\mathcal{S}$ is a finite set, by Thm. [5.2.9](#) there exists $\phi \in L^1_{loc}(G)$ such that

$$\int_G f(g) \phi(g) dg = \sum_{[\gamma] \in [\Gamma_1]_G \cup [\Gamma_2]_G} (a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2)) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) d(G_\gamma x).$$

Let E be the union of all conjugacy classes of hyperbolic and elliptic elements in $\Gamma_1 \cup \Gamma_2$. Then by Lem. [5.2.5](#), E is a closed subset of G with measure zero. Let $g \notin E$. There exists an open set $U \subseteq G_{\text{reg}}$ in containing g such that $U \cap E$ is empty. Therefore, for any $f \in C_c(U)$, we have

$$\int_G f(g) \phi(g) dg = 0.$$

This means that ϕ is essentially 0 on U . Note that U is a neighbourhood of an arbitrary point g away from E and E is a closed set of measure zero. Therefore, ϕ is essentially 0 on G_{reg} . Finally, ϕ is essentially 0 on G because the set $G \setminus G_{\text{reg}}$ of non-regular points is a measure zero set. Hence,

$$\sum_{\pi \in \mathcal{S}} t_\pi \chi_\pi(f) = 0, \quad \text{for all } f \in C_c^\infty(G).$$

From Thm. [5.2.8](#) on the linear independence of characters we have $t_\pi = 0$ for all $\pi \in \mathcal{S}$.

Hence, $L^2_{\text{disc}}(\Gamma_1 \backslash G) \cong L^2_{\text{disc}}(\Gamma_2 \backslash G)$.

5.5 Representation Equivalence for Congruence Lattices in $\mathrm{GL}(n)$.

5.5.1 Unramified Hecke Character

Let us introduce the following notation :

- F : a number field i.e. a finite extension of $\mathbb{Q}$
- $\mathcal{O}_F$: the ring of integers of F
- F_v : the completion of F at a place v of F and $\mathcal{O}_{F_v}$ is the ring of integers of F_v .
- S_∞ : the set of all archimedean places of F
- $S_\infty = S_r \cup S_c$ the real and complex places, respectively
- $\mathbb{A}_F$ be the adèle ring of F and $\mathbb{A}_{F,f}$ be the ring of finite adeles of F
- $\mathbb{I}_F$: the group of ideles of F i.e. $\mathbb{I}_F = \mathrm{GL}(1, \mathbb{A}_F)$.
- $F_\infty := F \otimes \mathbb{R} = \prod_{v \in S_r} F_v \times \prod_{w \in S_c} F_w = \prod_{v \in S_r} \mathbb{R} \times \prod_{w \in S_c} \mathbb{C}$

In the proof of Thm. [5.5.3](#), we need to consider the twist of a cuspidal automorphic representation with an unramified Hecke character. So, we discuss some basics on Hecke character. Reader is referred to [Rag22](#) for more detailed exposition.

Definition 5.5.1. *A Hecke character is a continuous character $\chi : \mathbb{I}_F / F^\times \rightarrow \mathbb{S}^1$. It is unitary if the image lands in $\mathbb{S}^1$. It is called unramified if $\chi(k) = 1$ for all $k \in \prod_v \mathcal{O}_{F_v}^\times$ (Here, the product runs over all places v of F).*

If $\|a\| := \prod_v \|a_v\|_v$; the product of normalized valuations over all places, then any Hecke character χ can be written as product $\chi = \chi^0 \otimes \|(\cdot)\|^\sigma$, for a unitary Hecke character χ^0 and a real number σ .

We now describe the character at infinity of an unramified Hecke character.

1) An unramified unitary character χ of $\mathbb{R}^\times$ is of the form $\chi(x) = (\frac{x}{|x|})^n |x|^\sigma$; for some $n \in \{0, 1\}$ and $\sigma \in i\mathbb{R}$.

2) An unramified unitary character χ of $\mathbb{C}^\times$ is of the form $\chi(z) = (\frac{z}{|z|})^n |z|^{2\sigma}$ for $n \in \mathbb{Z}$ and $\sigma \in i\mathbb{R}$; where $|z|$ is the absolute value of $z \in \mathbb{C}$.

For a unitary Hecke character χ , let $\chi_\infty := \chi|_{F_\infty^\times}$; where, $F_\infty^\times \hookrightarrow \mathbb{I}_F$. We write $\lambda \in S_\infty$ for any infinite place; $v \in S_r$ and $w \in S_c$ for real and complex places respectively. Note that, $F_v \cong \mathbb{R}$ canonically and $F_w \cong \mathbb{C}$ non-canonically. Finally, for any $x_\infty \in F_\infty$, let $|x_\infty| := \prod_{\lambda \in S_\infty} |x_\lambda|_\lambda$.

Now we can describe χ_∞ as follows:

$$\chi_\infty(x_\infty) = \prod_{\lambda \in S_\infty} \left(\frac{x_\lambda}{|x_\lambda|} \right)^{n_\lambda} |x_\lambda|_\lambda^{i\varphi_\lambda}$$

where, $n_v \in \{0, 1\}$; $n_w \in \mathbb{Z}$ and $\varphi_\lambda \in \mathbb{R}$.

5.5.2 Proof of Thm. 1.2.1

In this subsection, we describe a representation equivalence of cuspidal spectrum for congruence lattices in $GL(n)$. The idea is twisting by unramified Hecke character while keeping the infinity part of a cuspidal automorphic representation of $GL(n, \mathbb{A}_F)$.

Let G be a connected reductive group over F . Then $G(F_\infty)$ is a reductive Lie group. Let K be a compact open subgroup of $G(\mathbb{A}_{F,f})$. We have the congruence lattice $\Gamma_K := G(F) \cap K$ inside $G(F_\infty)$. By strong approximation we have

$$\Gamma_K \backslash G(F_\infty) \cong G(F) \backslash G(\mathbb{A}_F) / K.$$

Recall the spectral decomposition

$$L^2(G(F) \backslash G(\mathbb{A}_F)) = L_{\text{cusp}}^2(G(F) \backslash G(\mathbb{A}_F)) \bigoplus L_{\text{res}}^2(G(F) \backslash G(\mathbb{A}_F)) \bigoplus L_{\text{cont}}^2(G(F) \backslash G(\mathbb{A}_F))$$

We are interested in the discrete decomposition : $L_{\text{cusp}}^2(G(F) \backslash G(\mathbb{A}_F)) \cong \bigoplus_{\Pi \in \mathcal{A}_{\text{cusp}}} m(\Pi) \Pi$; where $\mathcal{A}_{\text{cusp}}$ denote the set of all cuspidal automorphic representations of $G(\mathbb{A}_F)$.

As representation of $G(F_\infty)$, we have the following isomorphism

$$L^2(\Gamma_K \backslash G(F_\infty)) \cong L^2(G(F) \backslash G(\mathbb{A}_F) / K).$$

Therefore, taking the K -fixed vectors of the cuspidal subspace we have

$$L_{\text{cusp}}^2(\Gamma_K \backslash G(F_\infty)) \cong \bigoplus_{\pi \in \widehat{G(F_\infty)}} m(\pi, \Gamma_K) \pi,$$

where $m(\pi, \Gamma_K)$ equals $\sum_{\Pi \in \mathcal{A}_{\text{cusp}} : \Pi_\infty = \pi} m(\Pi) \dim(\Pi^K)$.

Recall that an unramified Hecke character χ is a continuous character $\chi : \mathbb{I}_F / F^\times \longrightarrow S^1$ such that $\chi(k) = 1$ for all $k \in \widehat{O_F^\times} := \prod_p \mathcal{O}_p^\times$.

Let us now restrict to $G = GL(n)$. We state the main result of this section.

Theorem 5.5.3. *Let K_1 and K_2 be two compact open subgroups of $GL(n, \mathbb{A}_{F,f})$. Denote*

the congruence subgroup $GL(n, F) \cap K_i$ of $GL(n, F_\infty)$ by Γ_{K_i} for $i = 1, 2$. If $m(\pi, \Gamma_{K_1}) = m(\pi, \Gamma_{K_2})$ for all but finitely many $\pi \in \widehat{G(F_\infty)}$, then $L_{cusp}^2(\Gamma_{K_1} \backslash G(F_\infty)) \cong L_{cusp}^2(\Gamma_{K_2} \backslash G(F_\infty))$.

Proof. For any $x \in GL(n, \mathbb{A}_{F,f})$ and a compact open subgroup K of $GL(n, \mathbb{A}_{F,f})$, we have the compact open subgroup $K^x := xKx^{-1}$ of $GL(n, \mathbb{A}_{F,f})$. For a given cuspidal automorphic representation $\Pi \in \mathcal{A}_{cusp}$, it is easy to see that the space of K -fixed vectors in Π is isomorphic to the space of K^x -fixed vectors in Π . In particular, $m(\pi, \Gamma_K) = m(\pi, \Gamma_{K^x})$ for all $\pi \in \widehat{G(F_\infty)}$. Therefore, without loss of generality, we assume that both of the compact open subgroups K_1 and K_2 in the above theorem are contained in $GL(n, \widehat{\mathcal{O}_F})$. Here, $\widehat{\mathcal{O}_F} := \prod_v \mathcal{O}_{F_v}$. If Π is a cuspidal automorphic representation and K is a compact open subgroup, then for any unramified Hecke character χ , we have $(\Pi \otimes \chi)^K = \Pi^K$.

Note that the set $\{\Pi \otimes \chi : \chi \text{ is an unramified Hecke character}\}$ is an infinite set. Indeed, by the description of χ_∞ for a Hecke character χ in Sec. [5.5.1](#), there are infinitely many Hecke characters χ with distinct and nontrivial infinity part.

Let us denote $\mathcal{S}$ to be $\{\pi \in \widehat{GL_n(F_\infty)} : m(\pi, \Gamma_{K_1}) \neq m(\pi, \Gamma_{K_2})\}$. We fix a $\pi \in \mathcal{S}$. Denote the finite set $\{\Pi : \Pi_\infty = \pi\}$ by $\mathcal{A}(\pi)$. We can choose an unramified Hecke character χ such that χ_∞ is nontrivial and $\pi \otimes \chi_\infty \notin \mathcal{S}$. Now the set $\mathcal{A}(\pi \otimes \chi_\infty) = \{\Pi \otimes \chi : \Pi \in \mathcal{A}(\pi)\}$. Indeed, the association $\Pi \mapsto \Pi \otimes \chi$ defines a bijection between the sets $\mathcal{A}(\pi)$ and $\mathcal{A}(\pi \otimes \chi_\infty)$.

