Momentum Space Bootstrap of Conformal Correlation Functions

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Certificate

This is to certify that this dissertation entitled Momentum Space Bootstrap of Conformal Correlation Functions towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Bibhut Chandan Sahoo at Indian Institute of Science Education and Research under the supervision of Dr. Sachin Jain, Assistant Professor, Department of Physics , during the academic year 2021-2022.

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This thesis is dedicated to my parents.

Declaration

I hereby declare that the matter embodied in the report entitled Momentum Space Bootstrap of Conformal Correlation Functions are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Sachin Jain and the same has not been submitted elsewhere for any other degree.

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10.04.2022
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Abstract

Correlation functions are one of the most important objects in CFTs. Momentum space CFT correlation functions have wide range of applicability in contexts of cosmology, condensed matter etc. CFT correlators in momentum space also allow us to make a connection between conformal and S-matrix bootstrap. However, usual techniques of calculating the 3-point CFT correlators using conformal invariance is not very efficient for 4-point or higher point correlators having a spinning operator insertion. To gain insight we look for theories with more symmetry constraints. One such class of theories are theories with higher spin symmetry or with slightly broken higher spin symmetries. These are important classes of thoeries as they have a gravity dual, exhibit strong weak duality, non supersymmetric duality etc. In these theories calculating CFT correlation functions is quite difficult. However, we circumvent this problem by mapping the correlators in slightly broken higher spin theory correlators to free theory correlators. Thus, the higher spin approach allows us to analyse these correlators in a non-perturbative manner. Hence we map the slightly broken higher spin theories which are interacting CFTs to the free CFTs.

Contents

Abstract **xi**

1 Conformal Ward Identity **5**

1.1 Spinning 3-point correlator 7

1.2 Conformal Ward Identities in momentum space 9

1.3 Conformal Ward Identities in spinor helicity 13

1.4 Problems of using CWIs for higher point correlators 15

2 Higher Spin Equations **17**

2.1 Exact higher spin theories 17

2.2 Slightly broken higher spin theories 19

2.3 Higher Spin equation in momentum space 22

3 Using higher spin symmetry to compute correlators **27**

3.1 Three point functions 27

3.2 Four point functions 36

4 Conclusion and Discussion **51**

5 Appendix **53**

5.1 Spinor Helicity variables 53

Introduction

Quantum field theory is one of the most important developments in modern physics. Correlators play a central role in quantum field theories. Thus, it is only expected that studying the form of these correlators and constraining them is a task of importance and can allow us to infer fair amount of information regarding the theory. Here we focus on a specific subset of QFTs known as conformal field theories(CFTs), which are invariant under conformal transformations,and have important applications to high energy physics, condensed matter physics, quantum statistical physics etc. It also turns out that imposing conformal invariance can place a great amount of constraint on the structure of correlators in CFTs[5], commonly referred to as conformal correlators. These kind of analysis of using conformal invariance to constrain the form of correlators have been done extensively in position space to constrain 2- and 3- point scalar as well as spinning correlators upto some constant[18]. But, in momentum space this kind of analysis is fairly recent[3][7][8][23] and not as polished as it is position space.

Why momentum space analysis is important

There are many advantages of doing the above analysis of constraining the form of CFT correlators in momentum space over doing it in position space. One obvious plus is that the Feynman diagram computations are more natural in momentum space. This is also important in condensed matter theory where these momentum space correlators are useful in studying quantum critical transport[19]. Further, one of the most exciting prospects of such analysis is that, they could act as the stepping stones for developments towards momentum space formulations of conformal bootstrap.

Development of bootstrapping techniques is important because it allows us to use general symmetry arguments to constrain the correlation functions and allows a non-perturbative analysis. Momentum space analysis of the correlators are important in the context of the study of inflationary correlators in cosmology[12][16]. One interesting use of studying the momentum space bootstrap is that by analysing the singularity structure and the corresponding residues of the correlation functions in the flat space limit(refers to the total energy singularity i.e. $p_1 + p_2 + p_3 + p_4 \to 0$) one can decipher information regarding the corresponding flat space scattering amplitude. Thus, it allows us to make a **connection between conformal and S-matrix bootstrap**. Recent developments in momentum space study of 3-point correlators has also given us new and non trivial insights[9][20][6][15] such as the double copy structure[4]. An interesting review on connection of CFT correlator and scattering amplitudes can be found in [21], where they use widely used methods to extend the flat space correlators to their de Sitter counterparts.

Difficulties with momentum space analysis

The difficulty of working with momentum space compared to position space becomes clear from the start while imposing invariance under special conformal transformations which are first order differential equations in position space whereas these give rise to second order differential equations in momentum space[10]. As a remark one might think that the most straight forward way to get a representation in momentum space would be to just Fourier transform the position space expression of the correlator, but there is a subtlety involved here. The position space results are valid for separated points only, so naively Fourier transforming leads to encountering divergences which arise from integrating over configurations with coincident operator insertions. Thus, we first need to regulate and insert extra contact terms into the position space expression to get a renormalised expression, which we can then Fourier transform. Thus, the procedure requires considerable effort. Hence, imposing conformal invariance and solving the constraint equation seems to be a better strategy than blunt Fourier transforming. **In particular we map results from interacting theory CFT with slightly broken higher spin to free CFT results**.
As we will see in the upcoming sections, although imposing conformal invariance in the form of Conformal Ward Identities is a good strategy for the 3-point correlators, this approach is much more involved for higher point correlators. In case of 3-point correlators, we have

a parametric representation of the solutions in terms of Triple-K integrals. But, even there they are singular for specific operator dimensions and needs to be regularised[11]. However, in case of 4-point we don't have a well developed parametric representation. Also, for 4-point correlators, conformal invariance is not strong enough to completely fix the form of the correlators. Thus, to study the 4-point correlator, we took a different approach of using the higher spin symmetry which in principle provides us with infinite ladder of conservation equations[1]. Using the higher spin equations we show how we can constrain the form of correlators in not only theories with exact higher spin symmetry but also in theories which have slightly broken higher spin symmetry.

The outline of thesis is as follow: In the **chapter 1**, we introduce the methodology of using conformal Ward identities to constrain and decipher the form of 3-point correlators and explain the complications involved in using the same method for higher point correlators. In **Chapter 2**, we provide a brief introduction to the higher spin symmetry, describe how to write and give a general strategy to use higher spin equations to solve for correlators in momentum space. In **Chapter 3**, we provide detailed calculations involving the use of higher spin equations to solve for correlators in theories with slightly broken higher spin symmetry(quasi fermionic and quasi bosonic theories). In **Chapter 4**, we conclude with discussion and futute directions.

The work presented in this thesis is a part of an upcoming paper with Prof. Sachin Jain, Aashna Anil Zade, Prabhav Jain and Dhruva K.S.

Chapter 1

Conformal Ward Identity

First we start with the direct way of tackling the problem of getting the form of conformal correlators by imposing constraint of invariance under conformal transformations. As mentioned before these conditions can impose strong constraints on the form of the correlator. In particular for the case of 2-point and 3-point correlators, we can fix the form of the correlators with either scalar or spinning insertions upto a few constants just by imposing conformal invariance.

Conformal transformations consist of translation, rotations, dilatations and special conformal transformation. Invariance under translatio simply imposes momentum conservation. However invariance under dilatation and special conformal transformation impose strong constraints. We would go into the full details a bit later, but to illustrate this we start by considering scalar three point correlator, $\langle O_1(p_1)O_2(p_2)O_3(p_3)\rangle$ where O_i is a scalar operator of dimension Δ_i. Going step by step, we start by imposing translational invariance which implies momentum conservation that is-

$$\langle O_1(p_1)O_2(p_2)O_3(p_3)\rangle = \delta(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3)\langle\langle O_1(p_1)O_2(p_2)O_3(p_3)\rangle\rangle \tag{1.1}$$

where we would refer $\langle\langle O_1(p_1)O_2(p_2)O_3(p_3)\rangle\rangle$ as the reduced correlator. Then invariance under rotation or more generally Lorentz transformation implies that the reduced correlator $\langle\langle O_1(p_1)O_2(p_2)O_3(p_3)\rangle\rangle$ is a function of the magnitudes, $p_i = |\boldsymbol{p}_i|$. Invariance under dilatation

transformation imposes the condition that the correlator should be a homogenous function of the overall degree $(\Delta_t - 2d)$ where $\Delta_t = \sum \Delta_j$, where Δ_j is the conformal dimension of the operator O_j and d is the dimension in which we are working in.

Now, we focus our attention of the implications of imposing invariance under the special conformal transformation. Invariance under special conformal transformation means that the action of the special conformal generator on the correlator kills it. We can get the special confromal generator in momentum space by fourier transforming the corresponding position space generator which gives

$$K_i^\kappa = -2p_{i\mu}\partial_{p_{i\mu}}\partial_{p_{i\kappa}} + p_i^\kappa \partial_{p_{i\mu}}\partial_{p_i^\mu} - 2\partial_{p_{i\kappa}} \tag{1.2}$$

where i corresponds to the momentum label i.e. p_1, p_2, p_3. Thus, special conformal invariance gives us the constraint equation, also called Conformal Ward Identities:

$$\sum_{i=1}^{3} K_i^\kappa \langle O(p_1)O(p_2)O(p_3)\rangle = 0 \tag{1.3}$$

This can be brought to the form

$$0 = K_{12}\langle O_1(\boldsymbol{p_1})O_2(\boldsymbol{p_2})O_3(\boldsymbol{p_3})\rangle = K_{23}\langle O_1(\boldsymbol{p_1})O_2(\boldsymbol{p_2})O_3(\boldsymbol{p_3})\rangle \tag{1.4}$$

where

$$K_{ij} = K_i - K_j, \qquad K_i = \frac{\partial^2}{\partial p_i^2} + \frac{d+1-2\Delta_i}{p_i}\frac{\partial}{\partial p_i} \tag{1.5}$$

Thse solution to these conformal Ward Identitites has a parametric representation and are referred to in literature as Triple-K integrals:

$$\begin{aligned}&\langle\langle O_1(\boldsymbol{p_1})O_2(\boldsymbol{p_2})O_3(\boldsymbol{p_3})\rangle\rangle \\ &\qquad = C_{123}p_1^{\Delta_1-\frac{d}{2}}p_2^{\Delta_2-\frac{d}{2}}p_3^{\Delta_3-\frac{d}{2}}\int_0^\infty dx x^{\frac{d}{2}-1}K_{\Delta_1-\frac{d}{2}}(p_1x)K_{\Delta_2-\frac{d}{2}}(p_2x)K_{\Delta_3-\frac{d}{2}}(p_3x)\end{aligned} \tag{1.6}$$

where K_μ is the modified Bessels function and C_{123} is an over all constant. Thus, using conformal invariance it is possible to determine the correlator upto a constant.
Since, the solutions for the conformal Ward Identities, thus the form of the correlators can be expressed in terms of the Triple K integral, the use of these integrals is quite common in this approach. Thus, it is useful to use a shorthand notation for the Triple-K integrals, which is as follow:

$$I_{\alpha\beta_1\beta_2\beta_3}(p_1,p_2,p_3) = \int_0^\infty dx x^\alpha \prod_{i=1}^{3} p_i^{\beta_i} K_{\beta_i}(p_i x) \tag{1.7}$$

where α, β_i are parameter of the Triple-K inegrals which can be fixed by solving the conformal Ward identities.

Having briefly introduced the methodology of using conformal invariance by solving conformal Ward Identities to get the form of the correlators, we now look into the details.

1.1 Spinning 3-point correlator

1.1.1 Decomposition of tensor structures

Correlators of conserved currents are transverse and traceless upto some local terms. Thus, we try to find tensor decompositions of these correlators which satisfies the properties of transversality and tracelessness. First we write down general independent tensor structures applicable for a correlator.

Some specific operators which would be useful in defining the decompositions are the orthogonal projectors, so we define them first hand:

$$\pi^\mu_\alpha(\boldsymbol{p}) = \delta^\mu_\alpha - \frac{p^\mu p_\alpha}{p^2} \tag{1.8}$$

is a projection operator which is transverse to $\boldsymbol{p}$, so $p_\mu \pi^\mu_\alpha(\boldsymbol{p}) = 0$. Similarly, in d-dimensions

we define

$$\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = \frac{1}{2}\left(\pi^{\mu}_{\alpha}(\boldsymbol{p})\pi^{\nu}_{\beta}(\boldsymbol{p}) + \pi^{\mu}_{\beta}(\boldsymbol{p})\pi^{\nu}_{\alpha}(\boldsymbol{p}) - \frac{1}{d-1}\pi^{\mu\nu}(\boldsymbol{p})\pi_{\alpha\beta}(\boldsymbol{p})\right) \tag{1.9}$$

This projector is a symmetric tensor of rank 2 which is transverse with respect to $\boldsymbol{p}$ and is also traceless. That is

$$\delta_{\mu\nu}\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = 0, \quad p_{\nu}\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}) = 0 \tag{1.10}$$

Due to its properties Π is useful in depicting any transverse, traceless, symmetric tensor $t^{\mu\nu}$ of rank two as $t^{\mu\nu} = \Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p})X^{\alpha\beta}$, where $X^{\alpha\beta}$ can consist of independent tensor structures with free indices α and β.
Most our analysis that we did in the projects are focused on 3-dimensions. In 3 dimensions since there are three independent vectors, there is a notion of cross product. This allows us to write the metric tensor purely in terms of the momenta. If we take into account and deal with this degeneracy carefully, it reduces the number of independent tensor structures, since the metric is no longer an independent tensor. However, we can take another outlook and choose two out of three $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}$ to be independent and treat the metric to be independent. Usually we will choose $\boldsymbol{p_3}$ to be the independent vector and the other two to be independent.