By hypothesis,

$$\sum_{\Pi \in \mathcal{A}(\pi \otimes \chi_\infty)} m(\Pi)(\dim(\Pi^{K_1}) - \dim(\Pi^{K_2})) = 0.$$

In other words,

$$\sum_{\Pi \in \mathcal{A}(\pi)} m(\Pi \otimes \chi)(\dim((\Pi \otimes \chi)^{K_1}) - \dim((\Pi \otimes \chi)^{K_2})) = 0.$$

Now χ is unramified, so $\dim (\Pi \otimes \chi)^{K_i} = \dim (\Pi)^{K_i}$ for $i = 1, 2$. Moreover, by the multiplicity one theorem for $GL(n)$, we have $m(\Pi) = 1$ for all $\Pi \in \mathcal{A}_{\text{cusp}}$. So we have,

$$\sum_{\Pi \in \mathcal{A}(\pi)} (\dim(\Pi)^{K_1} - \dim(\Pi)^{K_2}) = 0.$$

Therefore, $m(\pi, \Gamma_{K_1}) = m(\pi, \Gamma_{K_2})$. Now this holds for every $\pi \in \mathcal{S}$. Hence,

$$L^2_{\text{cusp}}(\Gamma_{K_1} \backslash GL(n, F_\infty)) \cong L^2_{\text{cusp}}(\Gamma_{K_2} \backslash GL(n, F_\infty)).$$

□

Chapter 6

Discrete Series Multiplicities

6.1 Statements of Main Results

Let G be a semisimple connected Lie group whose complexification $G_{\mathbb{C}}$ is simply connected. We assume that G contains a compact Cartan subgroup, i.e. $\text{rank}(G) = \text{rank}(K)$, where K is a maximal compact subgroup in G . Consider Γ to be a lattice in G such that $\text{vol}(\Gamma \backslash G) < \infty$ with respect to the unique G -invariant Haar measure on $\Gamma \backslash G$. We have the right regular representation R_{Γ} of G on

$$L^2(\Gamma \backslash G) := \{ \phi : G \rightarrow \mathbb{C} \mid \phi(\gamma x) = \phi(x) \text{ for all } \gamma \in \Gamma, x \in G \text{ and } \int_{\Gamma \backslash G} |\phi(x)|^2 dx < \infty \}. \quad (6.1.1)$$

Let $L^2_{\text{disc}}(\Gamma \backslash G)$ be the maximal semisimple subrepresentation of $L^2(\Gamma \backslash G)$. Then

$$L^2_{\text{disc}}(\Gamma \backslash G) \cong \widehat{\bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) \pi},$$

where $\widehat{G}$ is the set of equivalence classes of unitary irreducible representations of G , and

$$m(\pi, \Gamma) = \dim \operatorname{Hom}_G(\pi, L_{\text{disc}}^2(\Gamma \backslash G)).$$

Since G has compact Cartan subgroups, it possesses the discrete series representations. It is well-known that discrete series representations are parametrized by the *Harish-Chandra* parameters λ , and denoted by π_λ (see Subsec. [6.2.1](#)). The purpose of this chapter is to present a linear dependence among the multiplicities $m(\pi_\lambda, \Gamma)$ of the discrete series representations π_λ of G . The multiplicity $m(\pi_\lambda, \Gamma)$ depends on $\operatorname{vol}(\Gamma \backslash G)$. For example, if Γ is a uniform torsion-free lattice in G then the multiplicity $m(\pi_\lambda, \Gamma)$ is equal to $d_{\pi_\lambda} \operatorname{vol}(\Gamma \backslash G)$, where d_{π_λ} is the formal degree of π_λ (see [Kna01](#) for the definition). Moreover, if Γ contains nontrivial elliptic elements (i.e. torsion elements) then the Γ -conjugacy classes of these elliptic elements also contribute to $m(\pi_\lambda, \Gamma)$.

Let us fix a maximal compact subgroup K in G . By the assumption on G , there exists a compact Cartan subgroup $T \subset K$. Let $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{t}_0$ be the Lie algebras of G, K, T respectively, and $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$ be their complexifications respectively. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ and $\Phi_c = \Phi(\mathfrak{k}, \mathfrak{t})$ be the root systems of (G, T) and (K, T) respectively such that $\Phi_c \subset \Phi$. Let Φ^+ be a fixed but arbitrary choice of a positive system of roots in Φ . A *Harish-Chandra* parameter λ is an integral regular linear form in the complex dual space $\mathfrak{t}^*$ of $\mathfrak{t}$. Let $\mathcal{D}(G)$ denote the set of all Harish-Chandra parameters in G (see Subsec. [6.2.1](#) for detailed definition). Every element $\lambda \in \mathcal{D}(G)$ determines a unique positive system of roots $\{\alpha \in \Phi : \langle \lambda, \alpha \rangle > 0\}$ which is denoted by P^λ . Any discrete series representation is uniquely determined by such a Harish-Chandra parameter λ up to W_c -conjugate (see Thm. [6.2.1](#) for details). Here, W_c denotes the Weyl group of the root system Φ_c .

Definition 6.1.1. Assume $\lambda_1, \lambda_2 \in \mathcal{D}(G)$ are given such that $P^{\lambda_1} = P^{\lambda_2}$. The infinite set $\mathcal{S}(\lambda_1, \lambda_2) := \{\lambda_1 + k\lambda_2 \in \mathfrak{t}^* : k \in \mathbb{N} \cup \{0\}\}$ of integral linear forms is called a string with base λ_1 and direction λ_2 . For each $k \in \mathbb{N} \cup \{0\}$, we denote $\lambda_1 + k\lambda_2$ by λ_k .

Note that every linear form $\lambda_k \in \mathcal{S}(\lambda_1, \lambda_2)$ is regular and therefore belongs to $\mathcal{D}(G)$. The discrete series representation determined by $\lambda_k \in \mathcal{D}(G)$ is denoted by π_{λ_k} . Each λ_k determines the same positive system of roots i.e. $P^{\lambda_k} = P^{\lambda_1} = P^{\lambda_2}$.

We need to impose a mild condition on $\lambda \in \mathcal{D}(G)$. We denote

$$\mathcal{D}^*(G) := \{\lambda \in \mathcal{D}(G) : \langle \lambda - \delta^\lambda, \alpha \rangle > 0 \text{ for all } \alpha \in P^\lambda \cap \Phi_n\},$$

where $\Phi_n = \Phi \setminus \Phi_c$ and $\delta^\lambda := \frac{1}{2} \sum_{\alpha \in P^\lambda} \alpha$. From Lem. 6.2.10, it can be seen that if the base or the direction of a string belongs to $\mathcal{D}^*(G)$ then every linear form in the string belongs to $\mathcal{D}^*(G)$.

Let G_e denote the set $\bigcup_{g \in G} gKg^{-1}$ of elliptic elements in G . Note that G_e does not depend on the choice of maximal compact subgroup K . The set $\Gamma \cap G_e$ of elliptic elements in Γ is a union of *finitely many* Γ -conjugacy classes i.e. $\Gamma \cap G_e = \bigcup_i [y_i]$ for some choice of representatives $y_i \in \Gamma \cap G_e$. Finally, we define the maximal order of elliptic elements in Γ to be the least common multiple of the orders of the elements $\{y_i\}_i$. Let us denote it by N_Γ . It is an invariant of Γ .

Now, we are ready to display our first main result.

Theorem 6.1.2. Let G be a semisimple connected Lie group whose complexification $G_{\mathbb{C}}$ is simply connected. Assume that G contains a compact Cartan subgroup. Let Γ be a uniform

lattice in G , and N_Γ be the maximal order of elliptic elements in Γ . For two Harish-Chandra parameters λ_1 and λ_2 with $P^{\lambda_1} = P^{\lambda_2}$, consider the string $\mathcal{S}(\lambda_1, \lambda_2)$ such that the base or the direction belongs to $\mathcal{D}^*(G)$. If a finite set $\mathcal{A} \subset \mathbb{N} \cup \{0\}$ satisfies $|\mathcal{A} \cap (j + N_\Gamma \mathbb{Z})| \geq |\Phi^+| + 1$ for all $0 \leq j \leq N_\Gamma - 1$, then for any $\lambda_\ell \in \mathcal{S}(\lambda_1, \lambda_2)$, the discrete series multiplicity $m(\pi_{\lambda_\ell}, \Gamma) = \sum_{k \in \mathcal{A}} n(\ell, k) m(\pi_{\lambda_k}, \Gamma)$ where $n(\ell, k)$ are integers and independent of Γ .

Since the integers $n(\ell, k)$ in the above linear expression do not depend on Γ , we have the following corollary:

Corollary 6.1.1. Assume the hypothesis on G as in Thm. 6.1.2. Let Γ_1 and Γ_2 be two uniform lattices in G , and $N = \text{l.c.m.}(N_{\Gamma_1}, N_{\Gamma_2})$. For two Harish-Chandra parameters λ_1 and λ_2 with $P^{\lambda_1} = P^{\lambda_2}$, consider the string $\mathcal{S}(\lambda_1, \lambda_2)$ such that the base or the direction belongs to $\mathcal{D}^*(G)$. Let us choose a finite set $\mathcal{A} \subset \mathbb{N} \cup \{0\}$ which satisfies $|\mathcal{A} \cap (j + N\mathbb{Z})| \geq |\Phi^+| + 1$ for all $0 \leq j \leq N - 1$. If $m(\pi_{\lambda_k}, \Gamma_1) = m(\pi_{\lambda_k}, \Gamma_2)$ for all $k \in \mathcal{A}$, then $m(\pi_{\lambda_k}, \Gamma_1) = m(\pi_{\lambda_k}, \Gamma_2)$ for all $\lambda_k \in \mathcal{S}(\lambda_1, \lambda_2)$.

In Subsec. 6.4.3, we obtain a generalised version of Thm. 6.1.2 for a multi-directed string (see Def. 6.4.1) of discrete series multiplicities. Even in this general case, it is sufficient to assume that only one direction belongs to $\mathcal{D}^*(G)$.

Remark 6.1.1. If G has one dimensional Cartan subgroup T i.e. $\dim \mathfrak{t} = 1$ (e.g. $G = \text{SL}(2, \mathbb{R})$), then for any Harish-Chandra parameter $\lambda \in \mathcal{D}^*(G)$ the set $\{k\lambda : k \in \mathbb{Z} \setminus \{0\}\}$ exhausts all but finitely many Harish-Chandra parameters in $\mathcal{D}^*(G)$. Therefore, Thm. 6.1.2 concludes that if G has a one-dimensional Cartan subgroup, then there exists a finite subset $\mathcal{F} \subset \mathcal{D}^*(G)$ such that for any $\mu \in \mathcal{D}^*(G)$, the discrete series multiplicity $m(\pi_\mu, \Gamma)$ is determined by (a linear expression of) the multiplicities $\{m(\pi_\lambda, \Gamma) : \lambda \in \mathcal{F}\}$.

As an application of Thm. [6.1.2](#), we obtain the following strong multiplicity one result for a string of discrete series representations. In fact, it holds for any infinite set of exceptions in the string with sufficiently small density.

Corollary 6.1.2. *Assume the hypothesis on G and K as in Thm. [6.1.2](#). Let Γ_1 and Γ_2 be two uniform lattices in G , and $N = \text{l.c.m.}(N_{\Gamma_1}, N_{\Gamma_2})$. For two Harish-Chandra parameters λ_1 and λ_2 with $P^{\lambda_1} = P^{\lambda_2}$, consider the string $\mathcal{S}(\lambda_1, \lambda_2)$ such that the base or the direction belongs to $\mathcal{D}^*(G)$. If the following condition holds:*

$$\limsup_{t \rightarrow \infty} \frac{|\{0 \leq k \leq t : m(\pi_{\lambda_k}, \Gamma_1) \neq m(\pi_{\lambda_k}, \Gamma_2)\}|}{t} < \frac{1}{N},$$

then $m(\pi_{\lambda_k}, \Gamma_1) = m(\pi_{\lambda_k}, \Gamma_2)$ for all $\lambda_k \in \mathcal{S}(\lambda_1, \lambda_2)$.

In particular, from Cor. [6.1.2](#) we can recover the following analogue of ([BR11](#); Thm. 1.1) for a large class of discrete series representations:

Corollary 6.1.3. *Assume the hypothesis on G and K as in Thm. [6.1.2](#). Let Γ_1 and Γ_2 be two uniform lattices in G . Assume that $m(\pi_\lambda, \Gamma_1) = m(\pi_\lambda, \Gamma_2)$ for all but finitely many discrete series representations π_λ with $\lambda \in \mathcal{D}^*(G)$. Then $m(\pi_\lambda, \Gamma_1) = m(\pi_\lambda, \Gamma_2)$ for all discrete series representations π_λ with $\lambda \in \mathcal{D}^*(G)$.*

Let us discuss the subset $\mathcal{D}^*(G)$ of $\mathcal{D}(G)$. The condition on λ in the definition of $\mathcal{D}^*(G)$ can be seen as a sufficiently regularity condition. If π_λ is an integrable discrete series representation i.e. all the K -finite matrix coefficients of π_λ are in $L^1(G)$, then λ satisfies the Trombi-Varadarajan estimate (see [TV72](#)):

$$|\langle \lambda, \beta \rangle| > \frac{1}{2} \sum_{\alpha \in P^\lambda} |\langle \alpha, \beta \rangle| \text{ for every } \beta \in P_n^\lambda.$$

But if λ satisfies the above estimate, then λ belongs to $\mathcal{D}^*(G)$. Therefore, Thm. 6.1.2 holds for all integrable discrete series representations. Moreover, our results hold for infinitely many non-integrable discrete series as well. For instance, in the case of $\mathrm{SL}(2, \mathbb{R})$, the set $\mathcal{D}(G) \setminus \mathcal{D}^*(G)$ is finite. In this sense, in the definition of $\mathcal{D}^*(G)$ the imposed condition on Harish-Chandra parameters is indeed a mild condition.

6.2 Preliminaries

6.2.1 Basic Setup

Let G be a connected semisimple non-compact Lie group with finite center. We fix once and for all a maximal compact subgroup K in G . Assume that G has a compact Cartan subgroup $T \subset K$; equivalently, $\mathrm{rank}(G) = \mathrm{rank}(K)$. Let $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{t}_0$ be the Lie algebras of G, K, T respectively, and $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$ be their complexifications respectively. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ and $\Phi_c = \Phi(\mathfrak{k}, \mathfrak{t})$ be the root systems of (G, T) and (K, T) respectively such that $\Phi_c \subset \Phi$, and let $W_c \subset W$ be their corresponding Weyl groups respectively. The set Φ_c is called the set of compact roots, and its complement Φ_n in Φ is called the set of non-compact roots. Let $\mathfrak{p}_0$ be the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to the killing form. Denote by $\mathfrak{g}_\alpha$ the root space of the root $\alpha \in \Phi$. Then $\alpha \in \Phi_c$ if and only if $\mathfrak{g}_\alpha \subset \mathfrak{k}$, and $\alpha \in \Phi_n$ if and only if $\mathfrak{g}_\alpha \subset \mathfrak{p}$. Moreover, $\dim \mathfrak{p}$ is even. We fix an arbitrary positive system Φ^+ in Φ with $\Phi_c^+ = \Phi^+ \cap \Phi_c, \Phi_n^+ = \Phi_n \cap \Phi^+, \delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \delta_c = \frac{1}{2} \sum_{\alpha \in \Phi_c^+} \alpha$, and $\delta_n = \frac{1}{2} \sum_{\alpha \in \Phi_n^+} \alpha = \delta - \delta_c$.

Let $\mathcal{F}$ be the set of all linear forms $\lambda \in \mathfrak{t}^*$ such that $\frac{2\langle \lambda - \delta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ is an integer. Such a linear form is called an integral linear form.

We define the set $\mathcal{D}(G) := \{\lambda \in \mathcal{F} : \langle \lambda, \alpha \rangle \neq 0 \text{ for all } \alpha \in \Phi; \text{ and } \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Phi_c^+\}$. An integral linear form λ satisfying $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Phi$ is called a *regular* integral linear form. Note that in the definition of $\mathcal{D}(G)$, the condition $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Phi_c^+$ ensures that no two $\lambda_1, \lambda_2 \in \mathcal{D}(G)$ are W_c -conjugates of each other. The following celebrated theorem of Harish-Chandra says that the set $\mathcal{D}(G)$ completely parameterizes the set of all equivalence classes of discrete series representations.

Theorem 6.2.1. ([HC68]) *Let $\lambda \in \mathcal{D}(G)$. Then there is a discrete series representation π_λ of G whose global character Θ_{π_λ} , as a locally L^1 -function on G , is given on T by*

$$\Theta_{\pi_\lambda}(e^X) = \frac{(-1)^m \prod_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle \sum_{w \in W_c} \det(w) e^{w\lambda(X)}}{\prod_{\alpha \in \Phi^+} (e^{\frac{\alpha(X)}{2}} - e^{-\frac{\alpha(X)}{2}})}$$

for $X \in \mathfrak{t}$, and $\dim G/K = 2m$. Every discrete series representation is unitarily equivalent to π_λ for some $\lambda \in \mathcal{D}(G)$.

Remark 6.2.1. Since no two linear forms in $\mathcal{D}(G)$ are W_c -conjugates to each other, the set $\mathcal{D}(G)$ completely parametrizes the set of equivalence classes of discrete series representations of G .

If an integral linear form λ is regular, then it gives rise to a positive system $P^\lambda = \{\alpha \in \Phi : \langle \lambda, \alpha \rangle > 0\}$. We denote $\delta^\lambda = \frac{1}{2} \sum_{\alpha \in P^\lambda} \alpha$.

Lemma 6.2.2. *For a given positive chamber (i.e. a given positive system) Σ^+ in Φ , there exists a basis $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of $\mathfrak{t}^*$ consisting of elements in $\mathcal{D}(G)$ such that $P^{\lambda_i} = \Sigma^+$.*

Proof. We know that $\{\lambda \in \mathfrak{t}^* : \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Phi^+\} = \{\lambda \in \mathfrak{t}^* : P^\lambda = \Phi^+\}$ is an open subset of $\mathfrak{t}^*$. Translating by a suitable Weyl group element we have the open set $\{\lambda \in$

$\mathfrak{t}^* : P^\lambda = \Sigma^+\}$. Now it is straightforward to see that any open set in a finite-dimensional vector space (with respect to standard Euclidean topology) contains a basis of the vector space. Therefore, we can choose a $\mathbb{Z}$ -basis $\{\lambda_1, \dots, \lambda_n\}$ with $\lambda_i \in \{\lambda \in \mathfrak{t}^* : P^\lambda = \Sigma^+\}$. $\square$

Consider Γ to be a uniform lattice in G . We have a right regular representation R_Γ of G on $L^2(\Gamma \backslash G)$. It decomposes as a Hilbert direct sum of unitary irreducible representations π of G with finite multiplicity;

$$L^2(\Gamma \backslash G) \cong \widehat{\bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) \pi}.$$

Here $m(\pi, \Gamma)$ is the multiplicity of π in $(R_\Gamma, L^2(\Gamma \backslash G))$; in other words, $m(\pi, \Gamma) = \dim \text{Hom}_G(\pi, R_\Gamma)$. The multiplicity of a discrete series representation π_λ with the Harish-Chandra parameter λ is denoted by $m(\pi_\lambda, \Gamma)$.

6.2.2 Lefschetz number and discrete series multiplicities

In this subsection, we define the Lefschetz number; and describe the relation between the Lefschetz number and the discrete series multiplicities. We need to introduce the notion of an invariant Dirac operator on G/K . The reader is referred to Chap. 6 of [Wil91] for detailed description.

We have $\mathfrak{p} = \sum_{\alpha \in \Phi_n} \mathfrak{g}_\alpha$. We choose a basis $\{X_i\}$ and $\{Y_j\}$ of $\mathfrak{k}$ and $\mathfrak{p}$ respectively, such that with respect to a bilinear form B on $\mathfrak{g}$ induced from the Killing form, we have $B(X_k, X_l) = -\delta_{kl}$ and $B(Y_m, Y_n) = \delta_{mn}$. Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to the non-degenerate bilinear form B . Since $\dim \mathfrak{p}$ is even, there is an algebra isomorphism ϵ between $C(\mathfrak{p})$ and the matrix algebra $\text{End}(S)$ for some vector space S with dimension $2^{(\frac{\dim \mathfrak{p}}{2})}$. The

restriction of ϵ to the spin group $\text{Spin}(\mathfrak{p}_0) \subset C(\mathfrak{p})$ gives a spin representation $\sigma := \epsilon|_{\text{Spin}(\mathfrak{p}_0)}$ on S . The spin representation σ turns out to be the direct sum of two ($\frac{1}{2}$ spin) irreducible representations $\sigma^\pm$ on $S^\pm$, and $S = S^+ \oplus S^-$. Let us denote the Casimir element by Ω , which is given by $\Omega = -\sum_i X_i^2 + \sum_j Y_j^2$, in the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$.