To illustrate this tensor decomposition, let's consdier a brief example, $\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)O(p_3)\rangle$, where t is a traceless transverse and symmetric raank 2 tensor. Thus, we can use the projector from 1.9 to write the

$$\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)O(p_3)\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)X^{\alpha_1\beta_1\alpha_2\beta_2} \tag{1.11}$$

where $X^{\alpha_1\beta_1\alpha_2\beta_2}$ is a tensor structure consisting of independent tensor structures constructed out of the momenta and the metric. Due to the presence of the projector, which is transverse, traceless and symmetric, the tensor structure $X^{\alpha_1\beta_1\alpha_2\beta_2}$ can have structures made out of $p_1^{\alpha_2}$, $p_1^{\beta_2}$, $p_2^{\alpha_1}$, $p_2^{\beta_1}$, $\delta^{\alpha_1\alpha_2}$, $\delta^{\alpha_1\alpha_2}$. Thus, the most general form of the correlator is

$$\begin{aligned}\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)o(p_3)\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \\ \left[A_1 p_2^{\alpha_1}p_2^{\beta_1}p_1^{\alpha_2}p_1^{\beta_2} + A_2 p_2^{\alpha_1}p_1^{\alpha_2}\delta^{\beta_1\beta_2} + A_3\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}\right]\end{aligned} \tag{1.12}$$

Here, $A_i's$ are are called form factors and are scalar functions of the momentas.
We can apply the tensor decomposition method to spin-1 and spin-2 conserved currents J^μ and stress energy tensor $T^{\mu\nu}$. These operators are however not transverse and traceless. So to proceed we define the transverse and traceless parts of these as

$$\begin{aligned} j^\mu &= \pi^\mu_\alpha J^\alpha, \qquad j^\mu_{loc} = J^\mu - j^\mu \\ t^{\mu\nu} &= \Pi^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}, \qquad t^{\mu\nu}_{loc} = T^{\mu\nu} - t^{\mu\nu} \end{aligned} \tag{1.13}$$

1.2 Conformal Ward Identities in momentum space

In position space, dilatation Ward identity imposes[5]

$$\left[\sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right] \langle T_1(x_1)...T_n(x_n)\rangle = \tag{1.14}$$

And the special conformal Ward identity for correlators with scalar operator insertions reads

$$\left[\sum_{j=1}^{n} \left(2\Delta_j x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_{j\kappa}}\right)\right] \langle O_1(x_1)...O_n(x_n)\rangle = 0 \tag{1.15}$$

where the expression inside the square bracket is that of the specal conformal generator, K^κ. Thus, the above equation just implies that the correlator remains invariant under special conformal transformations.
For the case of tensors T_j with Lorentz indices, we need an additional term in the equation for conformal Ward Identity. The extra term depends on the Lorentz structure or roughly saying the spin of the operator. Say the tensor T^j has r_j indices, then the extra term is [**?**]

$$2\sum_{j=1}^{n}\sum_{k=1}^{r_j} \left[(x_j)_{\alpha_{jk}} \delta^{\kappa\mu_{jk}} - x_j^{\mu_{jk}} \delta^\kappa_{\alpha_{jk}}\right] \langle T_1^{\mu_{11}...\mu_{1n_1}}(x_1)...T_j^{\mu_{j1}...\alpha_{jk}...\mu_{jr_j}}(x_j)...T_n^{\mu_{n1}...\mu_{nr_n}}(x_n)\rangle \tag{1.16}$$

We need to add this term to the 1.15 equation for the spinning operator insertions with Lorentz indices.

We can Fourier transform these equations to get the Ward identities in the momentum space. Also because of translational invariance, we have momentum conservation. Thus, we

can take

$$\boldsymbol{p}_n = -\sum_{i=1}^{n-1} \boldsymbol{p}_i \tag{1.17}$$

Keeping this in mind when we Fourier transform position space dilatation and conformal Ward identities, 1.141.15, we get

$$\Big[\sum_{j=1}^{n} \Delta_j - (n-1)d\sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\Big]\langle T_1(p_1)...T_n(p_n)\rangle = 0 \tag{1.18}$$

$$\Big[\sum_{j=1}^{n-1}\Big(2(\Delta_j - d)\frac{\partial}{\partial p_j^\kappa} - 2p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_j^\kappa} + p_{j\kappa}\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_{j\alpha}}\Big)\Big]\langle O_1(p_1)...O_n(p_n)\rangle = 0 \tag{1.19}$$

For correlators that have operator insertions having Lorentz indices, we need to add the Fourier transform of the extra term 1.16, which gives us

$$\sum_{j=1}^{n-1}\sum_{k=1}^{n_j}\Big(\delta^{\mu_{jk}\kappa}\frac{\partial}{\partial p_j^{\alpha_{jk}}} - \delta^\kappa_{\alpha_{jk}}\frac{\partial}{\partial p_{j\mu_{jk}}}\Big)\langle T_1^{\mu_{11}...\mu_{1r_1}}(\boldsymbol{p}_1)...T_j^{\mu_{j1}...\alpha_{jk}...\mu_{jr_j}}(\boldsymbol{p}_j)...T_n^{\mu_{n1}...\mu_{nr_n}}(\boldsymbol{p}_n)\rangle \tag{1.20}$$

We can add 1.19 and 1.20 expressions to denote the differential operator $\mathcal{K}^\kappa$. This allows us to write the conformal Ward identity for a correlator with spinning insertions succintly as

$$\mathcal{K}^\kappa\langle T_1(\boldsymbol{p}_1)...T_n(\boldsymbol{p}_n)\rangle = 0 \tag{1.21}$$

Now, that we have a definitive way of imposing invariance under special conformal transformations by writing the differential operators and write conformal Ward identity for both scalar as well as spinning operators, we can do the tensor decomposition of any 3-point correlator and use the conformal Ward identity to solve for the expressions of the form factors which would amount to solving for the correlators themselves since the independent tensor structures multiplying the form factors are known.

1.2.1 Dilatation Ward Identity

Using chain rule

$$\frac{\partial}{\partial p_{1\mu}} = \frac{p_1^\mu}{p_1}\frac{\partial}{\partial p_1} + \frac{p_1^\mu + p_2^\mu}{p_3}\frac{\partial}{\partial p_3}$$

we can write the dilatation Ward Identity 1.18 for any 3-point correlator $\langle T_1 T_2 T_3 \rangle$ in terms of its form factors as

$$\left[2d + N_n + \sum_{j=1}^{3}\left(p_j \frac{\partial}{\partial p_j} - \Delta_j\right)\right] A_n(p_1, p_2, p_3) = 0 \tag{1.22}$$

where A_n is a form factor in the expression of the concerned correlator and N_n is the number of momenta in the tensor multiplying the form factor. Thus, the dilatation ward identity imposes that the degree of form factors in the expression of correlator is

$$deg(A_n) = \Delta_t - 2d - N_n \tag{1.23}$$

where $\Delta_t = \Delta_1 + \Delta_2 + \Delta_3$

1.2.2 Special Conformal Ward Identities

To solve for the correlators using conformal Ward Identities, we first write the most general expression for the correlator in terms of projectors, tensor structures and form factors. After that we act the special conformal generator $\mathcal{K}^\kappa$ which has the scalar part as well the extra contribution for the spinning operators, on the correlator and set the action to be zero to get the equation for conformal invariance. This way we would get equations on the form factors of the correlator, which we can then solve.
For brevity consider the already familiar correlator $\langle TTO \rangle$ for which upon acting $\mathcal{K}^\kappa$ on its

expression we get a equation of the form:

$$\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2)\Bigg[p_1^\kappa\Bigg(C_{11}p_2^{\alpha_1}p_2^{\beta_1}p_1^{\alpha_2}p_1^{\beta_2}+C_{12}p_2^{\alpha_1}p_1^{\alpha_2}\delta^{\beta_1\beta_2}+C_{13}\delta^{\alpha\alpha_2}\delta^{\beta_1\beta_2}\Bigg)+$$
$$p_2^\kappa\Bigg(C_{21}p_2^{\alpha_1}p_2^{\beta_1}p_1^{\alpha_2}p_1^{\beta_2}+C_{22}p_2^{\alpha_1}p_1^{\alpha_2}\delta^{\beta_1\beta_2}+C_{23}\delta^{\alpha\alpha_2}\delta^{\beta_1\beta_2}\Bigg)$$
$$+\Bigg(C_{31}\delta^{\kappa\alpha_1}p_2^{\beta_1}p_1^{\alpha_2}p_1^{\beta_2}+C_{32}\delta^{\kappa\alpha_1}\delta^{\alpha_2\beta_1}p_1^{\beta_2}+C_{41}\delta^{\kappa\alpha_2}p_2^{\alpha_1}p_2^{\beta_1}p_1^{\beta_2}+C_{42}\delta^{\kappa\alpha_2}\delta^{\alpha_1\beta_2}p_2^{\beta_1}\Bigg)\Bigg]=0 \tag{1.24}$$

Each of the $C'_{ij}s$ represent differential equations involving the form factors of the decomposition of the correlator $\langle TTO\rangle$. Since each otf the tensor structures appearing in the above equations are linearly independent we ideally would have at most ten equations over three different form factors in this case. But, more often than not the there is an overlap between the ten equations.

For the 3-point case, the equations can be divided into two types: primary and secondary conformal Ward Identites. Primary CWis are the second order differential equations that appear along with the transverse and traceless tensors containing p_1^κ or p_2^κ. In the above equation for $\langle TTO\rangle$, the primary CWIs correspond to the coefficients C_{1j} or C_{2j} equated to zero. The remaining equations that don't come along with p_i^κ are first order differential equations and are called secondary conformal Ward Identities. In the above expression for $\langle TTO\rangle$, the secondary CWIs correspond to equating the coefficients C_{3j} or C_{4j} to zero.

Upon explicit calculation it can be shown that the primary CWIs for the form factors look similar to the CWIs for the scalar correlator[10] i.e.

$$K_{ij}A_K = fA_l \tag{1.25}$$

where

$$K_j = \frac{\partial^2}{\partial p_j^2}+\frac{d+1-2\Delta_j}{p_j}\frac{\partial}{\partial p_j} \tag{1.26}$$
$$K_{ij} = K_i - K_j \tag{1.27}$$

where f is a constant and $A_i's$ are the form factors. Thus, the difference from the CWIs for the scalar case is that here the differential equation can be non homogenous. However for the form factor with the most tensorial dimension(i.e. the form factor that comes along with most number of momentum in the tensor multiplying it, in the above case A_1) has homogenous differential equations as its CWIs. Thus, we can solve for A_1 and then we can use its expression for solve the other form differential equations for other form factors, kind of like a domino. Also, the solutions of the above differential equations were in the form of Triple-K integrals, as was noted in while stating the solution of scalar CWIs. Thus, the form factors are described in terms of the Triple-K integrals.

Solving each of the secondary CWIs fixes the form factors upto a constant.Hence this way using conformal Ward Identities we can get the form factors and likewise the expression for a 3-point correlator itself. This approach works well for 3-point functions but faces several complications while solving for higher point correlators.

1.3 Conformal Ward Identities in spinor helicity

While solving CWIs in momentum space one of the difficulty that one faces is choosing the independent tensor structures to work with. This is non trivial to do because of various Schouten identities and presence of various other degeneracies in momentum space. One of the common degeneracies is one involving the metric. Although naively it might seems that metric is an independent tensor from the momenta components, the metric can actually be written in terms of the momenta using inverse of gram matrix. There are other even more non trivial degeneracies that one has to deal with in momentum space.
However in the spinor helicity space such degeneracies are trivial and can be easily taken care of. Thus, we describe in this section how to solve the CWIs in spinor helicity variables. A brief introduction to the spinor helicity variables and the notations are given in the appendix. First we introduce the important conformal generators in spinor helicity variables. The special conformal generator becomes[9]-

$$\tilde{K}^{\kappa} = 2\sum_{i=1}^{n}(\sigma^{\kappa})_{\alpha}^{\beta}\frac{\partial^2}{\partial\lambda_{i\alpha}\partial\overline{\lambda}_i^{\beta}} \tag{1.28}$$

where σ is the Pauli matrix. Action of the special conformal generator on scalar operators with confromal dimension, $\Delta = 2$ is given by the usual action of special conformal generator K^κ as was in the momentum space. However for scalar correlator with $\Delta\neg 2$ and for spinning operators it is different

$$\begin{aligned}
&\tilde{K^\kappa} O_2 = K^\kappa O_2 \\
&\tilde{K^\kappa}\left(\frac{O_\Delta}{p^{\Delta-2}}\right) = -\frac{1}{p^{\Delta-2}} K^\kappa O_\Delta + \frac{O_\Delta}{p^\Delta} p^\kappa O_\Delta + \frac{O_\Delta}{p^\Delta} p^\kappa (\Delta-1)(\Delta-2) \\
&\tilde{K^\kappa} J^\pm = \left(- z_\pm^\alpha K^\kappa + 2 z_\pm^\kappa \frac{p^\alpha}{p^2} \right) J_\alpha \\
&\tilde{K^\kappa}\left(\frac{T^\pm}{p}\right) = \left(- \frac{1}{p} z_\pm^{(\alpha} z_\pm^{\beta)} K^\kappa + 12 z_\pm^\kappa \frac{z_\pm^{(\alpha} p^{\beta)}}{p^3} \right) T_{\alpha\beta}
\end{aligned} \tag{1.29}$$

Fot a general spinning correlator, pon acting the special conformal generator on the correlator, we get

$$\tilde{K^\kappa} \langle \frac{J_{s_1}}{p_1^{s_1-1}} \frac{J_{s_2}}{p_2^{s_2-1}} \frac{J_{s_3}}{p_3^{s_3-1}} \rangle = \text{contact terms} \tag{1.30}$$

From using transverse Ward Identity, the RHS contains contact terms and can be written in terms of lower point correlators. Now, being a differential equation, its solution can be separated into homogenos and non-homogenous components. That is the conformal correlator satisfying the above equation can be written as

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = \langle J_{s_1} J_{s_2} J_{s_3} \rangle_h + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh} \tag{1.31}$$

where h represents the homogenous solution of the conformal identity and nh represents the non homogenous soltion. The the non zero RHS contribution to the conformal identity is completely from the non homogenous part. Also, the homogenous part of the correlator is completely transverse and the local terms in the correlators make an appearance only in the non homogenous part.