Now fix a $\lambda \in \mathcal{D}(G)$. It can be easily seen that $\lambda - \delta_c$ is Φ_c^+ -dominant. Therefore, there exists a finite dimensional irreducible representation $\tau_{\lambda - \delta_c}$ of $\mathfrak{k}$ on a finite-dimensional vector space $V_{\lambda - \delta_c}$ with highest weight $\lambda - \delta_c$. Let us denote $\chi_\lambda^\pm := \sigma^\pm \otimes \tau_{\lambda - \delta_c}$ to be a representation of $\mathfrak{k}$ (note that $\mathfrak{k}$ is also the complexified Lie algebra of the spin group $\text{Spin } \mathfrak{p}_0$). This may not integrate to a representation of K in general. However under the assumption that $\lambda + \delta_n^{(\lambda)}$ exponentiates to a character of T , $\chi_\lambda^\pm$ does integrate to a representation of K . We consider the smooth homogeneous vector bundle $E_\lambda^\pm := G \times_K \chi_\lambda^\pm$ on G/K . Let $\Gamma^\infty E_\lambda^\pm$ be the space of smooth sections of a homogeneous vector bundle $E_\lambda^\pm$. Then there exist 1st order G -invariant elliptic differential operators $D_\lambda^\pm : \Gamma^\infty E_\lambda^\pm \rightarrow \Gamma^\infty E_\lambda^\mp$. These are Dirac operators twisted by $\tau_{\lambda - \delta_c}$. With respect to appropriate metric, D_λ^+ and D_λ^- are adjoints of each other, and their composition $D_\lambda^\pm D_\lambda^\mp$ has the following simple description due to R. Parthasarathy:

Theorem 6.2.3. ([\[Par72\]](#)) For all $\lambda \in \mathcal{D}(G)$, we have $D_\lambda^\pm D_\lambda^\mp = -\Omega + \langle \lambda - \delta_c, \lambda + \delta + \delta_n \rangle$, where Ω is the Casimir element.

Finally, we define $H^\pm(E_\lambda, \Gamma)$ to be the space of all Γ -invariant sections $s \in \Gamma^\infty E_\lambda^\pm$ such that $D_\lambda^\pm s = 0$. Now we are ready to define the *Lefschetz number* and state the theorem that relates it with the discrete series multiplicities.

Definition 6.2.4. For any $\lambda \in \mathcal{D}(G)$ and a uniform lattice Γ in G , the difference $\dim H^+(E_\lambda, \Gamma) -$

$\dim H^-(E_\lambda, \Gamma)$ is called the Lefschetz number, and is denoted by $\chi(\lambda, \Gamma)$.

Remark 6.2.2. If Γ is a uniform torsion-free lattice in G , then the elliptic operator D_λ^+ projects to a first order elliptic differential operator $D_{\Gamma, \lambda}^+$ on the locally symmetric space $\Gamma \backslash G/K$. In this case, the above Lefschetz number $\chi(\lambda, \Gamma)$ is called the index of the differential operator $D_{\Gamma, \lambda}^+$.

Theorem 6.2.5. (Thm. 3.3; [Wil84]) Suppose that $\lambda \in \mathcal{D}^*(G)$, and $\pi_\lambda \in \widehat{G}$ is the associated discrete series representation of G . Let $w_0 \in W$ be the unique Weyl group element such that $w_0\Phi^+ = P^\lambda$, and let $l(w_0) = |w_0(-\Phi^+) \cap \Phi^+|$ be the length of w_0 . Then for any uniform lattice Γ in G , the multiplicity $m(\pi_\lambda, \Gamma)$ of π_λ in $L^2(\Gamma \backslash G)$ is given by

$$m(\pi_\lambda, \Gamma) = (-1)^{l(w_0)} \{ \dim H^+(E_\lambda, \Gamma) - \dim H^-(E_\lambda, \Gamma) \} = (-1)^{l(w_0)} \chi(\lambda, \Gamma).$$

6.2.3 The results of Hotta-Parthasarathy on Lefschetz number

In this subsection, we review a couple of results of Hotta and Parthasarathy on the Lefschetz number $\chi(\lambda, \Gamma)$ from section 2 of [HP73]. For any $y \in T$, let G_y^0 be the identity component of the centraliser G_y of y in G . Let W_y be the Weyl group for the root system $\Phi(G_y^0, T)$. Let K_y be the centraliser of y in K and $d = \dim G/K$, $d_y = \dim G_y/K_y$. We use the convention that e^α is the character of T corresponding to an element $\alpha \in \mathfrak{t}^*$. Let $\Phi_y = \{\alpha \in \Phi : e^\alpha(y) = 1\}$. Then $\Phi_y^+ = \Phi^+ \cap \Phi_y$ is a positive system of $\Phi(G_y^0, T)$.

For any $\lambda \in \mathcal{D}(G)$, we define the function Ψ_λ^0 on T as follows:

$$\Psi_\lambda^0(y) = \frac{\sum_{w \in W_c/W_y} (-1)^{l(w)} e^{w\lambda - \delta}(y) \prod_{\alpha \in \Phi_y^+} \langle w\lambda, \alpha \rangle}{\prod_{\alpha \in \Phi^+ \setminus \Phi_y^+} (1 - e^{-\alpha}(y))}.$$

Here $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$, and W_y is considered as a subgroup of W_c . The function Ψ_λ^0 is W_c -invariant in λ , and it extends to an $\text{Ad}(G)$ -invariant function on the set G_e of elliptic elements of G i.e. the set of elements of G conjugate to some element of K .

Remark 6.2.3. If y is a regular element in G , then $\Phi_y = \emptyset$. So we take $\prod_{\alpha \in \Phi_y^+} \langle w\lambda, \alpha \rangle = 1$.

Finally, for any $\lambda \in \mathcal{D}(G)$ we define the function $\Psi_\lambda(y)$ in y as follows:

$$\Psi_\lambda(y) = \frac{(-1)^{\frac{d+d_y}{2}} \prod_{\alpha \in \Phi_y^+} (\langle \delta_y, \alpha \rangle)^{-1} \Psi_\lambda^0(y)}{|W_y| [G_y : G_y^0]},$$

For any element $y \in \Gamma$, we denote Γ_y as the centralizer of y in Γ . Then we have the following result of Hotta and Parthasarathy:

Theorem 6.2.6. (Thm. 3; [HP73]) Let Γ be a uniform lattice in G . For an arbitrary $\lambda \in \mathcal{D}(G)$ we have

$$\chi(\lambda, \Gamma) = (-1)^{\ell(w_0)} \sum_{[y]} \text{vol}(\Gamma_y \backslash G_y) \Psi_\lambda(y),$$

where the sum runs over the (finite) set of Γ -conjugacy classes of elliptic elements in Γ and w_0 be the unique element in W such that $w_0\Phi^+ = P^\lambda$.

Remark 6.2.4. Note that the above theorem does not require $\lambda \in \mathcal{D}^*(G)$.

Combining Thm. 6.2.5 and Thm. 6.2.6, we deduce the following:

Theorem 6.2.7. Let $\lambda \in \mathcal{D}^*(G)$ and $w_0 \in W$ such that $w_0\Phi^+ = P^{(\lambda)}$. Then

$$m(\pi_\lambda, \Gamma) = \sum_{[y]} \text{vol}(\Gamma_y \backslash G_y) \Psi_\lambda(y),$$

where the sum runs over the set of Γ -conjugacy classes of elliptic elements in Γ .

Corollary 6.2.1. *Assume the hypothesis on $\lambda \in \mathcal{D}^*(G)$ as in Thm. 6.2.7. If Γ is uniform torsion-free lattice in G , then $m(\pi_\lambda, \Gamma) = \text{vol}(\Gamma \backslash G) d_{\pi_\lambda}$.*

Proof. The proof follows from the identity: $\Psi_\lambda(1) = \frac{\prod_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle}{\prod_{\alpha \in \Phi^+} \langle \delta, \alpha \rangle |W|} = d_{\pi_\lambda}$. □

Remark 6.2.5. *The Thm. 7.47 of [Wil91] is an extension of Thm. 6.2.5 for the lattices Γ which are torsion-free, of finite $\text{vol}(\Gamma \backslash G)$ and satisfy some Langlands assumptions. These latter assumptions on Γ are automatic either when Γ is an arithmetic lattice and G/K is a rank one symmetric space or when G has no compact simple factor. Since the multiplicity of discrete series representation for torsion free lattices Γ is very explicit by Cor. 6.2.1, the only interesting cases in our context are for Γ having nontrivial elliptic elements.*

6.2.4 String of regular integral linear forms

In this subsection we introduce the notion of a string of regular integral linear forms in $\mathcal{D}(G)$ and $\mathcal{D}^*(G)$.

Definition 6.2.8. *Given two integral regular linear forms $\lambda_1, \lambda_2 \in \mathcal{D}(G)$, the countably infinite set $\mathcal{S}(\lambda_1, \lambda_2) := \{\lambda_1 + k\lambda_2 : k \in \mathbb{N} \cup \{0\}\}$ is called a string of integral regular linear forms with base λ_1 and direction λ_2 .*

For $\lambda_1, \lambda_2 \in \mathcal{D}(G)$, let P^{λ_1} and P^{λ_2} be the corresponding positive root systems for the regular linear forms λ_1 and λ_2 respectively. If $P^{\lambda_1} \neq P^{\lambda_2}$, then there exists $\sigma \in W$ such that $\sigma P^{\lambda_1} = P^{\lambda_2}$. On the other hand, $\sigma P^{\lambda_1} = P^{\sigma \lambda_1}$. Therefore without loss of generality, we can consider $\lambda_1, \lambda_2 \in \mathcal{D}(G)$ such that $P^{\lambda_1} = P^{\lambda_2}$.

Lemma 6.2.9. *If $\lambda \in \mathcal{D}^*(G)$, then $k\lambda$ belongs to $\mathcal{D}^*(G)$ for every $k \in \mathbb{N}$.*

Proof. For any $k \in \mathbb{N}$, it is easy to see that $k\lambda \in \mathcal{D}(G)$ and $P^\lambda = P^{k\lambda}$ i.e. λ and $k\lambda$ determine the same positive chamber. We have

$$\begin{aligned} \langle k\lambda - \delta^{k\lambda}, \alpha \rangle &= \langle (k-1)\lambda, \alpha \rangle + \langle \lambda - \delta^\lambda, \alpha \rangle \\ &= (k-1)\langle \lambda, \alpha \rangle + \langle \lambda - \delta^\lambda, \alpha \rangle. \end{aligned} \tag{6.2.1}$$

Now for any $\alpha \in P^\lambda \cap \Phi_n$, $\langle \lambda, \alpha \rangle > 0$ and by the hypothesis, $\langle \lambda - \delta^\lambda, \alpha \rangle > 0$. Hence $k\lambda \in \mathcal{D}^*(G)$. $\square$

Before describing the next lemma, we record a useful observation. If $\lambda_1, \lambda_2 \in \mathcal{D}(G)$, then one can easily check that $P^{(\lambda_1)} \cap P^{(\lambda_2)} \subset P^{(\lambda_1 + \lambda_2)} \subset P^{(\lambda_1)} \cup P^{(\lambda_2)}$.

Lemma 6.2.10. *Consider two linear forms $\lambda_1, \lambda_2 \in \mathcal{D}(G)$ with $P^{\lambda_1} = P^{\lambda_2}$. If λ_1 or λ_2 belongs to $\mathcal{D}^*(G)$, then every linear form in the string $S(\lambda_1, \lambda_2)$ belongs to $\mathcal{D}^*(G)$.*

Proof. Indeed, $\lambda_1 + k\lambda_2 \in \mathcal{D}(G)$ for any $k \in \mathbb{N}$ and $P^{(\lambda_1)} \cap P^{(k\lambda_2)} \subset P^{(\lambda_1 + k\lambda_2)} \subset P^{(\lambda_1)} \cup P^{(k\lambda_2)}$. Therefore all integral linear forms of the string determine the same positive system as the one determined by the base and the direction of the string. Without loss of generality, we assume that $\lambda_2 \in \mathcal{D}^*(G)$. Let us write $P_n^\lambda = P^\lambda \cap \Phi_n$. Then for any $\alpha \in P_n^{(\lambda_1 + k\lambda_2)} = P_n^{(k\lambda_2)} = P_n^{(\lambda_1)} \subset P^{(\lambda_1)}$, we have

$$\langle \lambda_1 + k\lambda_2 - \delta^{(\lambda_1 + k\lambda_2)}, \alpha \rangle = \langle \lambda_1, \alpha \rangle + \langle k\lambda_2 - \delta^{(k\lambda_2)}, \alpha \rangle > 0.$$

Therefore, every member of the string $S(\lambda_1, \lambda_2)$ belongs to $\mathcal{D}^*(G)$. $\square$

Proposition 6.2.11. *Let π_λ be the discrete series representation with Harish-Chandra parameter $\lambda \in \mathcal{D}(G)$. Assume that $\lambda_1, \lambda_2 \in \mathcal{D}(G)$ with $P^{\lambda_1} = P^{\lambda_2}$. Then $\{\pi_{\lambda_1 + k\lambda_2} : k \in \mathbb{N}\}$ is an infinite family of inequivalent discrete series representations of G .*

Proof. We know that two discrete series representations π_{λ_1} and π_{λ_2} are equivalent if and only if λ_1 and λ_2 are W_c -conjugate. We need to show that for two distinct positive integers k_1 and k_2 , the linear forms $\lambda_1 + k_1\lambda_2$ and $\lambda_1 + k_2\lambda_2$ are not W_c -conjugate.

If $w \in W_c$ and $\mu \in \mathcal{D}(G)$ such that $\langle \mu, \alpha \rangle > 0$ for all $\alpha \in \Phi_c^+$, then by the definition of the length of w there exists $\alpha \in \Phi_c^+$ (which depends on w) such that $\langle w\mu, \alpha \rangle < 0$. In other words, if $\mu_1, \mu_2 \in \mathcal{D}(G)$ with $\langle \mu_1, \alpha \rangle > 0$ and $\langle \mu_2, \alpha \rangle > 0$ for all $\alpha \in \Phi_c^+$ then μ_1 and μ_2 are *not* W_c -conjugate.

Thus it is enough to show that $\langle \lambda_1 + k\lambda_2, \alpha \rangle > 0$ for all $\alpha \in \Phi_c^+$. For any $\alpha \in \Phi_c^+$, we have $\langle \lambda_1 + k\lambda_2, \alpha \rangle = \langle \lambda_1, \alpha \rangle + k\langle \lambda_2, \alpha \rangle$. Now $\lambda_1, \lambda_2 \in \mathcal{D}(G)$ implies that $\Phi_c^+ \subset P^{(\lambda_2)} = P^{(k\lambda_2)} = P^{(\lambda_1)}$. Therefore, $\langle \lambda_1, \alpha \rangle > 0$ and $\langle \lambda_2, \alpha \rangle > 0$ for all $\alpha \in \Phi_c^+$. Hence the proof follows. $\square$

Remark 6.2.6. *In fact the above proposition shows that no two linear forms in a string give rise to equivalent discrete series representations. In other words, each member of a string determines a discrete series representation which is inequivalent to the rest.*

6.3 Generating Function for a String

For two Harish-Chandra parameters λ_1, λ_2 , let $\mathcal{S}(\lambda_1, \lambda_2)$ be a string such that the base λ_1 or the direction λ_2 belongs to $\mathcal{D}^*(G)$. For this string, we formally define the generating function in complex variable z as

$$F_{\lambda_1, \lambda_2, \Gamma}(z) := \sum_{k \in \mathbb{N}} m(\pi_k, \Gamma) z^{k-1},$$

where $\pi_k = \pi_{\lambda_1 + k\lambda_2}$ with $k \in \mathbb{N}$. Let $\lambda_k = \lambda_1 + k\lambda_2$ and $w_0 \in W$ such that $w_0\Phi^+ = P^{\lambda_k} =$

$P^{\lambda_1} = P^{\lambda_2}$. Then by Thm. 6.2.5, the generating function $F_{\lambda_1, \lambda_2, \Gamma}(z)$ can be rewritten as

$$F_{\lambda_1, \lambda_2, \Gamma}(z) = \sum_{k \in \mathbb{N}} (-1)^{l(w_0)} \chi(\lambda_k, \Gamma) z^{k-1} = (-1)^{l(w_0)} \sum_{k \in \mathbb{N}} \chi(\lambda_k, \Gamma) z^{k-1}.$$

The number of Γ -conjugacy classes in the set $\Gamma \cap G_e$ of elliptic elements in Γ is finite. We fix a (finite) set of representatives $\{y_i\}$ of these classes. The least common multiple of the order of the elements $\{y_i\}_i$ is called the maximal order of the elliptic elements in Γ , and it is denoted by N_Γ .

Proposition 6.3.1. *Let N_Γ be the maximal order of elliptic elements in Γ . Let $S(\lambda_1, \lambda_2)$ be a string with $P^{\lambda_1} = P^{\lambda_2}$, and $F_{\lambda_1, \lambda_2, \Gamma}$ be its associated generating function in the complex variable z . Then there exists a complex polynomial $p_{\lambda_1, \lambda_2, \Gamma}$ with degree less than $N_\Gamma(|\Phi^+| + 1)$ such that*

$$F_{\lambda_1, \lambda_2, \Gamma}(z) = \frac{p_{\lambda_1, \lambda_2, \Gamma}(z)}{(1 - z^{N_\Gamma})^{|\Phi^+| + 1}}.$$

In particular, $F_{\lambda_1, \lambda_2, \Gamma}(z)$ is a complex rational function whose poles are the N_Γ -th roots of unity, each of order $|\Phi^+| + 1$.

Proof. By Thm. 6.2.7, we have

$$\begin{aligned} F_{\lambda_1, \lambda_2, \Gamma}(z) &= \sum_{k \in \mathbb{N}} \sum_{[y]} \text{vol}(\Gamma_y \backslash G_y) \Psi_{\lambda_k}(y) z^{k-1} \\ &= \sum_{[y]} \text{vol}(\Gamma_y \backslash G_y) \sum_{k \in \mathbb{N}} \Psi_{\lambda_k}(y) z^{k-1}, \end{aligned} \tag{6.3.1}$$

where $[y]$ runs over the Γ -conjugacy classes of elliptic elements in Γ . Now by definition of

$\Psi_{\lambda_k}(y)$, we get

$$\begin{aligned}
 F_{\lambda_1, \lambda_2, \Gamma}(z) &= \sum_{[y]} \text{vol}(\Gamma_y \backslash G_y) \sum_{k \in \mathbb{N}} \frac{(-1)^{\frac{d+d_y}{2}} \prod_{\alpha \in \Phi_y^+} (\langle \delta_y, \alpha \rangle)^{-1} \Psi_{\lambda_k}^0(y)}{|W_y|[G_y : G_y^0]} z^{k-1} \\
 &= \sum_{[y]} \frac{\text{vol}(\Gamma_y \backslash G_y) (-1)^{\frac{d+d_y}{2}}}{|W_y|[G_y : G_y^0]} \sum_{k \in \mathbb{N}} \prod_{\alpha \in \Phi_y^+} (\langle \delta_y, \alpha \rangle)^{-1} \Psi_{\lambda_k}^0(y) z^{k-1}.
 \end{aligned} \tag{6.3.2}$$

Let us denote $\frac{\text{vol}(\Gamma_y \backslash G_y) (-1)^{\frac{d+d_y}{2}}}{|W_y|[G_y : G_y^0]}$ by C_y . Finally, from the definition of $\Psi_{\lambda_k}^0(y)$, the generating function $F_{\lambda_1, \lambda_2, \Gamma}(z)$ equals

$$\sum_{[y]} C_y \sum_{k \in \mathbb{N}} \prod_{\alpha \in \Phi_y^+} (\langle \delta_y, \alpha \rangle)^{-1} \frac{\sum_{w \in W_c/W_y} (-1)^{l(w)} e^{w\lambda_k - \delta}(y) \prod_{\alpha \in \Phi_y^+} \langle w\lambda_k, \alpha \rangle}{\prod_{\alpha \in \Phi^+ \setminus \Phi_y^+} (1 - e^{-\alpha}(y))} z^{k-1}.$$

We write this expression as

$$F_{\lambda_1, \lambda_2, \Gamma}(z) = \sum_{[y]} C_y \prod_{\alpha \in \Phi^+ \setminus \Phi_y^+} (1 - e^{-\alpha}(y))^{-1} \sum_{w \in W_c/W_y} (-1)^{l(w)} a_{y,w}(z),$$

where $a_{y,w}(z) = \sum_{k \in \mathbb{N}} e^{w\lambda_k - \delta}(y) \prod_{\alpha \in \Phi_y^+} \frac{\langle w\lambda_k, \alpha \rangle}{\langle \delta_y, \alpha \rangle} z^{k-1}$. Therefore, it is enough to show that $a_{y,w}(z)$ is a rational function in z .