As we did in the case of momentum space CWIs, now we will look at a concrete example of solving the CWIs in spinor helicity.

- $\langle J_s J_s O\rangle$ for general spin s

It can be checked that $\langle J_s J_s O\rangle$is transverse and hence the only ontribution is from the homogenous part. First we need to write the ansatz for the correlator. The ansatz for a general correlator can be written as[9]-

$$\langle O^{h_1} O^{h_2} O^{h_3}\rangle = (c_1 F_1(p_1,p_3,p_3) + i c_2 F_2(p_1,p_3,p_3))\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h-1-h_3} \tag{1.32}$$

where $F_i's$ are the form factors. Using this general ansatz we can write the ansatz for $\langle J_s J_s O\rangle$ for all helicities as

$$\langle \frac{J^{s-}}{p_1^{s_1-1}} \frac{J^{s-}}{p_2^{s_2-1}} \frac{O}{p_3^{\Delta-2}}\rangle = (c_1 F_1(p_1,p_2,p_3) + i c_2 F_2(p_1,p_2,p_3))\langle 12\rangle^{2s} \tag{1.33}$$

$$\langle \frac{J^{s+}}{p_1^{s_1-1}} \frac{J^{s+}}{p_2^{s_2-1}} \frac{O}{p_3^{\Delta-2}}\rangle = (c_1 F_1(p_1,p_2,p_3) + i c_2 F_2(p_1,p_2,p_3))\langle \overline{12}\rangle^{2s} \tag{1.34}$$

$$\langle \frac{J^{s+}}{p_1^{s_1-1}} \frac{J^{s-}}{p_2^{s_2-1}} \frac{O}{p_3^{\Delta-2}}\rangle = (c_1 F_1(p_1,p_2,p_3) + i c_2 F_2(p_1,p_2,p_3))\langle \overline{12}\rangle^{2s} \tag{1.35}$$

Using the action of conformal generators on several operators from 1.29, we can write the differential equation for the form factors. The form of the solutions for the form factors for general Δ turns out to be[9]-

$$F_1(p_1,p_2,p_3) = F_2(p_1,p_2,p_3) = p_3^{2-\Delta} I_{(\frac{1}{2})\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}} \tag{1.36}$$

$$G_1(p_1,p_2,p_3) = G_2(p_1,p_2,p_3) = 0 \tag{1.37}$$

In spinor helicity, solving conformal Ward identity turns out to be easier because the degenracy structure is trivial for 3-point functions and can be easily deciphered using Schouten identities.

1.4 Problems of using CWIs for higher point correlators

The reason why dealing with 4-point functions is particularly difficult compared to the 3-point correlators is that in the case of 4-point correlators, constraint of conformal invariance

can no longer provide suffiecient information to completely constrain the form of the correlator. In position space it constrains the form of the 4-point crrelator only upto a function of the conformal cross ratios. In momentum space it is even harder because the momentum space analog of conformal cross ratios are not well defined.

More particularly, the problem of using the above approach of using conformal Ward identities to solve for higher point correlators is that the tensor decomposition of the correlators become more complicated. To show this, let's try to write the decomposition for 4-point correlator, $\langle TTOO\rangle$. Using the similar idea as to the 3-point case, we can write the decomposition of the transverse traceless part as[10]

$$\langle t^{\mu_1\nu_1}(\boldsymbol{p}_1)t^{\mu_2\nu_2}(\boldsymbol{p}_2)O(\boldsymbol{p}_3)O(\boldsymbol{p}_4)\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(\boldsymbol{p}_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(\boldsymbol{p}_2) \tag{1.38}$$

$$\Bigg[\sum_{n_{1,2}\in\{2,3\}}\sum_{n_{3,4}\in\{1,3\}} A_{n_1n_2n_3n_4}p^{\alpha_1}_{n_1}p^{\beta_1}_{n_2}p^{\alpha_2}_{n_3}p^{\beta_2}_{n_4} + \delta^{\alpha_1\alpha_2}\sum_{n_1\in\{2,3\}}\sum_{n_2\in\{1,3\}} A_{n_1n_2}p^{\beta_1}_{n_1}p^{\beta_2}_{n_2} + A_0\delta^{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}\Bigg] \tag{1.39}$$

As before, $A_{n_1n_2n_3n_4}, A_{n-1n_2}, A_0$ are the different form factors. Thus, we would have $2^4 = 16$ possible tensor structures which survive after contraction with the transverse, traceless and symmetric projectors. For the case of $d = 3$ dimensions, we can remove the terms that have matric tensor in their tensor structures as they are not independent.

However, if we consider 4-point correlators which higher spin operator insertions, the decomposition becomes even more complicated. For example in the case of $\langle TTTT\rangle$, we would have 47,868 number of possible tensor structures. Using the methods as described for thrre point functions, we can infer that the number of independent tensor structures which survive after contraction with the transverse, traceless, symmetric projectors can be reduced to 382. So, solving the conformal Ward identities for the form factors can becomes quite tedious.

Also, in the case of 3-point correlators, we had a parametric solution to the confromal Ward identities in terms of the triple-K integral. However such parametric solutions for the case of 4-point correlators have not been well developed yet. Due, these problems we use an alternative approach of using Higher spin symmetries to tame the form of the conformal correlators.

Chapter 2

Higher Spin Equations

2.1 Exact higher spin theories

For the case of theories with higher spin symmetries there is an infinite ladder of conserved current J_s corresponding to a spin s . With each of these conserved currents we can associate a conserved charge Q_S defined as[1]-

$$Q_s = \int_{x^+=constant} dx^- dy J_{-\ldots-} \tag{2.1}$$

in lightcone coordinates where the metric is $ds^2 = dx^- dx^+ + dy^2$. As these are the conserved charges associated with theese theories, hence upon acting these charges on the correlator $\langle J_{s_1}(x_1)...J_{s_n}(x_n)\rangle$ kills the correlator.

$$\sum_{i=1}^{n} \langle J_{s_1}(x_1)...[Q_s, J_{s_i}(x_i)]...J_{s_n}(x_n)\rangle = 0 \tag{2.2}$$

This equation holds for theories with higher spin symmetries as mentioned above. However, for more general case of theories with slightly broken symmetries the RHS of the above equation is non-zero and are explained further below.

Now, for the theories with exact higher spin symmetry, since there exists infinite number

of conserved higher spin charges for all spin $s > 0$, we have in principle infinite number of equations on the correlator, called higher spin equations and thus, higher spin symmetry proves crucial in fixing the form of these correlators. The form of these higher spin equations are obtained by using the algebra of these charges Q_s on the various current operators $J_{s'}$. These algebras can be constrained by using several general arguments like dimensionality, covariance, parity, spin conjugations etc. In this thesis, we will be focusing our attention on the conserved higher spin charges, Q_3 and Q_4.

2.1.1 Algebra

In this thesis for the exact higher spin symmettric theories we consider free boson and free fermion theories.
First we write the conservation algebra for free boson theory in which we have a scalar operator J_0 with dimension $\Delta = 1$ and infinite ladder of conserved currents J_s with spin s and dimension $\Delta = s + 1$. Here, the algebra for Q_3 and Q_4 on J_0 and J_1 are :-

$$\begin{aligned}
&[Q_3, O] = [Q_{\mu\nu}, O] = c_1 \partial_{(\mu} J_{\nu)} \\
&[Q_3, J] = [Q_{\mu\nu}, J_\rho] = c_2 \partial_\mu \partial_\nu \partial_\rho O + c_3 g_{\mu\nu} \partial_\rho \Box O + c_4 \partial_{(\mu} T_{\nu)\rho} + c_5 \partial_\rho T_{\mu\nu} \\
&[Q_4, O] = [Q_{\mu\nu\rho}, O] = d_1 \partial_\mu \partial_\nu \partial_\rho O + d_2 g_{(\mu\nu} \partial_{\rho)} \Box O + d_3 \partial_{(\mu} T_{\nu\rho)} \\
&[Q_4, J] = [Q_{\mu\nu\rho}, J_\alpha] = d_4\, \partial_{(\mu} \partial_\nu \partial_{\rho)} J_\alpha + d_5\, \partial_\alpha \partial_{(\mu} \partial_\nu J_{\rho)} + d_6 g_{(\mu\nu} \partial_{\rho)} \Box J_\alpha \\
&\qquad\qquad + d_7 g_{\mu\nu} \partial_\alpha \Box J_\rho + d_8 g_{\alpha(\mu} \partial_\nu \Box J_{\rho)} + d_9 \partial_\alpha J_{(\mu\nu\rho)}
\end{aligned} \tag{2.3}$$

There are many unknown constants in the above algebra. These can be fixed by using the higher spin equation in 2- and 3-point cases where we know the form of the correlators.

Next, we write the algebra for the case of free fermion theory in three dimensions, where we have pseudo-scalar operator J_0 of dimension $\Delta = 2$ and the infinite ladder of conserved spinning currents J_s are operators of dimension $\Delta = s + 1$. Here the algebra for Q_3 and Q_4

on J_0, J_1 and J_2 is[1]

$$
\begin{aligned}
&[Q_{\mu\nu}, O] = \epsilon_{\mu ab}\, \partial^a \partial_\nu J^b \\
&[Q_{\mu\nu}, J_\rho] = \epsilon_{\sigma\rho(\mu}\partial_{\nu)}\partial_\sigma O + i\partial_{(\mu}T_{\nu)\rho} + \partial_\rho T_{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_{\mu\nu\rho}, O] = \partial_\mu\partial_\nu\partial_\rho O + g_{\mu\nu}\partial_\rho \Box O + \epsilon_{\mu ab}\partial_\nu\partial_a T_{b\rho} \\
&[Q_{\mu\nu\rho}, J_\alpha] = a\,\partial_{(\mu}\partial_\nu\partial_{\rho)}J_\alpha + b\,\partial_\alpha\partial_{(\mu}\partial_\nu J_{\rho)} + \partial_\alpha J_{(\mu\nu\rho)} \\
&\quad + g_{(\mu\nu}\partial_{\rho)}\Box J_\alpha + g_{\mu\nu}\partial_\alpha \Box J_\rho + c g_{\alpha(\mu}\partial_\nu \Box J_{\rho)} \\
&[Q_{\mu\nu\rho}, T_{\alpha\beta}] = b_1\partial_\mu\partial_\nu\partial_\rho T_{\alpha\beta} + b_2\partial_\mu\partial_\nu\partial_\alpha T_{\rho\beta} + b_3\partial_\mu J_{\nu\rho\alpha\beta} + b_4\partial_\alpha J_{\mu\nu\rho\beta}
\end{aligned}
\tag{2.4}
$$

As in the above case of free boson the coefficients of free fermion algebra can also be fixed by writing the higher spin equation for the two point and three point cases and using the known form of the correlators. These kind of algebras for higher spin charges can also be obtained by using parity and charge conjugation arguments.

2.2 Slightly broken higher spin theories

In the case of theories with slightly broken higher spin symmetry, the current J_s are not exactly conserved and obey a non conservation equations[13] and the higher spin charges corresponding to them are not exactly conserved either. Thus, the action of these charges on the correlators doesn't kill it, rather gives the following non conservation equation which has a contribution proportional to the coupling streangth λ:-

$$
\sum_{i=1}^{n}\langle J_{s_1}(x_1)...[Q_s, J_{s_i}(x_i)]...J_{s_n}(x_n)\rangle = \int_x \langle \partial_\mu J^\mu_{-...-}(x) J_{s_1}(x_1)...J_{s_i}(x_i)...J_{s_n}(x_n)\rangle \tag{2.5}
$$

Here, $J^\mu_{-...-}$ is a spin s current. Thus the non conservation for Q_s with spin s is governed by the divergence of J_s which is a corresponding current with spin s.

In our work, we have focused on interacting conformal field theories with slightly broken

[1]It is to be noted that the indices of the charge, in case of Q_3 here, μ and ν are symmetrised. Symmetrisation is to be assumed implicitly in the LHS of every higher spin equation even though we dont mention it explicity to avoid cluttering.

higher spin symmetry. These include critical bosonic, quasi bosonic and quasi fermionic theories. Quasi fermionic theory refers to ususual fermion coupled to Chern Simons theory. But, it can also refer to critical boson coupled to Chern-Simons theory. It is to be noted that usual fermion coupled to Chern-Simons and critical fermion coupled to Chern-Simons are dual to each other. The strong coupling limit of the quasi fermionic theory gives us the critical bosonic theory. In similar manner, quasi bosonic theory refers to usual boson coupled to Chern-Simons and critical fermion coupled to Chern-Simons both of which are dual to each other. Also, in the large N limit, the usual fermion coupled to Chern-Simons and critical fermion coupled to Chern-Simons are related to each other by Legendre transformation. Similarly, critical boson and usual boson are related by Legendre transformation.

2.2.1 Quasi-fermionic theory

In quasi-fermionic theory we have a scalar operator J_0 with a dimension $\Delta = 2+O(\frac{1}{N})$, which in large N limit just becomes J_0 with dimension $\Delta = 2$, that is it has same dimensions as the J_0 of the usual fermionic theory. And as in the free femrion case we also have an infinite ladder of spinning conserved currents of spin s which have dimension $\Delta = s + 1 + O(\frac{1}{N})$ which as can be expected, has the same dimension as the J_s of free fermionic theory for the large N limit. The algebra for Q_3 and Q_4 in quasi fermionic theory is given as

$$[Q_{\mu\nu}, O] = c_1 \epsilon_{\mu ab}\, \partial^a \partial_\nu J^b \tag{2.6}$$

$$[Q_{\mu\nu}, J_\rho] = c_2 \epsilon_{\sigma\rho(\mu}\partial_{\nu)}\partial_\sigma O + c_3 \partial_{(\mu} T_{\nu)\rho} + c_4 \partial_\rho T_{\mu\nu} \tag{2.7}$$

$$[Q_{\mu\nu\rho}, O] = a_1 \partial_\mu\partial_\nu\partial_\rho O + a_2 g_{\mu\nu}\partial_\rho \Box O + a_3 \epsilon_{\mu ab}\partial_\nu \partial_a T_{b\rho} \tag{2.8}$$

$$\begin{aligned}[Q_{\mu\nu\rho}, J_\alpha] = {} & a\, \partial_{(\mu}\partial_\nu\partial_{\rho)} J_\alpha + b\, \partial_\alpha \partial_{(\mu}\partial_\nu J_{\rho)} + \partial_\alpha J_{(\mu\nu\rho)} \\ & + g_{(\mu\nu}\partial_{\rho)}\Box J_\alpha + g_{\mu\nu}\partial_\alpha \Box J_\rho + c g_{\alpha(\mu}\partial_\nu \Box J_{\rho)}\end{aligned} \tag{2.9}$$

$$[Q_{\mu\nu\rho}, T_{\alpha\beta}] = b_1 \partial_\mu\partial_\nu\partial_\rho T_{\alpha\beta} + b_2 \partial_\mu\partial_\nu\partial_\alpha T_{\rho\beta} + b_3 \partial_\mu J_{\nu\rho\alpha\beta} + b_4 \partial_\alpha J_{\mu\nu\rho\beta} \tag{2.10}$$

Since, in quasi fermionic theory is modified, the algebra coefficients in this case can have dependence on the coupling λ_{qf}. We define

$$\tilde{\lambda}_{qf} = \frac{\tan(\lambda_{qf}\pi)}{2} \tag{2.11}$$

The algebra and the non conservation has a polynomial dependence on $\tilde{\lambda}_{qf}$.
The current non-conservation equation for J_3 and J_4 which dictates the non zero RHS of the slightly broken higher spin equation for quasi fermionic theory is given as[2]-

$$\partial_a J^a_{\mu\nu} = -16i(r_0\partial_\mu J_0 J_\nu + r_1\partial_\mu J_\nu J_0) \tag{2.12}$$

$$\partial_a J^a_{\mu\nu\rho} = \frac{480}{7}(c_0\partial_\mu J_0 T_{\nu\rho} + c_1\partial_\mu T_{\nu\rho} J_0) \tag{2.13}$$

Here r_0, r_1,c_0, c_1 have dependence on $\tilde{\lambda}$ as given by

$$r_0 = \frac{3\tilde{\lambda}_{qf}}{5\tilde{N}(1+\tilde{\lambda}^2_{qf})}, \quad r_1 = -\frac{2\tilde{\lambda}_{qf}}{5\tilde{N}(1+\tilde{\lambda}^2_{qf})} \tag{2.14}$$

$$c_0 = \frac{\tilde{\lambda}_{qf}}{\tilde{N}(1+\tilde{\lambda}^2_{qf})}, \quad c_1 = -\frac{2\tilde{\lambda}_{qf}}{5\tilde{N}(1+\tilde{\lambda}^2_{qf})} \tag{2.15}$$

To fix the coefficients of the algebra equations 2.6-2.10, we can explicitly work out no conservation HSE for 2-point and 3-point functions. In the limit $\tilde{\lambda}_{qf} \to 0$, the quasi fermionic algebra and hence the coefficients should reduce to the free fermionic algebra. Also, for the limit $\tilde{\lambda}_{qf} \to \infty$, the algebra reduces to the critical boson algebra.

2.2.2 Quasi-bosonic theory

In the spectrum of the quasi bosonic theory we have a scalar operator J_0 with conformal dimension $\Delta = 1 + O(\frac{1}{N})$, which in large N limit has dimension $\Delta = 1$. Then, we also have infinite ladder of conserved spinning currents J_s of spin s and of dimension $\Delta = s+1+O(\frac{1}{N})$ has the same dimension as the free bosonic J_s in the large N limit.
The algebra for Q_3 and Q_4 in quasi bosonic theory is given by

$$[Q_3, O] = [Q_{\mu\nu}, O] = c_1\partial_{(\mu}J_{\nu)} \tag{2.16}$$

$$[Q_3, J] = [Q_{\mu\nu}, J_\rho] = c_2\partial_\mu\partial_\nu\partial_\rho O + c_3 g_{\mu\nu}\partial_\rho\Box O + c_4\partial_{(\mu}T_{\nu)\rho} + c_5\partial_\rho T_{\mu\nu} \tag{2.17}$$

$$[Q_4, O] = [Q_{\mu\nu\rho}, O] = d_1\partial_\mu\partial_\nu\partial_\rho O + d_2 g_{(\mu\nu}\partial_{\rho)}\Box O + d_3\partial_{(\mu}T_{\nu\rho)} \tag{2.18}$$

$$\begin{aligned}[Q_4, J] = [Q_{\mu\nu\rho}, J_\alpha] = {} & d_4\,\partial_{(\mu}\partial_\nu\partial_{\rho)}J_\alpha + d_5\,\partial_\alpha\partial_{(\mu}\partial_\nu J_{\rho)} + d_6 g_{(\mu\nu}\partial_{\rho)}\Box J_\alpha \\ & + d_7 g_{\mu\nu}\partial_\alpha\Box J_\rho + d_8 g_{\alpha(\mu}\partial_\nu\Box J_{\rho)} + d_9\partial_\alpha J_{(\mu\nu\rho)}\end{aligned} \tag{2.19}$$

Here as in QF case, e define

$$\tilde{\lambda}_{qb} = \frac{\tan(\lambda_{qb}\pi)}{2} \tag{2.20}$$

where λ_{qb} is the coupling that comes in the lagrangian. The free theory limit or the limit $\tilde{\lambda}_{qf} \to 0$, the algebra reduces to the free bosonic algebra. And for the limit $\tilde{\lambda}_{qf} \to \infty$ of the above quasi bosonic algebra we would get the critical fermion algebra. The current non-conservation equations which governs the RHS of the slightly broken higher spin equation for J_3 in quasi bosonic theory at large N limit is given by

$$\partial_a J^a_{\mu\nu} = -\frac{\epsilon_{cd\mu}}{10}(f_0 J^c \partial^d \partial_\nu J_0 + b_1 \partial^c J^d \partial_\nu J_0 + f_1 \partial_\nu J^c \partial^d J_0 + b_2 J_0 \partial_\nu \partial^c J^d) \tag{2.21}$$

where the b_i and f_i are dependent on the coupling $\tilde{\lambda}_{qb}$ and are given as

$$b_1 = -\frac{512i\tilde{\lambda}_{qb}}{3\tilde{N}(1+\tilde{\lambda}^2_{qb})},\ f_1 = -\frac{128i\tilde{\lambda}_{qb}}{3\tilde{N}(1+\tilde{\lambda}^2_{qb})},\ f_0 = \frac{64i\tilde{\lambda}}{\tilde{N}(1+\tilde{\lambda}^2_{qb})},\ b_2 = \frac{128i\tilde{\lambda}}{3\tilde{N}(1+\tilde{\lambda}^2_{qb})} \tag{2.22}$$

The divergence of J_3 current makes an appearance in the RHS of the non conservation equation for the Q_3 charge. Similarly for the Q_4 charge in the RHS of the non conservation higher spin equation, we have divergence of J_4 current which in the leading order in the large N limit is given by[2]

$$\partial_a J^a_{\mu\nu\rho} = \epsilon_{cd\mu}(f_0 T^c_\nu \partial^d \partial_\rho J_0 + b_1 \partial^c T^d_\nu \partial_\rho J_0 + f_1 \partial_\nu T^c_\rho \partial^d J_0 + b_2 J_0 \partial_\nu \partial^c T^d_\rho) \tag{2.23}$$

Here the b_i and f_i are given by

$$b_1 = -\frac{192i\tilde{\lambda}_{qb}}{\tilde{N}(1+\tilde{\lambda}^2_{qb})},\ f_1 = -\frac{64i\tilde{\lambda}_{qb}}{\tilde{N}(1+\tilde{\lambda}^2_{qb})},\ f_0 = \frac{160i\tilde{\lambda}}{\tilde{N}(1+\tilde{\lambda}^2_{qb})},\ b_2 = \frac{32i\tilde{\lambda}}{\tilde{N}(1+\tilde{\lambda}^2_{qb})} \tag{2.24}$$

2.3 Higher Spin equation in momentum space

Since the main aim of our project was to study CFT correlators in momentum space, after the introduction to higher spin equations, we shift our focus to studying these higher spin equations in momentum space and to see how these equations prove useful in constraining the form of these correlators. Momentum space higher spin equations are obtained by doing

a Fourier transform of the position space equations. For brevity we consider an example of higher spin equation that we get acting Q_4 on $\langle TOO\rangle$ in quasi fermionic theory

$$\begin{aligned}&\langle[Q_{\mu\nu\rho},T_{\alpha\beta}(x_1)]J_0(x_2)J_0(x_3)\rangle_{QF}+\Big[\langle T_{\alpha\beta}[Q_{\mu\nu\rho},J_0(x_2)]J_0(x_3)\rangle_{QF}+(2\leftrightarrow 3)\Big]\\&=\int_x\langle\partial^{\sigma}_{\mu\nu\rho}(x)T_{\alpha\beta}(x_1)J_0(x_2)J_0(x_3)\rangle_{QF}\end{aligned}\tag{2.25}$$

Now, using the algebra for quasi fermion theory from 2.8, 2.10 and the current equation 2.23, we can write the above equation as

$$\begin{aligned}&\Big[b_1\langle\partial_\mu\partial_\nu\partial_\rho T_{\alpha\beta}(x_1)O(x_2)O(x_3)\rangle_{QF}+b_2\langle\partial_\mu\partial_\nu\partial_\alpha T_{\rho\beta}(x_1)O(x_2)O(x_3)\rangle_{QF}\\&+b_3\langle\partial_\mu J_{\nu\rho\alpha\beta}(x_1)O(x_2)O(x_3)\rangle_{QF}+b_4\langle\partial_\alpha J_{\nu\rho\mu\beta}(x_1)O(x_2)O(x_3)\rangle_{QF}\Big]\\&+\Big[a_1\langle T_{\alpha\beta}(x_1)\partial_\mu\partial_\nu\partial_\rho O(x_2)O(x_3)\rangle_{QF}+a_2\langle T_{\alpha\beta}(x_1)g_{\mu\nu}\partial_\rho\Box O(x_2)O(x_3)\rangle_{QF}\\&+a_3\epsilon_{\mu ab}\langle T_{\alpha\beta}(x_1)\partial_\nu\partial_a T_{b\rho}(x_2)O(x_3)\rangle_{QF}+(2\leftrightarrow 3)\Big]\\&=\frac{96\tilde{\lambda}_{qf}}{1+\tilde{\lambda}_{qf}^2}\Big[\langle\partial_\mu OT_{\nu\rho}(x)T_{\alpha\beta}(x_1)O(x_2)O(x_3)\rangle_{QF}\Big]\end{aligned}\tag{2.26}$$

Here the first bracket in the LHS is from the action of Q_4 on $T_{\alpha\beta}(x_1)$ and the second bracket is from the action of Q_4 on $O(x_2)$. The $(2\leftrightarrow 3)$ signifies 2-3 exchange since to get the action of Q_4 on $O(x_3)$, we can take the Q_4 action on $O(x_2)$ and then do $x_2\leftrightarrow x_3$ exchange. The RHS can be obtained by taking the current equation for J_4 and then doing integration by parts on the terms accordingly. Now, we can Fourier transform the the above equation to

get the corrsponding higher spin equation in momentum space:-

$$\begin{aligned}
&\Big[b_1 p_{1\mu}p_{1\nu}p_{1\rho}\langle T_{\alpha\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} + b_2 p_{1\mu}p_{1\nu}p_{1\alpha}\langle T_{\rho\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} \\
&+ b_3 p_{1\mu}\langle J_{\nu\rho\alpha\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} + b_4 p_{1\alpha}\langle J_{\nu\rho\mu\beta}(p_1)O(p_2)O(p_3)\rangle_{QF}\Big] \\
&+ \Big[a_1 p_{2\mu}p_{2\nu}p_{2\rho}\langle T_{\alpha\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} + a_2 p_{2\rho}g_{\mu\nu}p_2^2\langle T_{\alpha\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} \\
&+ a_3\epsilon_{\mu ab}p_{2\nu}p_{2a}\langle T_{\alpha\beta}(p_1)T_{b\rho}(p_2)O(p_3)\rangle_{QF}\Big] \\
&= \frac{96\tilde{\lambda}_{qf}}{1+\tilde{\lambda}_{qf}^2}\Big[p_{1\mu}\langle T_{\nu\rho}(p_1)T_{\alpha\beta}(-p_1)\rangle_{QF}\langle O(p_1)O(p_2)O(p_3)\rangle_{QF} \\
&+ (p_{2\mu}\langle O(p_2)O(-p_2)\rangle_{QF}\langle T_{\alpha\beta}(p_1)T_{\nu\rho}(p_2)O(p_3)\rangle_{QF} + (2\leftrightarrow 3))\Big]
\end{aligned} \tag{2.27}$$

Here the $(2 \leftrightarrow 3)$ implies $p_2 \leftrightarrow p_3$ exchange. Here, we have also used large N factorisation for the five-point correlator that appears in the RHS. Since the only non-vanishing two-point correlators that can be obtained from this factorisations are $\langle OO\rangle$ and $\langle TT\rangle$, thus the only possible factorisation that can be done of the five-point correlator $\langle OTTOO\rangle$ are the ones that are written above in the RHS. In our project we have restricted ourself to large N limit which allows us to factorise the higher point correlators which appear in the RHS of the non-conservation equation, thus making the analysis a bit simpler.