We observe that

$$\begin{aligned}
\prod_{\alpha \in \Phi_y^+} \frac{\langle w\lambda_k, \alpha \rangle}{\langle \delta_y, \alpha \rangle} &= \prod_{\alpha \in \Phi_y^+} \frac{1}{\langle \delta_y, \alpha \rangle} \prod_{\alpha \in \Phi_y^+} \langle w(\lambda_1 + k\lambda_2), \alpha \rangle \\
&= \prod_{\alpha \in \Phi_y^+} \frac{1}{\langle \delta_y, \alpha \rangle} \prod_{\alpha \in \Phi_y^+} (\langle w\lambda_1, \alpha \rangle + \langle kw\lambda_2, \alpha \rangle) \\
&= \prod_{\alpha \in \Phi_y^+} \frac{1}{\langle \delta_y, \alpha \rangle} \prod_{\alpha \in \Phi_y^+} (\langle w\lambda_1, \alpha \rangle + k\langle w\lambda_2, \alpha \rangle).
\end{aligned} \tag{6.3.3}$$

Therefore, $\prod_{\alpha \in \Phi_y^+} \frac{\langle w\lambda_k, \alpha \rangle}{\langle \delta_y, \alpha \rangle}$ is a polynomial in k of degree at most $|\Phi_y^+|$, and let us write it as

$$\sum_{j=0}^{|\Phi_y^+|} b_j k^j \text{ for some complex numbers } b_j.$$

Then

$$a_{y,w}(z) = \sum_{k \in \mathbb{N}} e^{w\lambda_1 - \delta}(y) e^{kw\lambda_2}(y) \sum_{j=0}^{|\Phi_y^+|} b_j k^j z^{k-1}.$$

For the sake of simplicity, we show that $z \cdot a_{y,w}(z) = e^{w\lambda_1 - \delta}(y) \sum_{j=0}^{|\Phi_y^+|} b_j \sum_{k \in \mathbb{N}} k^j (e^{w\lambda_2}(y)z)^k$ is a rational function in z .

We first claim that for any $0 \leq j \leq |\Phi_y^+|$,

$$\sum_{k \in \mathbb{N}} k^j (e^{w\lambda_2}(y)z)^k = \frac{p_j(z)}{(1 - e^{w\lambda_2}(y)z)^{|\Phi_y^+|+1}} \tag{6.3.4}$$

for some polynomial $p_j(z)$ of degree at most $|\Phi_y^+|$.

We will show claim 6.3.4 by induction on j . Clearly, it is true for $j = 0$. Assume that it is true for $\{1, 2, \dots, j-1\}$. Observe that $j! \binom{k+j}{k} = k^j + \sum_{l=0}^{j-1} c_l k^l$ for some $c_l \in \mathbb{Z}$. Using this we have,

$$\sum_{k \in \mathbb{N}} k^j x^k = \frac{j!}{(1-x)^{j+1}} - \sum_{l=0}^{j-1} c_l \sum_{k \in \mathbb{N}} k^l x^k.$$

Since $(1-x)^{-(j+1)} = \frac{(1-x)^{|\Phi_y^+|-j}}{(1-x)^{|\Phi_y^+|+1}}$, the claim 6.3.4 follows by substituting $x = e^{w\lambda_2}(y)z$.

We write

$$p_{y,w}(z) = e^{w\lambda_1-\delta}(y) \sum_{j=0}^{|\Phi_y^+|} b_j p_j(z).$$

Then by the claim 6.3.4, we have

$$z \cdot a_{y,w}(z) = \frac{p_{y,w}(z)}{(1 - e^{w\lambda_2}(y)z)^{|\Phi_y^+|+1}}.$$

By the definition of N_Γ , $e^{w\lambda_2}(y)$ is a N_Γ -th root of unity (not necessarily primitive). Therefore, we can write

$$\frac{p_{y,w}(z)}{(1 - e^{w\lambda_2}(y)z)^{|\Phi_y^+|+1}} = \frac{p_{y,w}(z)(1 - z^{N_\Gamma})^{|\Phi^+|-|\Phi_y^+|} \prod_{\xi^{N_\Gamma}=1, \xi \neq e^{w\lambda_2}(y)} (1 - \xi z)^{|\Phi_y^+|+1}}{(1 - z^{N_\Gamma})^{|\Phi^+|+1}}.$$

Now $|\Phi^+| \geq |\Phi_y^+|$. Hence, the degree of the numerator of the above expression is less than or equal to $|\Phi_y^+| + N_\Gamma(|\Phi^+| - |\Phi_y^+|) + (N_\Gamma - 1)(|\Phi_y^+| + 1) < N_\Gamma(|\Phi^+| + 1)$. $\square$

Remark 6.3.1. The Prop. 6.3.1 also implies that for a given string $S(\lambda_1, \lambda_2)$ with $P^{\lambda_1} = P^{\lambda_2}$, and λ_1 or λ_2 belonging to $\mathcal{D}^*(G)$, there are infinitely many discrete series representations π_λ with Harish-Chandra parameters $\lambda \in S(\lambda_1, \lambda_2)$ such that π_λ occurs as a subrepresentation in $L^2(\Gamma \backslash G)$ with non-zero multiplicity.

Remark 6.3.2. Note that for a given string, the generating function can be defined by simply taking the coefficient to be the Lefschetz number instead of the multiplicity of the discrete series. In that case, the proposition holds without assuming that the base or the direction of the string belongs to $\mathcal{D}^*(G)$. Since we are concerned with only the multiplicity of discrete series, we

need the Harish-Chandra parameters in $\mathcal{D}^*(G)$ which is sufficient to relate the multiplicities and Lefschetz numbers up to sign (e.g. Thm. [6.2.5](#)).

6.4 Proof of the Main Results

6.4.1 Proof of Thm. [6.1.2](#)

For simplicity, we write $N := N_\Gamma$. We have the string $\mathcal{S}(\lambda_1, \lambda_2)$ with base λ_1 , direction λ_2 and $P^{\lambda_1} = P^{\lambda_2}$; and by Prop. [6.3.1](#) its generating function $F_{\lambda_1, \lambda_2, \Gamma}$ is of the form

$$F_{\lambda_1, \lambda_2, \Gamma}(z) = \frac{p(z)}{(1 - z^N)^{|\Phi^+|+1}},$$

with $\deg p(z) < N(|\Phi^+| + 1)$. Let us write $p(z) = \sum_{k=0}^{N(|\Phi^+|+1)-1} b_k z^k$ for some $b_k \in \mathbb{C}$. We claim that for any $0 \leq j \leq N - 1$ and $m \geq 0$,

$$m(\pi_{\lambda_{mN+j}}, \Gamma) = \sum_{h=0}^{|\Phi^+|} b_{hN+j} \binom{m - h + |\Phi^+|}{|\Phi^+|},$$

where $\binom{m - h + |\Phi^+|}{|\Phi^+|} = 0$ when $m < h$. Therefore, the sum essentially runs over $0 \leq h \leq \min(|\Phi^+|, m)$. We use the following binomial expansion:

$$\frac{1}{(1 - y)^{j+1}} = \sum_{k \geq 0} \binom{k + j}{k} y^k \quad \text{for any } j \in \mathbb{N}.$$

Then, we have

$$\begin{aligned}
 F_{\lambda_1, \lambda_2, \Gamma}(z) &= p(z)(1 - z^N)^{-(|\Phi^+|+1)} \\
 &= \left(\sum_{j=0}^{N-1} \sum_{h=0}^{|\Phi^+|} b_{hN+j} z^{hN+j} \right) \left(\sum_{k \geq 0} \binom{k + |\Phi^+|}{|\Phi^+|} z^{kN} \right) \\
 &= \sum_{j=0}^{N-1} \sum_{k \geq 0} \sum_{h=0}^{|\Phi^+|} b_{hN+j} \binom{k + |\Phi^+|}{|\Phi^+|} z^{(k+h)N+j} \\
 &= \sum_{j=0}^{N-1} \sum_{m \geq 0} \left(\sum_{h=0}^{|\Phi^+|} b_{hN+j} \binom{m - h + |\Phi^+|}{|\Phi^+|} \right) z^{mN+j}.
 \end{aligned}$$

Let us fix a j with $0 \leq j \leq N-1$. Since $|\mathcal{A} \cap (j + N\mathbb{Z})| \geq |\Phi^+| + 1$, there are $m_0, \dots, m_{|\Phi^+|} \in \mathbb{Z}^+$ such that $m_i N + j \in \mathcal{A}$ for all $0 \leq i \leq |\Phi^+|$.

Therefore,

$$m(\pi_{\lambda_{m_i N+j}}, \Gamma) = \sum_{h=0}^{|\Phi^+|} b_{hN+j} \binom{m_i - h + |\Phi^+|}{|\Phi^+|} \quad \text{with } 0 \leq i \leq |\Phi^+|$$

is a system of $|\Phi^+| + 1$ many linear equations in $|\Phi^+| + 1$ many variables $\{b_{hN+j}\}_{h=0}^{|\Phi^+|}$. It is not difficult to check that the square matrix

$$\left\{ \binom{m_i - h + |\Phi^+|}{|\Phi^+|} \right\}_{0 \leq h, i \leq |\Phi^+|}$$

of order $|\Phi^+| + 1$ is non-singular. Hence, it has a unique solution. Consequently, for any $0 \leq h \leq |\Phi^+|$, $b_{hN+j} = \sum_i n(hN+j, m_i N+j) m(\pi_{\lambda_{m_i N+j}}, \Gamma)$ for some integers $n(hN+j, m_i N+j)$ independent of Γ . In other words, the solution $\{b_{hN+j}\}_{h=0}^{|\Phi^+|}$ can be expressed linearly in

terms of $\{m(\pi_{\lambda_{m_i N+j}}, \Gamma) : m_i N + j \in \mathcal{A}\}$.