After writing the non-conservation HSE, we use the form of three point correlators in quasi fermionic theory in terms of the contributions from the free boson(FB), free fermion(FF) and parity odd piece(odd)[22]-

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{QF} = \alpha_{s_1,s_2,s_3}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{FB} + \beta_{s_1,s_2,s_3}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{FF} + \gamma_{s_1,s_2,s_3}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{odd} \tag{2.28}$$

which we then plug back in the expression for QF correlator in the higher spin equation. More details regarding the specifics of 2.28 and the coefficients α, β, γ are provided later in the section for 3-point correlators, where we work out the higher spin equations in detail. One thing to note is that the coefficients α, β, γ are dependent on the coupling $\tilde{\lambda}$. Thus, after plugging back 2.28 in the higher spin equation, we look at the equation at various orders in $\tilde{\lambda}$ and using these equations, solve for the unknown parts of the quasi fermionic correlators in terms of the known free fermionic and critical bosonic correlators.

This strategy is general for any interacting theory with slightly broken higher spin symmetry that we considered in the project. We use the same principle to work with 4-point functions as well. The difference for the case of 4-point functions is that the analogous decomposition of the correlators at different orders of $\tilde{\lambda}$ 2.28 is more involved and contains more terms than just the ones in the 3-point case. Also, the factorisation of the correlators in the RHS in the large N limit is more complicated.

General Strategy for using higher spin equation to solve for correlators

- Impose symmetry for the charge operator Q_s on the concerned correlator and using the higher spin algebra for Q_s in the theory that we are concerned with to write the LHS of the higher spin equation.
- If the theory concerned has an exact higher spin symmetry(for example:- free boson or free fermionic theory) then set the RHS to be zero. If the theory has slightly broken higher spin symmetry(interacting theories like quasi fermionic theory) then use the divergence of current J_s corrsponding to Q_s to write the non conservation RHS as described above.
- For the large N limit and considering only leading order terms, we can factorise the correlator in the RHS.
- Then we Fourier transform the higher spin equation to get the equation in momentum space.
- After writing the HSE in momentum space, plug in the expansion for the quasi fermion correlator at different order of the coupling $\tilde{\lambda}$ in terms of known(free theory) and unknown contributions to the correlator.
- After expanding the interacting correlators into different orders in coupling $\tilde{\lambda}$, we look at various orders of the higher spin equations separately. We also do a series expansion around $\tilde{\lambda} = i$ to get the pole equations.
- Usually the lowest and highest order equation in $\tilde{\lambda}$ correspond to the free theory and critical theory of the respectively. For example, equation in lowest order in $\tilde{\lambda}$ of the

higher spin equation for a quasi fermionic correlator usually correspond to free fermion higher spin equation and the highest order usually to the critical bosonic higher spin equation.

- We use the pole equations and the equations from different orders in $\tilde{\lambda}$ to solve for the correlator in the theories with slightly broken higher spin symmetry in terms of free and critical theory correlators.

Chapter 3

Using higher spin symmetry to compute correlators

3.1 Three point functions

3.1.1 $\langle TTO\rangle_{QF}$

Above we wrote the non conservation HSE for $\langle TTO\rangle_{QF}$ by acting Q_4 on $\langle TOO\rangle_{QF}$. The HSE was:

$$\begin{aligned}
&\Big[b_1 p_{1\mu}p_{1\nu}p_{1\rho}\langle T_{\alpha\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} + b_2 p_{1\mu}p_{1\nu}p_{1\alpha}\langle T_{\rho\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} \\
&+ b_3 p_{1\mu}\langle J_{\nu\rho\alpha\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} + b_4 p_{1\alpha}\langle J_{\nu\rho\mu\beta}(p_1)O(p_2)O(p_3)\rangle_{QF}\Big] \\
&+ \Big[a_1 p_{2\mu}p_{2\nu}p_{2\rho}\langle T_{\alpha\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} + a_2 p_{2\rho}g_{\mu\nu}p_2^2\langle T_{\alpha\beta}(p_1)O(p_2)O(p_3)\rangle_{QF} \\
&+ a_3\epsilon_{\mu ab}p_{2\nu}p_{2a}\langle T_{\alpha\beta}(p_1)T_{b\rho}(p_2)O(p_3)\rangle_{QF}\Big] \\
&= \frac{96\tilde{\lambda}_{qf}}{1+\tilde{\lambda}_{qf}^2}\Big[p_{1\mu}\langle T_{\nu\rho}(p_1)T_{\alpha\beta}(-p_1)\rangle_{QF}\langle O(p_1)O(p_2)O(p_3)\rangle_{QF} \\
&+ (p_{2\mu}\langle O(p_2)O(-p_2)\rangle_{QF}\langle T_{\alpha\beta}(p_1)T_{\nu\rho}(p_2)O(p_3)\rangle_{QF} + (2\leftrightarrow 3))\Big]
\end{aligned} \tag{3.1}$$

Next we look at different orders in coupling $\tilde{\lambda}_{qf}$ of the higher spin equation:
O(1):

$$\begin{aligned}
&b_1 p_{1\mu} p_{1\nu} p_{1\rho} \langle T_{\alpha\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} + b_2 p_{1\mu} p_{1\nu} p_{1\alpha} \langle T_{\beta\rho}(p_1) O(p_2) O(p_3)\rangle_{FF} \\
&+ b_3 p_{1\mu} \langle J_{\nu\rho\alpha\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} + b_4 p_{1\alpha} \langle J_{\nu\rho\mu\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} \\
&+ \Big[a_1 p_{2\mu} p_{2\nu} p_{2\rho} \langle T_{\alpha\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} + a_2 p_{2\rho} g_{\mu\nu} p_2^2 \langle T_{\alpha\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} \\
&+ a_3 \epsilon_{\mu ab} p_{2\nu} p_2^a \langle T_{\alpha\beta}(p_1) T^b_\rho(p_2) O(p_3)\rangle_0 + (2 \leftrightarrow 3) \Big] = 0
\end{aligned} \tag{3.2}$$

Since, this is the lowest order equation in $\tilde{\lambda}_{qf}$, this would correspond to the free fermion higher spin equation which we obtain by using free fermionic algebra of Q_4 on $\langle TOO\rangle$. Thu, we can identify

$$\langle TTO\rangle_0 = \langle TTO\rangle_{FF} \tag{3.3}$$

O($\tilde{\lambda}_{qf}$)

$$a_3 \epsilon_{\mu ab} p_{2\nu} p_2^a \langle T_{\alpha\beta}(p_1) T^b_\rho(p_2) O(p_3)\rangle_1 + (2 \leftrightarrow 3) = -12 p_{2\mu} p_2 \langle T_{\alpha\beta}(p_1) T_{\nu\rho}(p_2) O(p_3)\rangle_0 + (2 \leftrightarrow 3) \tag{3.4}$$

O($\tilde{\lambda}_{qf}^2$):

$$\begin{aligned}
&b_1 p_{1\mu} p_{1\nu} p_{1\rho} \langle T_{\alpha\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} + b_2 p_{1\mu} p_{1\nu} p_{1\alpha} \langle T_{\beta\rho}(p_1) O(p_2) O(p_3)\rangle_{FF} \\
&+ b_3 p_{1\mu} \langle J_{\nu\rho\alpha\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} + b_4 p_{1\alpha} \langle J_{\nu\rho\mu\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} \\
&+ \Big[a_1 p_{2\mu} p_{2\nu} p_{2\rho} \langle T_{\alpha\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} + a_2 p_{2\rho} g_{\mu\nu} p_2^2 \langle T_{\alpha\beta}(p_1) O(p_2) O(p_3)\rangle_{FF} \\
&+ (2 \leftrightarrow 3) \Big] = -12 p_{2\mu} p_2 \langle T_{\alpha\beta}(p_1) T_{\nu\rho}(p_2) O(p_3)\rangle_1 + (2 \leftrightarrow 3)
\end{aligned} \tag{3.5}$$

This is the highest order equation. Thus, this is the equation that we get from the non conservation higher spin equation at the limit $\tilde{\lambda}_{qf} \to \infty$ as it would have the dominant contribution. Thus, this would correspond to the critical boson higher spin equation. So, we can identify

$$\langle TTO\rangle_1 = \langle TTO\rangle_{CB} \tag{3.6}$$

Thus, we get

$$\boxed{\langle T(p_1)T(p_2)O(p_3)\rangle_{QF} = \langle T(p_1)T(p_2)O(p_3)\rangle_{FF} + \tilde{\lambda}_{qf}\langle T(p_1)T(p_2)O(p_3)\rangle_{CB}} \tag{3.7}$$

It can be also checked that the $O(\tilde{\lambda}_{qf})$ equation maps to $O(\tilde{\lambda}_{qf}^2)$-$O(1)$ if we dot the expression with null polarization tensors and convert them to spinor helicity variables. Thus in spinor helicity variables we can map all order equations to free and critical higher spin equations. We can also do a series expansion around $\tilde{\lambda}_{qf} = i$ to get the pole equation corresponding to the above higher spin spin equation. Doing this exercise gives us the following two pole equations

$$\begin{aligned} a_3\epsilon_{\mu ab}p_{2\nu}p_2^a\langle T_{\alpha\beta}(p_1)T_\rho^b(p_2)O(p_3)\rangle_0 + (2\leftrightarrow 3) &= 12p_{2\mu}p_2\langle T_{\alpha\beta}(p_1)T_{\nu\rho}(p_2)O(p_3)\rangle_1 + (2\leftrightarrow 3) \\ a_3\epsilon_{\mu ab}p_{2\nu}p_2^a\langle T_{\alpha\beta}(p_1)T_\rho^b(p_2)O(p_3)\rangle_1 + (2\leftrightarrow 3) &= -12p_{2\mu}p_2\langle T_{\alpha\beta}(p_1)T_{\nu\rho}(p_2)O(p_3)\rangle_0 + (2\leftrightarrow 3) \end{aligned} \tag{3.8}$$

3.1.2 $\langle JJJ\rangle_{QF}$

Next we consider the example of the higher spin equation for $\langle JJJ\rangle$ in quasi fermionic theory. To get the higher spin equation for $\langle JJJ\rangle$ in the case of slightly broken interacting quasi fermionic theory, we act $Q_{\mu\nu}$ on $\langle J_\alpha J_\beta O\rangle$. The quasi fermionic algebra for action of Q_3 on scalar operator is:

$$[Q_{\mu\nu}, O(p)] = c_1\epsilon_{\mu ab}p^a p_\nu J^a(p) \tag{3.9}$$

$$[Q_{\mu\nu}, J_\alpha(p)] = c_2\frac{\epsilon_{a\alpha\mu}p_\nu p^a}{1+\tilde{\lambda}_{qf}^2}O(p) + c_3 p_\mu T_{\nu\rho}(p) + c_4 p_\alpha T_{\mu\nu}(p) \tag{3.10}$$

By using this algebra we can write the LHS of the non-conservation higher spin equation. To write the RHS we use the divergence of J_3 to write the combined correlator $\langle JOJJO\rangle$ which in the large N limit can be factorised into $\langle OO\rangle\langle JJJ\rangle$ and $\langle JJ\rangle\langle JOO\rangle$. Then we can

fourier transform the RHS to get the higher spin equation in the momentum space as:

$$\begin{aligned}
&\Bigg[c_2\frac{\epsilon_{a\alpha\mu}p_{1\nu}p_1^a}{1+\tilde{\lambda}_{qf}^2}\langle O(p_1)J_\beta(p_2)O(p_3)\rangle_{QF}+c_3p_{1\mu}\langle T_{\nu\alpha}(p_1)J_\beta(p_2)O(p_3)\rangle_{QF}\\
&+c_4p_{1\alpha}\langle T_{\mu\nu}(p_1)J_\beta(p_2)O(p_3)\rangle_{QF}+(1\leftrightarrow 2)\Bigg]+c_1\epsilon_{\mu ab}p_3^ap_{3\nu}\langle J_\alpha(p_1)J_\beta(p_2)J^b(p_3)\rangle_{QF}\\
&=-\frac{16i\tilde{\lambda}_{qf}}{1+\tilde{\lambda}_{qf}^2}\Bigg[\Bigg(p_{1\mu}\langle J_\nu(p_1)J_\alpha(-p_1)\rangle_{QF}\langle O(p_1)J_\beta(p_2)O(p_3)\rangle_{QF}+(1\leftrightarrow 2)\Bigg)\\
&+p_{3\mu}\langle O(p_3)O(-p_3)\rangle_{QF}\langle J_\alpha(p_1)J_\beta(p_2)J_\nu(p_3)\rangle_{QF}\Bigg]
\end{aligned}\tag{3.11}$$

Then, we plug in the quasi fermionic correlator expansion in terms of free boson correlators:

$$\begin{aligned}
&\langle JOO\rangle_{QF}=(1+\tilde{\lambda}_{qb}^2)\langle JOO\rangle_{FF}\\
&\langle TJO\rangle_{QF}=0\\
&\langle JJJ\rangle_{QF}=\frac{1}{1+\tilde{\lambda}_{qf}^2}\Bigg(\langle JJJ\rangle_0+\tilde{\lambda}_{qf}\langle JJJ\rangle_1+\tilde{\lambda}_{qf}^2\langle JJJ\rangle_2\Bigg)
\end{aligned}\tag{3.12}$$

After plugging in the expansion for the quasi bosonic correlator, we get the HSE to be:

$$\begin{aligned}
&\Bigg(c_2\epsilon_{a\alpha\mu}p_{1\nu}p_1^a\langle O(p_1)J_\beta(p_2)O(p_3)\rangle_{FF}+(1\leftrightarrow 2)\Bigg)\\
&+\frac{c_1}{1+\tilde{\lambda}_{qf}^2}\epsilon_{\mu ab}p_3^ap_{3\nu}\Bigg(\langle J_\alpha(p_1)J_\beta(p_2)J^b(p_3)\rangle_0+\tilde{\lambda}_{qf}\langle J_\alpha(p_1)J_\beta(p_2)J^b(p_3)\rangle_1\\
&+\tilde{\lambda}_{qf}^2\langle J_\alpha(p_1)J_\beta(p_2)J^b(p_3)\rangle_2\Bigg)\\
&=-16i\Bigg[\Bigg(\tilde{\lambda}_{qf}p_{1\mu}\langle J_\nu(p_1)J_\alpha(-p_1)\rangle_{FF}\langle O(p_1)J_\beta(p_2)O(p_3)\rangle_{FF}+(1\leftrightarrow 2)\Bigg)\\
&+p_{3\mu}\frac{\langle O(p_3)O(-p_3)\rangle_{QF}}{1+\tilde{\lambda}_{qf}^2}\Bigg(\tilde{\lambda}_{qf}\langle J_\alpha(p_1)J_\beta(p_2)J_\nu(p_3)\rangle_0+\tilde{\lambda}_{qf}^2\langle J_\alpha(p_1)J_\beta(p_2)J_\nu(p_3)\rangle_1\\
&+\tilde{\lambda}_{qf}^3\langle J_\alpha(p_1)J_\beta(p_2)J_\nu(p_3)\rangle_2\Bigg)\Bigg]
\end{aligned}\tag{3.13}$$

Now, we look at equations at various orders in the coupling $\tilde{\lambda}_{qf}$:-

O(1):

$$\Big[c_2\epsilon_{a\alpha\mu}p_{1\nu}p_1^a\langle O(p_1)J_\beta(p_2)O(p_3)\rangle_{FF}+(1\leftrightarrow 2)\Big]+c_1\epsilon_{\mu ab}p_3^a p_{3\nu}\langle J_\alpha(p_1)J_\beta(p_2)J^b(p_3)\rangle_0=0 \tag{3.14}$$

This is the equation for the limit $\tilde{\lambda}_{qf}\to 0$. Also, this is exactly the higher spin equation that we would get for $\langle JJJ\rangle_{FF}$ if we use the free fermion higher spin algebra. Thus, we can identify

$$\langle JJJ\rangle_0=\langle JJJ\rangle_{FF} \tag{3.15}$$

Next we look at O($\tilde{\lambda}_{qf}$) equation:

O($\tilde{\lambda}_{qf}$):

$$\begin{aligned}&c_1\epsilon_{\mu ab}p_3^a p_{3\nu}\langle J_\alpha(p_1)J_\beta(p_2)J^b(p_3)\rangle_1\\&=-16i\Big[\Big(p_{1\mu}\langle J_\nu(p_1)J_\alpha(-p_1)\rangle_{FF}\langle O(p_1)J_\beta(p_2)O(p_3)\rangle_{FF}+(1\leftrightarrow 2)\Big)\\&+p_{3\mu}\langle O(p_3)O(-p_3)\rangle_{FF}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)\rangle_0\Big]\end{aligned} \tag{3.16}$$

O($\tilde{\lambda}_{qf}^2$):

$$\begin{aligned}&\Big[c_2\epsilon_{a\alpha\mu}p_{1\nu}p_1^a\langle O(p_1)J_\beta(p_2)O(p_3)\rangle_{FF}+(1\leftrightarrow 2)\Big]+c_1\epsilon_{\mu ab}p_3^a p_{3\nu}\langle J_\alpha(p_1)J_\beta(p_2)J^b(p_3)\rangle_2\\&=-16i\Big[p_{3\mu}\langle O(p_3)O(-p_3)\rangle_{FF}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)\rangle_1\Big]\end{aligned} \tag{3.17}$$

O($\tilde{\lambda}_{qf}^3$):

$$0=-16i\Big[\Big(p_{1\mu}\langle J_\nu(p_1)J_\alpha(-p_1)\rangle_{FF}\langle O(p_1)J_\beta(p_2)O(p_3)\rangle_{FF}+(1\leftrightarrow 2)\Big) \tag{3.18}$$

$$+p_{3\mu}\langle O(p_3)O(-p_3)\rangle_{FF}\langle J_\alpha(p_1)J_\beta(p_2)J_\nu(p_3)\rangle_2\Big] \tag{3.19}$$

This is the highest order equation in $\tilde{\lambda}_{qf}$ of the higher spin equation of $\langle JJJ\rangle_{QF}$. This is the equation that corresponds to the higher spin equation for critical bosonic $\langle JJJ\rangle$. We can verify this by using the critical bosonic algebra for Q_3 on $\langle JJO\rangle$:

$$[Q_{\mu\nu}, O] = 0$$
$$[Q_{\mu\nu}, J_\alpha] = c_3 p_\mu T_{\nu\rho} + c_4 p_\alpha T_{\mu\nu}$$

So the only terms in the LHS that we would get in the critical bosonic equation is $\langle JTO\rangle_{CB}$, which is zero. Thus, the LHS is just zero. Hence we can verify that the $\mathrm{O}(\tilde{\lambda}_{qf})$ is indeed the critical bosonic higher spin equation. And hence we can identify

$$\langle JJJ\rangle_2 = \langle JJJ\rangle_{CB} \tag{3.20}$$

We can also do an series expansion of the entire higher spin equation about $\tilde{\lambda}_{qf} = i$ to get the pole equations. Doing this exercise gives us the following equations

$$c_1 \epsilon_{\mu ab} p_3^a p_{3\nu} \Big(\langle J_\alpha(p_1) J_\beta(p_2) J^b(p_3)\rangle_0 - \langle J_\alpha(p_1) J_\beta(p_2) J^b(p_3)\rangle_2 \Big) = -2ip_{3\mu} p_3 \langle J_\alpha(p_1) J_\beta(p_2) J_\nu(p_3)\rangle_1 \tag{3.21}$$

and

$$c_1 \epsilon_{\mu ab} p_3^a p_{3\nu} \langle J_\alpha(p_1) J_\beta(p_2) J^b(p_3)\rangle_1 = 2ip_{3\mu} p_3 \Big(\langle J_\alpha(p_1) J_\beta(p_2) J_\nu(p_3)\rangle_0 - \langle J_\alpha(p_1) J_\beta(p_2) J_\nu(p_3)\rangle_2 \Big) \tag{3.22}$$

From the solution of these pole equations we can get the expression for $\langle JJJ\rangle_1$ as the epsilon transform of $\langle JJJ\rangle_{FF} - \langle JJJ\rangle_{CB}$. If we dot 3.21 with p_3^μ, we can get an expression for $\langle JJJ\rangle_1$. Thus, we know the expression for $\langle JJJ\rangle_{QF}$ in terms of $\langle JJJ\rangle_{FF}$ $\langle JJJ\rangle_{CB}$ and epsilon transform of ($\langle JJJ\rangle_{FF}$-$\langle JJJ\rangle_{CB}$):-

$$\boxed{\langle J_\alpha J_\beta J_\nu\rangle_{QF} = \frac{1}{1+\tilde{\lambda}_{qf}^2} \Big(\langle J_\alpha J_\beta J_\nu\rangle_{FF} + \tilde{\lambda}_{qf} \frac{ic_1}{2} \epsilon_{\nu ab} \frac{p_3^a}{p_3} (\langle J_\alpha J_\beta J^b\rangle_{FF} - \langle J_\alpha J_\beta J^b\rangle_{CB}) + \tilde{\lambda}_{qf}^2 \langle JJJ\rangle_{CB} \Big)} \tag{3.23}$$

3.1.3 $\langle J_{s_1} J_{s_2} T \rangle$

In this section we do a rough analysis on constraining a general 3-point correlator $\langle J_{s_1} J_{s_2} T \rangle_{QB}$. We start by acting Q_3 on $\langle J_{s_1} J_{s_2} O \rangle_{QB}$. First we write the relevant algebra

$$\begin{aligned} [Q_4, J_s] &= c_{s,s-2}\partial^5 J_{s-2} + c_{s,s}\partial^3 J_s + c_{s,s+2}\partial J_{s+2} \\ [Q_4, O] &= \partial^3 O + \partial T \end{aligned} \tag{3.24}$$

Then we act Q_4 on the correlator $\langle J_{s_1} J_{s_2} O \rangle$ and get the following HSE

$$\begin{aligned} &\Big(c_{s_1,s_1-2}\langle \partial^5 J_{s_1-2} J_{s_2} O \rangle + c_{s_1,s_1}\langle \partial^3 J_{s_1} J_{s_2} O \rangle + c_{s_1,s_1+2}\langle \partial J_{s_1+2} J_{s_2} O \rangle \\ &(1 \leftrightarrow 2) \Big) + \langle J_{s_1} J_{s_2} \partial^3 O \rangle + \langle J_{s_1} J_{s_2} \partial T \rangle = \frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} r_0 \int \langle \epsilon_{-ab} T_{a-}(x) \partial_b \partial_- O(x) J_{s_1} J_{s_2} O \rangle \end{aligned} \tag{3.25}$$

The correlator $\langle T O J_{s_1} J_{s_2} O \rangle$ can be factorised into $\langle OO \rangle \langle J_{s_1} J_{s_2} O \rangle$ in the large N limit and looking at just the leading order. Then Fourier transforming the RHS, we get

$$\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} r_0 \epsilon_{-ab} p_{3b} p_{3-} \langle OO \rangle \langle J_{s_1} J_{s_2} T_{a-} \rangle \tag{3.26}$$

Now, Fourier transform the LHS and we get the non conservtion higher spin equation in momentum space as

$$\Big(c_{s_1,s_1-2} p_{1-}^5 \langle J_{s_1-2} J_{s_2} O \rangle + c_{s_1,s_1} p_{1-}^3 \langle J_{s_1} J_{s_2} O \rangle + c_{s_1,s_1+2} p_{1-} \langle J_{s_1+2} J_{s_2} O \rangle \tag{3.27}$$

$$(1 \leftrightarrow 2) \Big) + p_{3-}^3 \langle J_{s_1} J_{s_2} O \rangle + p_{3-} \langle J_{s_1} J_{s_2} T \rangle = \frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} r_0 \epsilon_{-ab} p_{3b} p_{3-} \langle OO \rangle \langle J_{s_1} J_{s_2} T_{a-} \rangle \tag{3.28}$$

Now again we use the following expansions

$$\langle OO\rangle_{\rm QB} = (1+\tilde{\lambda}^2)\langle OO\rangle_{\rm FB} \tag{3.29}$$

$$\langle J_{s1}J_{s_2}O\rangle_{\rm QB} = \langle J_{s1}J_{s_2}O\rangle_{\rm FB} + \tilde{\lambda}\langle J_{s1}J_{s_2}O\rangle_{\rm odd} \tag{3.30}$$

$$\langle J_{s1}J_{s_2}T\rangle_{\rm QB} = \frac{1}{1+\tilde{\lambda}^2}\big(\langle J_{s1}J_{s_2}T\rangle_{\rm FB} + \tilde{\lambda}\langle J_{s1}J_{s_2}T\rangle_{\rm odd} + \tilde{\lambda}^2\langle J_{s1}J_{s_2}T\rangle_{\rm FF}\big) \tag{3.31}$$

plugging these back in the (3.27) and collecting terms at various orders in $\tilde{\lambda}$ we get the following equations
O(1):

$$\Big[c_{s_1,s_1-2}p_{1-}^5\langle J_{s_1-2}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{FB} + c_{s_1,s_1}p_{1-}^3\langle J_{s_1}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{FB}$$
$$+ c_{s_1,s_1+2}p_{1-}\langle J_{s_1+2}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{FB} + (1\leftrightarrow 2)\Big] + p_{3-}^3\langle J_{s_1}(p_1)J_{s_1}(p_2)T(p_3)\rangle_{FB} = 0 \tag{3.32}$$