Since the finite set $\mathcal{A}$ satisfies the condition $|\mathcal{A} \cap (j + N\mathbb{Z})| \geq |\Phi^+| + 1$ for all $j \in \{0, 1, \dots, N-1\}$, we have $b_\ell = \sum_{k \in \mathcal{A}} n(\ell, k) m(\pi_{\lambda_k}, \Gamma)$ for all $0 \leq \ell \leq N(|\Phi^+| + 1)$. This implies that the polynomial $p(z)$ and hence the generating function $F_{\lambda_1, \lambda_2, \Gamma}$ is determined by the same. Therefore for any $\ell \geq 0$, the discrete series multiplicity $m(\pi_{\lambda_\ell}, \Gamma) = \sum_{k \in \mathcal{A}} n(\ell, k) m(\pi_{\lambda_k}, \Gamma)$ for some integers $n(\ell, k)$ independent of Γ .

Remark 6.4.1. *Thm. 6.1.2 implies the set of multiplicities of infinitely many discrete series representations parametrized by the string $\mathcal{S}(\lambda_1, \lambda_2)$ is determined by a finite subset of multiplicities with cardinality $N(|\Phi^+| + 1)$.*

6.4.2 Proof of Cor. 6.1.2

Let $\mathcal{A} = \{k \in \mathbb{N} \cup \{0\} : m(\pi_{\lambda_k}, \Gamma_1) = m(\pi_{\lambda_k}, \Gamma_2)\}$. We need to show that $\mathcal{A}$ satisfies

$$|\mathcal{A} \cap (N\mathbb{Z} + j)| \geq |\Phi^+| + 1 \quad \text{for all } 0 \leq j \leq N-1.$$

By the hypothesis, we have

$$\liminf_{t \rightarrow \infty} \frac{|\{0 \leq k \leq t : k \in \mathcal{A}\}|}{t} > \frac{N-1}{N}.$$

This means, for every $\epsilon > 0$ there is $r_0 > 0$ such that

$$\frac{|\{k \in \mathcal{A} : k < rN(|\Phi^+| + 1)\}|}{rN(|\Phi^+| + 1)} > \frac{N-1+\epsilon}{N} \quad \text{for all } r \geq r_0.$$

Equivalently,

$$|\{k \in \mathcal{A} : k < rN(|\Phi^+| + 1)\}| > r(N - 1 + \epsilon)(|\Phi^+| + 1).$$

Let $r \geq r_0$. Fix a j_0 such that $0 \leq j_0 \leq N - 1$. Then the cardinality

$$\begin{aligned} |\{k \in \mathcal{A} : k < rN(|\Phi^+| + 1)\}| &= \sum_{j=0}^{N-1} |\{k \in \mathcal{A} : k < rN(|\Phi^+| + 1)\} \cap (j + N\mathbb{Z})| \\ &\leq |\mathcal{A} \cap (j_0 + N\mathbb{Z})| + (N - 1)r(|\Phi^+| + 1). \end{aligned}$$

In other words,

$$|\mathcal{A} \cap (j_0 + N\mathbb{Z})| > r\epsilon(|\Phi^+| + 1).$$

This holds for all $r \geq r_0$. In particular for $r \geq \frac{1}{\epsilon}$, we have $|\mathcal{A} \cap (j_0 + N\mathbb{Z})| \geq (|\Phi^+| + 1)$. This holds for all $j_0 \in \{0, 1, \dots, N - 1\}$.

6.4.3 Exhaustion of Harish-Chandra parameters in $\mathcal{D}^*(G)$

In this subsection, we describe the notion of a multi-directed string of Harish-Chandra parameters and prove analogous result to Thm. [6.1.2](#) in this generality. Let us denote the cartesian product $\mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N}$ of n copies of $\mathbb{N}$ by $\mathbb{N}^n$.

Definition 6.4.1. *Given a finite set of linear forms $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{D}(G)$, the infinite set $\mathcal{S}(\lambda_1, \lambda_2, \dots, \lambda_n) := \{k_1\lambda_1 + k_2\lambda_2 + \dots + k_n\lambda_n : (k_1, k_2, \dots, k_n) \in \mathbb{N}^n \setminus \{(0, 0, \dots, 0)\}\}$ is called a multi-directed string with directions $\lambda_1, \lambda_2, \dots, \lambda_n$. For every $\mathbf{k} := (k_1, k_2, \dots, k_n) \in \mathbb{N}^n \setminus \{(0, 0, \dots, 0)\}$, the linear form $\sum_{i=1}^n k_i\lambda_i$ is denoted by $\lambda_{\mathbf{k}}$.*

It is not difficult to see the following generalisation of Lem. [6.2.10](#):

Lemma 6.4.2. Assume $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{D}(G)$ such that $P^{\lambda_r} = P^{\lambda_s}$ for all $1 \leq r, s \leq n$, and $\lambda_1 \in \mathcal{D}^*(G)$. Then every linear form in the multi-directed string $\mathcal{S}(\lambda_1, \lambda_2, \dots, \lambda_n)$ belongs to $\mathcal{D}^*(G)$.

Now we can generalise the Thm. 6.1.2 for a multi-directed string. First, we need a few notations: Let $p_i : \mathbb{N}^n \rightarrow \mathbb{N}$ be the projection onto i -th component of an n -tuple of natural numbers. For a subset $\mathcal{A}$ of $\mathbb{N}^n$, let $\mathcal{A}_i$ denote the image of $\mathcal{A}$ under p_i for $1 \leq i \leq n$.

Theorem 6.4.3. Assume the hypothesis on G and K as in Thm. 6.1.2. Let N_Γ be the maximal order of elliptic elements in Γ . For given $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{D}(G)$ such that $P^{\lambda_r} = P^{\lambda_s}$ for all $1 \leq r, s \leq n$ and at least one of the λ_r 's belongs to $\mathcal{D}^*(G)$, consider the multi-directed string $\mathcal{S}(\lambda_1, \lambda_2, \dots, \lambda_n)$. If $\mathcal{A}$ is a finite subset of $\mathbb{N}^n$ such that each $\mathcal{A}_i$ satisfies $|\mathcal{A}_i \cap (j + N_\Gamma \mathbb{Z})| \geq |\Phi^+| + 1$ for all $0 \leq j \leq N_\Gamma - 1$, then for any $\lambda_k \in \mathcal{S}(\lambda_1, \lambda_2, \dots, \lambda_n)$ the discrete series multiplicity $m(\pi_{\lambda_k}, \Gamma)$ can be expressed linearly in terms of the finitely many discrete series multiplicities $\{m(\pi_{\lambda_k}, \Gamma) : \mathbf{k} \in \mathcal{A}\}$.

Proof. By Lem. 6.4.2, we have $\lambda_k \in \mathcal{D}^*(G)$ for every $\mathbf{k} \in \mathbb{N}^n \setminus \{(0, 0, \dots, 0)\}$. By induction on n , we are reduced to the case for $n = 2$. Let us fix $(t_1, t_2) \in \mathbb{N}^2$. We need to show that $m(\pi_{t_1\lambda_1+t_2\lambda_2}, \Gamma)$ can be linearly expressed in terms of the finite set $\{m(\pi_{\lambda_k}, \Gamma) : \mathbf{k} \in \mathcal{A}\}$. We have the string $\mathcal{S}(t_1\lambda_1, \lambda_2)$ with base $t_1\lambda_1$ and direction λ_2 . Since $\mathcal{A}_2$ satisfies $|\mathcal{A}_2 \cap (j + N_\Gamma \mathbb{Z})| \geq |\Phi^+| + 1$ for all $0 \leq j \leq N_\Gamma - 1$, by Thm. 6.1.2 the multiplicity $m(\pi_{t_1\lambda_1+t_2\lambda_2}, \Gamma)$ can be expressed linearly in terms of the finite set $\{m(\pi_{\lambda_k}, \Gamma) : p_1(\mathbf{k}) = t_1, p_2(\mathbf{k}) \in \mathcal{A}_2\}$. For any $\ell \in \mathcal{A}_2$, we have the string $\mathcal{S}(\ell\lambda_2, \lambda_1)$ with base $\ell\lambda_2$ and direction λ_1 . Since $\mathcal{A}_1$ satisfies $|\mathcal{A}_1 \cap (j + N_\Gamma \mathbb{Z})| \geq |\Phi^+| + 1$, again by Thm. 6.1.2 the multiplicity $m(\pi_{\ell\lambda_2+t_1\lambda_1}, \Gamma)$ can be expressed linearly in terms of the finite set $\{m(\pi_{\lambda_k}, \Gamma) : p_1(\mathbf{k}) \in \mathcal{A}_1, p_2(\mathbf{k}) = \ell\}$. This holds for every $\ell \in \mathcal{A}_2$. Therefore, $m(\pi_{t_1\lambda_1+t_2\lambda_2}, \Gamma)$ can be expressed linearly in terms of the finite set $\{m(\pi_{\lambda_k}, \Gamma) : p_1(\mathbf{k}) \in \mathcal{A}_1, p_2(\mathbf{k}) \in \mathcal{A}_2\}$. $\square$

It is tempting to ask whether there exists a finite set $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathcal{D}(G)$, with the property $P^{\lambda_r} = P^{\lambda_s}$ for all $1 \leq r, s \leq n$ and at least one of the λ_r 's belongs to $\mathcal{D}^*(G)$, so that the set equality $\{\mu \in \mathcal{D}^*(G) : P^\mu = P^{\lambda_r}\} = \mathcal{S}(\lambda_1, \lambda_2, \dots, \lambda_n)$ holds. The inclusion $\mathcal{S}(\lambda_1, \lambda_2, \dots, \lambda_n) \subset \{\mu \in \mathcal{D}^*(G) : P^\mu = P^{\lambda_r}\}$ is obvious, whereas the reverse containment is not true by the following proposition:

Proposition 6.4.4. *Let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a $\mathbb{Z}$ -linearly independent subset of $\mathcal{D}(G)$ with at least one of the λ_r 's belonging to $\mathcal{D}^*(G)$ and $P^{\lambda_r} = P^{\lambda_s}$ for all $1 \leq r, s \leq n$. Then there exist infinitely many $\mu \in \mathcal{D}^*(G)$ with $P^\mu = P^{\lambda_r}$ such that $\mu \notin \mathcal{S}(\lambda_1, \lambda_2, \dots, \lambda_n)$.*