This is the lowest order of the higher spin equation in orders of $\tilde{\lambda}_{qb}$. Thus, we would get the same equation if we used the free bosonic algebra for Q_4 on $\langle J_{s_1}J_{s_2}O\rangle_{FB}$ and write the conservation higher spin equation. Thus we can identify

$$\langle JJT\rangle_0 = \langle JJT\rangle_{FB} \tag{3.33}$$

O($\tilde{\lambda}_{qb}$):

$$\Big[c_{s_1,s_1-2}p_{1-}^5\langle J_{s_1-2}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{odd} + c_{s_1,s_1}p_{1-}^3\langle J_{s_1}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{odd}$$
$$+ c_{s_1,s_1+2}p_{1-}\langle J_{s_1+2}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{odd} + (1\leftrightarrow 2)\Big] + p_{3-}^3\langle J_{s_1}(p_1)J_{s_1}(p_2)T(p_3)\rangle_1 \tag{3.34}$$
$$= r_0\epsilon_{-ab}p_{3b}p_{3-}\langle OO\rangle_{FB}\langle J_{s_1}J_{s_2}T\rangle_0$$

$O(\tilde{\lambda}_{qb}^2)$:

$$\begin{aligned}&\Big[c_{s_1,s_1-2}p_{1-}^5\langle J_{s_1-2}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{FB}+c_{s_1,s_1}p_{1-}^3\langle J_{s_1}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{FB}\\&+c_{s_1,s_1+2}p_{1-}\langle J_{s_1+2}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{FB}+(1\leftrightarrow 2)\Big]+p_{3-}^3\langle J_{s_1}(p_1)J_{s_1}(p_2)T(p_3)\rangle_{0+2}\\&=r_0\epsilon_{-ab}p_{3b}p_{3-}\langle OO\rangle_{FB}\langle J_{s_1}J_{s_2}T\rangle_1\end{aligned}\tag{3.35}$$

$O(\tilde{\lambda}_{qb}^3)$:

$$\begin{aligned}&\Big[c_{s_1,s_1-2}p_{1-}^5\langle J_{s_1-2}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{odd}+c_{s_1,s_1}p_{1-}^3\langle J_{s_1}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{odd}\\&+c_{s_1,s_1+2}p_{1-}\langle J_{s_1+2}(p_1)J_{s_1}(p_2)O(p_3)\rangle_{odd}+(1\leftrightarrow 2)\Big]\\&=r_0\epsilon_{-ab}p_{3b}p_{3-}\langle OO\rangle_{FB}\langle J_{s_1}J_{s_2}T\rangle_2\end{aligned}\tag{3.36}$$

This is the highest order of the quasi bosonic higher spin equation for $\langle JJT\rangle$. Thus, at the $\tilde{\lambda}_{qb}\to\infty$ limit these will be the dominating terms of the equation. Thus, this would correspond to the critical fermionic higher spin equation that we can write using the critical fermionic algebra. Hence we can identify

$$\langle JJT\rangle_2=\langle JJT\rangle_{CF}\tag{3.37}$$

We can also do a series expansion about the pole $\tilde{\lambda}_{qb}$ to get the pole euations. Upon doing the exercise we get the two following equations from equating the real and imaginary parts of the resulting equation

$$\begin{aligned}&b(\langle J_{s_1}J_{s_2}T\rangle_{\rm FB}-\langle J_{s_1}J_{s_2}T\rangle_{\rm CF})+r_0\epsilon_{-ab}p_{3b}\langle J_{s_1}J_{s_2}T\rangle_{\rm odd}\langle OO\rangle_{\rm FB}=0\\&b\langle J_{s_1}J_{s_2}T\rangle_{\rm odd}-r_0\epsilon_{-ab}p_{3b}\langle OO\rangle_{\rm FB}(\langle J_{s_1}J_{s_2}T\rangle_{\rm FB}-\langle J_{s_1}J_{s_2}T\rangle_{\rm CF})=0\end{aligned}\tag{3.38}$$

From the second pole equation we can read off that $\langle J_{s_1}J_{s_2}T\rangle_{odd}$ can be written as the epsilon transform of $\langle J_{s_1}J_{s_2}T\rangle_{FB}-\langle J_{s_1}J_{s_2}T\rangle_{CF}$

$$\langle J_{s_1} J_{s_2} T\rangle_{\text{odd}} = c\frac{\epsilon_{-ab}p_{3b}}{p_3}(\langle J_{s_1} J_{s_2} T\rangle_{\text{FB}} - \langle J_{s_1} J_{s_2} T\rangle_{\text{CF}}) \tag{3.39}$$

Thus, by solving the higher spin equation we could solve for all the components in the expansion of $\langle J_{s_1} J_{s_2} T\rangle_{QB}$ in terms of $\langle J_{s_1} J_{s_2} T\rangle_{FB}$, $\langle J_{s_1} J_{s_2} T\rangle_{CF}$ and epsilon transform of ($\langle J_{s_1} J_{s_2} T\rangle_{FB}$-$\langle J_{s_1} J_{s_2} T\rangle_{CF}$) :-

$$\boxed{\langle J_{s_1} J_{s_2} T\rangle_{QF} = \frac{1}{1+\tilde{\lambda}_{qb}^2}\left(\langle J_{s_1} J_{s_2} T\rangle_{FF} + \tilde{\lambda}_{qb} c\frac{\epsilon_{-ab}p_{3b}}{p_3}(\langle J_{s_1} J_{s_2} T^a_-\rangle_{\text{FB}} - \langle J_{s_1} J_{s_2} T^a_-\rangle_{\text{CB}}) + \tilde{\lambda}_{qb}^2\langle J_{s_1} J_{s_2} T\rangle_{CB}\right)} \tag{3.40}$$

where, $c = \frac{r_0}{b}$.

3.2 Four point functions

In this section we give some concrete calculations of how we use higher spin equations to solve for 4-point correlators in theories with slightly broken higher spin symmetry.

3.2.1 $\langle JJOO\rangle_{QF}$

To get HSE for $\langle JJOO\rangle_{QF}$, we act $Q_{\mu\nu}$ on $\langle J_\alpha OOO\rangle_{QF}$. Then we use the algebra from 2.6 for Q_3 in quasi fermion thoery to write the LHS of the corresponding non conservation higher spin equation. Then we use the divergence of J_3 from 2.12 to write the RHS for the non conservation equations. As mentioned before in the prototypical example of $\langle TTO\rangle$, we factorise the RHS in the large N limit and Fourier transform the resulting higher spin

equation to get:-

$$
\begin{aligned}
&c1\epsilon_{(\mu|\alpha a}p_{1|\nu)}p_1^a\frac{\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{QF}}{(1+\tilde{\lambda}^2)}+c2p_{1(\mu|}\langle T_{|\nu)\alpha}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{QF}\\
&\quad+c3p_{1\alpha}\langle T_{\mu\nu}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{QF}\\
&\quad+c4\{\epsilon_{(\mu|ab}p_2^ap_{2|\nu)}\langle J_\alpha(p_1)J^b(p_2)O(p_3)O(p_4)\rangle_{QF}+(2\leftrightarrow 3)+(2\leftrightarrow 4)\}\\
&\quad=\frac{-16i\tilde{\lambda}}{(1+\tilde{\lambda}^2)}\Bigg[p_{1(\mu|}\langle J_\alpha J_{|\nu)}\rangle_{QF}\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{QF}\\
&\quad+\{p_{2(\mu|}\langle O(p_2)O(-p_2)\rangle_{QF}\langle J_\alpha(p_1)J_{|\nu)}(p_2)O(p_3)O(p_4)\rangle_{QF}+(2\leftrightarrow 3)+(2\leftrightarrow 4)\}\Bigg]
\end{aligned}
\tag{3.41}
$$

Now, inputing the ansatz for the QF correlators at various orders of the coupling $\tilde{\lambda}_{qf}$ in terms of the unkown and unknown contributions, we get,

$$
\begin{aligned}
&c1\epsilon_{(\mu|\alpha a}p_{1|\nu)}p_1^a(\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF}+\tilde{\lambda}^2\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB})\\
&\quad+c2p_{1(\mu|}(1+\tilde{\lambda}^2)(\langle T_{|\nu)\alpha}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF}+\tilde{\lambda}\langle T_{|\nu)\alpha}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB})\\
&\quad+c3p_{1\alpha}(1+\tilde{\lambda}^2)(\langle T_{\mu\nu}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF}+\tilde{\lambda}\langle T_{\mu\nu}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB})+\\
&\quad c4\{\epsilon_{(\mu|ab}p_2^ap_{2|\nu)}(\langle J_\alpha(p_1)J^b(p_2)O(p_3)O(p_4)\rangle_0+\tilde{\lambda}\langle J_\alpha(p_1)J^b(p_2)O(p_3)O(p_4)\rangle_1\\
&\quad+\tilde{\lambda}^2\langle J_\alpha(p_1)J^b(p_2)O(p_3)O(p_4)\rangle_2)+(2\leftrightarrow 3)+(2\leftrightarrow 4)\}\\
&\quad=-16i\tilde{\lambda}\Bigg[p_{1(\mu|}\langle J_\alpha J_{|\nu)}\rangle_{even}(\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF}+\tilde{\lambda}^2\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB})\\
&\quad+\{\frac{p_{2(\mu|}p_2}{8}(\langle J_\alpha(p_1)J_{|\nu)}(p_2)O(p_3)O(p_4)\rangle_0+\tilde{\lambda}\langle J_\alpha(p_1)J_{|\nu)}(p_2)O(p_3)O(p_4)\rangle_1\\
&\quad+\tilde{\lambda}^2\langle J_\alpha(p_1)J_{|\nu)}(p_2)O(p_3)O(p_4)\rangle_2)+(2\leftrightarrow 3)+(2\leftrightarrow 4)\}\Bigg]
\end{aligned}
\tag{3.42}
$$

Now we can look at different orders of $\tilde{\lambda}_{qf}$ of the HSE:-
O(1):

$$
\begin{aligned}
&c_1\epsilon_{(\mu|\alpha a}p_{1|\nu)}p_1^a\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF}+c_2p_{1(\mu|}\langle T_{|\nu)\alpha}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF}\\
&+c_3p_{1\alpha}\langle T_{\mu\nu}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF}+c_4\{\epsilon_{(\mu|ab}p_2^ap_{2|\nu)}\langle J_\alpha(p_1)J^b(p_2)O(p_3)O(p_4)\rangle_0\\
&+(2\leftrightarrow 3)+(2\leftrightarrow 4)\}=0
\end{aligned}
\tag{3.43}
$$

O($\tilde{\lambda}$):

$$\begin{aligned}
&c_2 p_{1(\mu|}\langle T_{|\nu)\alpha}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB} + c_3 p_{1\alpha}\langle T_{\mu\nu}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB} + \\
&\{c_4\epsilon_{(\mu|ab}p_2^a p_{2|\nu)}\langle J_\alpha(p_1)J^b(p_2)O(p_3)O(p_4)\rangle_1 + (2\leftrightarrow 3) + (2\leftrightarrow 4)\} \\
&= -16i\Bigg[p_{1(\mu|}\langle J_\alpha J_{|\nu)}\rangle_{even}\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF} \\
&+ \{\frac{p_{2(\mu|}p_2}{8}(\langle J_\alpha(p_1)J_{|\nu)}(p_2)O(p_3)O(p_4)\rangle_0) + (2\leftrightarrow 3) + (2\leftrightarrow 4)\}\Bigg]
\end{aligned} \tag{3.44}$$

O($\tilde{\lambda}^2$)

$$\begin{aligned}
&c_1\epsilon_{(\mu|\alpha a}p_{1|\nu)}p_1^a\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB} + c_2 p_{1(\mu|}\langle T_{|\nu)\alpha}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF} \\
&+ c_3 p_{1\alpha}\langle T_{\mu\nu}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF} + c_4\{\epsilon_{(\mu|ab}p_2^a p_{2|\nu)}\langle J_\alpha(p_1)J^b(p_2)O(p_3)O(p_4)\rangle_2 \\
&+ (2\leftrightarrow 3) + (2\leftrightarrow 4)\} \\
&= -16i\Bigg[\frac{p_{2(\mu|}p_2}{8}\langle J_\alpha(p_1)J_{|\nu)}(p_2)O(p_3)O(p_4)\rangle_1 + (2\leftrightarrow 3) + (2\leftrightarrow 4)\Bigg]
\end{aligned} \tag{3.45}$$