Proof. Without loss of generality, let us assume that $\lambda_1 \in \mathcal{D}^*(G)$. We show that $m\lambda_1 - \lambda_2 \in \mathcal{D}^*(G)$ for sufficiently large positive integer m . Clearly $\Phi = P^{\lambda_1} \cup P^{-\lambda_1} = P^{\lambda_2} \cup P^{-\lambda_2}$. For any $\alpha \in P^{\lambda_1} = P^{\lambda_2}$, we have $\langle \lambda_1, \alpha \rangle > 0$ and $\langle \lambda_2, \alpha \rangle > 0$. So, there exists $k \in \mathbb{N}$ such that for each $\alpha \in P^{\lambda_1} = P^{\lambda_2}$, $m\langle \lambda_1, \alpha \rangle > \langle \lambda_2, \alpha \rangle$ for all $m \geq k$. Equivalently, for every $\alpha \in P^{-\lambda_1} = P^{-\lambda_2}$, $m\langle \lambda_1, \alpha \rangle < \langle \lambda_2, \alpha \rangle$. In other words, if $\alpha \in \Phi$ then for every $m \geq k$, we get $\langle m\lambda_1 - \lambda_2, \alpha \rangle > 0$ if and only if $\alpha \in P^{\lambda_1} = P^{\lambda_2}$. Therefore, $P^{m\lambda_1 - \lambda_2} = P^{\lambda_1} = P^{\lambda_2}$. For a fixed $m > k$, let us choose $\alpha \in P^{m\lambda_1 - \lambda_2} \cap \Phi_n$. Then we have $\langle m\lambda_1 - \lambda_2 - \delta^{m\lambda_1 - \lambda_2}, \alpha \rangle = \langle m\lambda_1 - \delta^{\lambda_1}, \alpha \rangle - \langle \lambda_2, \alpha \rangle = \langle \lambda_1 - \delta^{\lambda_1}, \alpha \rangle + \langle (m-1)\lambda_1 - \lambda_2, \alpha \rangle$. Now the first summand is positive because $\lambda_1 \in \mathcal{D}^*(G)$, and the second summand is positive by the choice of m . Therefore, $m\lambda_1 - \lambda_2 \in \mathcal{D}^*(G)$ for all $m > k$. Finally, by the $\mathbb{Z}$ -linear independence of λ_i 's, the linear form $\mu := m\lambda_1 - \lambda_2 \notin \mathcal{S}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Hence, the proposition follows. $\square$

Remark 6.4.2. *From the above proof it can be seen easily that for any $1 < i \leq n$, the linear form $m\lambda_1 - \lambda_i$ with sufficiently large m provides an infinite collection of such μ which determines the same positive system but does not belong to the multi-string. The Prop. 6.4.4 also holds for any arbitrary finite set of n linear forms in which any subset of $n-1$ linear forms is $\mathbb{Z}$ -linearly independent.*

6.5 Dimensions of the space of cusp forms

In this section, we showcase an application of Thm. [6.1.2](#) to the dimension of the space of cusp forms. We need to write the explicit dictionary between the dimension of the space of cusp forms and discrete series multiplicities.

Let B be a division quaternion algebra over a totally real number field F of degree n . Assume that B splits at exactly one real place v of F i.e. $B_v \cong M(2, \mathbb{R})$. Let $|\cdot|$ be the standard norm on B . Let $B^1 = \{x \in B : |x| = 1\}$. Then $B_v^1 \cong SL(2, \mathbb{R})$. Consider a maximal order $\mathcal{O}$ in B . Let $\mathcal{O}^1 = \{x \in \mathcal{O} : |x| = 1\}$. Let $\mathcal{H}$ denote the Hamiltonian algebra over $\mathbb{R}$. We have the map

$$\iota : B^1 \longrightarrow B_{\mathbb{R}}^1 \cong (SL(2, \mathbb{R}) \otimes \mathcal{H}^{n-1}) \longrightarrow SL(2, \mathbb{R}),$$

where the first map is the usual inclusion and the second map is the projection onto the first component. Let Γ be the image of $\mathcal{O}^1$ under the map ι in $SL(2, \mathbb{R})$. Then Γ is a uniform lattice in $SL(2, \mathbb{R})$. In fact, every uniform lattice in $SL(2, \mathbb{R})$ arises in this way (see [\[Wei60\]](#)). Let us recall the definition of cusp forms.

Definition 6.5.1. *Let $\Gamma \subset SL(2, \mathbb{R})$ be a uniform lattice and fix an integer k . A holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called a cusp form of weight k and level Γ if*

1. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $z \in \mathbb{H}$;
2. $\int_{\Gamma \backslash \mathbb{H}} |f(z)|^2 y^k \frac{dx dy}{y^2} < \infty$.

The space of cusp forms of weight k for a uniform lattice $\Gamma \subset SL(2, \mathbb{R})$ is denoted by $\mathcal{S}_k(\Gamma)$.

For each $f \in \mathcal{S}_k(\Gamma)$, we define $\phi_f : \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathbb{C}$ by

$$\phi_f(g) = (ci + d)^{-k} f\left(\frac{ai + b}{ci + d}\right) \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

where $i = \sqrt{-1}$. It can be seen easily that $\phi_f(\gamma g) = \phi_f(g)$ for all $\gamma \in \Gamma$ and $g \in \mathrm{SL}(2, \mathbb{R})$.

Also, for any $s_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \mathrm{SO}(2)$ and $g \in \mathrm{SL}(2, \mathbb{R})$, $\phi_f(g s_\theta) = e^{-ik\theta} \phi_f(g)$.

Moreover, $\phi_f \in L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$. Therefore, we have the following linear map:

$$T : \mathcal{S}_k(\Gamma) \longrightarrow L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R})) \text{ defined by } T(f) = \phi_f.$$

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ be the complexified Lie algebra of $\mathrm{SL}(2, \mathbb{R})$. The Cartan subalgebra $\mathfrak{h}$ equals $\mathbb{C}X$, where $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The set Φ of roots is $\{\pm\alpha\}$, where α is defined by $\alpha(X) = 2i$. We fix the positive system $\Phi^+ = \{\alpha\}$.

Any $\lambda \in \mathcal{D}(\mathrm{SL}(2, \mathbb{R}))$ is of the form $\lambda = \frac{k-1}{2}\alpha$ for some $k \in \mathbb{Z}$ with $|k| \geq 2$. Let π_k be the discrete series representation associated to the Harish-Chandra parameter $\lambda = \frac{k-1}{2}\alpha$. If $k \geq -2$, then π_k is called a holomorphic discrete series representation of $\mathrm{SL}(2, \mathbb{R})$.

Proposition 6.5.2. *Let Γ be a uniform lattice in $\mathrm{SL}(2, \mathbb{R})$. Then $\mathcal{S}_k(\Gamma) \cong \mathrm{Hom}_G(\pi_{-k}, L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R})))$ for all $k \geq 2$.*

Proof. For any $k \in \mathbb{Z}$, let τ_k be the character of $\mathrm{SO}(2)$ defined by $\tau_k(s_\theta) = s_\theta^k$. It is well known that for every $f \in \mathcal{S}_k(\Gamma)$, the image $T(f) = \phi_f$ in $L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$ generates an irreducible subrepresentation of $L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$, which is isomorphic to the holomorphic discrete series π_{-k} of $\mathrm{SL}(2, \mathbb{R})$. Under the action of $\mathrm{SO}(2)$, ϕ_f transforms via the character

τ_{-k} . Since τ_{-k} is the minimal $\mathrm{SO}(2)$ -type of the holomorphic discrete series representation π_{-k} of $\mathrm{SL}(2, \mathbb{R})$, ϕ_f belongs to the one-dimensional τ_{-k} -isotypic component of this copy of π_{-k} in $L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$. Therefore, if f_1 and f_2 are two linearly independent cusp forms in $\mathcal{S}_k(\Gamma)$ then the irreducible subrepresentations $\langle \phi_{f_1} \rangle$ and $\langle \phi_{f_2} \rangle$ in $L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$ generated by ϕ_{f_1} and ϕ_{f_2} respectively, are linearly independent; in fact, those are orthogonal to each other. In other words, the linear map T is injective and it induces the isomorphism between $\mathcal{S}_k(\Gamma)$ and $\mathrm{Hom}_G(\pi_{-k}, L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R})))$. $\square$

Remark 6.5.1. *The reader is referred to Thm. 2.10 of [Gel75] for a more detailed proof.*

In particular, from Prop. 6.5.2 we have $\dim \mathcal{S}_k(\Gamma) = m(\pi_{-k}, \Gamma)$, where $m(\pi_{-k}, \Gamma)$ is the multiplicity of π_{-k} in $L^2(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$. Using this, we deduce the following consequence of Thm. 6.1.2.

Corollary 6.5.1. *Let Γ be a uniform lattice in $\mathrm{SL}(2, \mathbb{R})$, and let N_Γ be the maximal order of elliptic elements in Γ . Let $\mathcal{A}$ be a finite subset of $\mathbb{N}$, which satisfies $|\mathcal{A} \cap (j + N_\Gamma \mathbb{Z})| \geq 2$ for all $0 \leq j \leq N_\Gamma - 1$. Then for any $\ell \geq 2$, $\dim \mathcal{S}_\ell(\Gamma) = \sum_k n(\ell, k) \dim \mathcal{S}_k(\Gamma)$, where the summation runs over the set $\{k \geq 2 : \frac{k}{2} \in \mathcal{A} \text{ or } \frac{k+1}{2} \in \mathcal{A}\}$ and $n(\ell, k)$ are integers which are independent of Γ .*

Proof. By Prop. 6.5.2 the set $\{\dim \mathcal{S}_k(\Gamma) : k \geq 2\}$ equals to the set $\{m(\pi_{-k}, \Gamma) : k \geq 2\}$. The corresponding set of Harish-Chandra parameters is $\{-\frac{k+1}{2}\alpha : k \geq 2\}$. Note that $\{-\frac{k+1}{2}\alpha : k \geq 2\} = \{-\frac{\alpha}{2} + m(-\alpha) : m \geq 1\} \cup \{n(-\alpha) : n \geq 2\}$. Moreover, it is easy to check that $-\alpha \in \mathcal{D}^*(\mathrm{SL}(2, \mathbb{R}))$. Therefore, the corollary follows from Thm. 6.1.2. $\square$

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