O($\tilde{\lambda}^3$)

$$\begin{aligned}
&c_2 p_{1(\mu|}\langle T_{|\nu)\alpha}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB} + c_3 p_{1\alpha}\langle T_{\mu\nu}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB} \\
&= -16i\Bigg[p_{1(\mu|}\langle J_\alpha J_{|\nu)}\rangle_{even}(\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB}) \\
&+ \{\frac{p_{2(\mu|}p_2}{8}(\langle J_\alpha(p_1)J_{|\nu)}(p_2)O(p_3)O(p_4)\rangle_2) + (2\leftrightarrow 3) + (2\leftrightarrow 4)\}\Bigg]
\end{aligned} \tag{3.46}$$

We can see that the lowest order O(1) and the highest order O($\tilde{\lambda}^3$) resemble the free fermion and critical boson HSE for $\langle JJOO\rangle$ respectively. Thus we can identify:-

$$\begin{aligned}
\langle JJOO\rangle_0 &= \langle JJOO\rangle_{FF} \\
\langle JJOO\rangle_2 &= \langle JJOO\rangle_{CB}
\end{aligned} \tag{3.47}$$

Now, to get the pole equations of the HSE at $\tilde{\lambda} = i$ we do a series expansion of the higher spin equation around $\tilde{\lambda} = i$ which gives us:-

$$c_4 e_{(\mu|ab} p_2^a p_{2|\nu)} (\langle J_\alpha(p_1) J^b(p_2) O(p_3) O(p_4)\rangle_0 - \langle J_\alpha(p_1) J^b(p_2) O(p_3) O(p_4)\rangle_2) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)$$
$$= 2ip_2 p_{2(\mu|} \langle J_\alpha(p_1) J_{|\nu)}(p_2) O(p_3) O(p_4)\rangle_1 + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \tag{3.48}$$

and

$$c_4 \epsilon_{(\mu|ab} p_2^a p_{2|\nu)} \langle J_\alpha(p_1) J^b(p_2) O(p_3) O(p_4)\rangle_1 + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) =$$
$$- 2ip_2 p_{2(\mu|} (\langle J_\alpha(p_1) J_{|\nu)}(p_2) O(p_3) O(p_4)\rangle_0 - \langle J_\alpha(p_1) J_{|\nu)}(p_2) O(p_3) O(p_4)\rangle_2) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \tag{3.49}$$

One possible solution of the above pole equations is:-

$$\begin{aligned} &c_4 \epsilon_{(\mu|ab} p_2^a p_{2|\nu)} (\langle J_\alpha(p_1) J^b(p_2) O(p_3) O(p_4)\rangle_{FF} - \langle J_\alpha(p_1) J^b(p_2) O(p_3) O(p_4)\rangle_{CB}) = \\ &2ip_2 p_{2(\mu|} \langle J_\alpha(p_1) J_{|\nu)}(p_2) O(p_3) O(p_4)\rangle_1 \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} &c_4 \epsilon_{(\mu|ab} p_2^a p_{2|\nu)} \langle J_\alpha(p_1) J^b(p_2) O(p_3) O(p_4)\rangle_1 = \\ &- 2ip_2 p_{2(\mu|} (\langle J_\alpha(p_1) J_{|\nu)}(p_2) O(p_3) O(p_4)\rangle_{FF} - \langle J_\alpha(p_1) J_{|\nu)}(p_2) O(p_3) O(p_4)\rangle_{CB}) \end{aligned} \tag{3.51}$$

As mentioned before the O(1) and O(λ^3) equaions are the free fermion and critical boson HSE. Now, if we plug the above solutions for the pole equation into the O(λ) equations, we get:-
O(λ):-

$$\begin{aligned} &c_2 p_{1\mu} \langle T_{\nu\alpha}(p_1) O(p_2) O(p_3) O(p_4)\rangle_{CB} + c_3 p_{1\alpha} \langle T_{\mu\nu}(p_1) O(p_2) O(p_3) O(p_4)\rangle_{CB} \\ &+ (2 \leftrightarrow 3) + (2 \leftrightarrow 4) = \\ &- 2ip_2 p_{2\mu} \langle J_\alpha(p_1) J_\nu(p_2) O(p_3) O(p_4)\rangle_{CB} - 16i \langle J_\alpha J_\nu\rangle_{even} \langle O(p_1) O(p_2) O(p_3) O(p_4)\rangle_{FF} \\ &+ (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \end{aligned} \tag{3.52}$$

which is exactly the critical boson HSE. Now, plugging the solution of the HSE into the O($\tilde{\lambda}^2$) equation, we get:-

$O(\tilde{\lambda}^2)$

$$
\begin{aligned}
&c_1\epsilon_{(\mu|\alpha a}p_{1|\nu)}p_1^a\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle_{CB} + c_2 p_{1(\mu|}\langle T_{|\nu)\alpha}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF} \\
&+ c_3 p_{1\alpha}\langle T_{\mu\nu}(p_1)O(p_2)O(p_3)O(p_4)\rangle_{FF} + c_4\epsilon_{(\mu|ab}p_2^a p_{2|\nu)}(\langle J_\alpha(p_1)J^b(p_2)O(p_3)O(p_4)\rangle_{FF}) \\
&+ (2\leftrightarrow 3) + (2\leftrightarrow 4) = 0
\end{aligned} \tag{3.53}
$$

which is exactly the free fermionic equation. Thus, using the solutions of the pole equations we can map the $O(\tilde{\lambda})$ and $O(\tilde{\lambda}^2)$ and hence the entire HSE to the free fermionic and critical bosonic HSE.

Taking solution of pole equation 3.50, we can dot p_2^μ. This gives us an expression for $\langle JJOO\rangle_1$. Thus, we can write the QF correlator as

$$
\boxed{\langle J_\alpha J_\nu OO\rangle_{QF} = \langle J_\alpha J_\nu OO\rangle_{FF} - \tilde{\lambda}_{qf}c_4 i\frac{\epsilon_{\nu ab}p_2^a}{2p_2}(\langle J_\alpha J^b OO\rangle_{FF} - \langle J_\alpha J^b OO\rangle_{CB}) + \tilde{\lambda}_{qf}^2\langle J_\alpha J_\nu OO\rangle_{CB}} \tag{3.54}
$$

3.2.2 $\langle JJJJ\rangle_{QB}$

To get the form for the correlator $\langle JJJJ\rangle$ in the quasi-bosonic theory we act $Q_{\mu\nu}$ on $\langle J_\alpha J_\beta J_\gamma O\rangle_{QB}$. This gives us the higher spin equation:-

$$
\begin{aligned}
&\Bigg[\frac{a_1}{1+\tilde{\lambda}_{qb}^2}p_{1\mu}p_{1\nu}p_{1\alpha}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{QB} - \frac{a_3}{1+\tilde{\lambda}_{qb}^2}g_{\mu\alpha}p_{1\nu}p_1^2\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{QB} \\
&+ a_2 p_{1\mu}\langle T_{\nu\alpha}(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{QB} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Bigg] \\
&+ b_1 p_{4\mu}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{QB} = \\
&- \epsilon_{\mu ab}\frac{32i\tilde{\lambda}_{qb}}{1+\tilde{\lambda}_{qb}^2}\Bigg[p_4^a p_{4\nu}\langle O(p_4)O(-p_4)\rangle_{QF}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{QB} \\
&+ \Bigg(p_1^b p_{1\nu}\langle J^a(p_1)J_\alpha(-p_1)\rangle_{QB}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{QB} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Bigg)\Bigg]
\end{aligned} \tag{3.55}
$$

Now we plug in the expressions for the QB correlators:-

$$\begin{aligned}
&\langle JJOO\rangle_{QB} = \langle JJOO\rangle_{FB} + \tilde{\lambda}_{qb}\langle JJOO\rangle_{odd} + \tilde{\lambda}^2_{qb}\langle JJOO\rangle_{CF}\\
&\langle TJJO\rangle_{QB} = \frac{1}{1+\tilde{\lambda}^2_{qb}}\bigg(\langle TJJO\rangle_{Y0} + \tilde{\lambda}_{qb}\langle TJJO\rangle_{Y1} + \tilde{\lambda}^2_{qb}\langle TJJO\rangle_{Y2} + \tilde{\lambda}^3_{qb}\langle TJJO\rangle_{Y3}\bigg)\\
&\langle JJJJ\rangle_{QB}\frac{1}{(1+\tilde{\lambda}^2_{qb})^2}\bigg(\langle JJJJ\rangle_{X0} + \tilde{\lambda}_{qb}\langle JJJJ\rangle_{X1} + \tilde{\lambda}^2_{qb}\langle JJJJ\rangle_{X2} + \tilde{\lambda}^3_{qb}\langle JJJJ\rangle_{X3} + \tilde{\lambda}^4_{qb}\langle JJJJ\rangle_{X4}\bigg)\\
&\langle OO\rangle_{QB} = (1+\tilde{\lambda}^2_{qb})\langle OO\rangle_{FB}\\
&\langle J_a(p)J_b(-p)\rangle_{QB} = \bigg[\langle J_aJ_b\rangle_{FB} + \tilde{\lambda}_{qb}\langle J_aJ_b\rangle_{odd}\bigg]
\end{aligned} \tag{3.56}$$

After plugging in the above relations for QB correlators, we do a series expansion around $\tilde{\lambda}_{qb} = i$ to get two leading order pole equations

O($\frac{1}{(1+\tilde{\lambda}^2_{qb})^2}$):-

$$\begin{aligned}
&b_1 p_{4\mu}\bigg(\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X0} - \langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X2}\\
&+\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X4}\bigg)\\
&= -32i\epsilon_{\mu ab}p_4^b p_{4\nu}\langle O(p_4)O(-p_4)\rangle_{FB}\\
&\bigg(\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X3} - \langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X1}\bigg)
\end{aligned} \tag{3.57}$$

O($\frac{\tilde{\lambda}_{qb}}{(1+\tilde{\lambda}^2_{qb})^2}$):-

$$\begin{aligned}
&b_1 p_{4\mu}\bigg(\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X3} - \langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X1}\\
&= 32i\epsilon_{\mu ab}p_4^b p_{4\nu}\langle O(p_4)O(-p_4)\rangle_{FB}\bigg(\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X0}\\
&-\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X2} + \langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X4}\bigg)
\end{aligned} \tag{3.58}$$

Solution of the two above pole equations is

$$\begin{aligned}\langle JJJJ\rangle_{X3} &= \langle JJJJ\rangle_{X1}\\ \langle JJJJ\rangle_{X2} &= \langle JJJJ\rangle_{X0} + \langle JJJJ\rangle_{X4}\end{aligned} \tag{3.59}$$

Now, we look at different order equations of the HSE:-

O(1):

$$\begin{aligned}&\Big(a_1 p_{1\mu}p_{1\nu}p_{1\rho}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{FB} + a_2 p_{1\mu}\langle T_{\nu\alpha}(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{Y0}\\ &- a_3 g_{\mu\alpha}p_{1\nu}p_1^2\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{FB} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big)\\ &+ b_1 p_{4\mu}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{FB} = 0\end{aligned} \tag{3.60}$$

This is the higher spin equation that we would get at the limit of $\tilde{\lambda}_{qb} \to 0$. So, we can identify

$$\boxed{\langle JJJJ\rangle_{X0} = \langle JJJJ\rangle_{FB}} \tag{3.61}$$

Now, if we consider the $O(\tilde{\lambda}_{qb}^5)$ equation, we have

$O(\tilde{\lambda}_{qb}^5)$:

$$\begin{aligned}&a_2 p_{1\mu}\langle T_{\nu\alpha}(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{Y3} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\\ &= -32i\epsilon_{\mu ab}\Big(p_4^b p_{4\nu}\langle O(p_4)O(-p_4)\rangle_{FB}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X4}\\ &+ (p_1^b p_{1\nu}\langle J^a(p_1)J_\alpha(-p_1)\rangle_{FB}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{CF} + (1\leftrightarrow 2) + (1\leftrightarrow 3))\Big)\end{aligned} \tag{3.62}$$

From the pole equation of $\langle JJTO\rangle$, we know that $\langle JJTO\rangle_{Y3} = \langle JJTO\rangle_{CF}$. Since, we would get the above equation at the limit $\tilde{\lambda}_{qb} \to \infty$, we can identify

$$\boxed{\langle JJJJ\rangle_{X4} = \langle JJJJ\rangle_{CF}} \tag{3.63}$$

O($\tilde{\lambda}_{qb}$):

$$\Big(a_1 p_{1\mu}p_{1\nu}p_{1\rho}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{odd} + a_2 p_{1\mu}\langle T_{\nu\alpha}(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{Y1}$$
$$- a_3 g_{\mu\alpha}p_{1\nu}p_1^2\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{odd} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big)$$
$$+ b_1 p_{4\mu}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X1} = \tag{3.64}$$
$$- 32i\epsilon_{\mu ab}\Big(p_4^b p_{4\nu}\langle O(p_4)O(-p_4)\rangle_{FB}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X0}$$
$$+ (p_1^b p_{1\nu}\langle J^a(p_1)J_\alpha(-p_1)\rangle_{FB}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{FB} + (1\leftrightarrow 2) + (1\leftrightarrow 3))\Big)$$

O($\tilde{\lambda}_{qb}^2$):

$$\Big(a_1 p_{1\mu}p_{1\nu}p_{1\rho}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{FB+CF} + a_2 p_{1\mu}\langle T_{\nu\alpha}(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{Y0+Y2}$$
$$- a_3 g_{\mu\alpha}p_{1\nu}p_1^2\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{FB+CF} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big)$$
$$+ b_1 p_{4\mu}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X2} =$$
$$- 32i\epsilon_{\mu ab}\Big(p_4^b p_{4\nu}\langle O(p_4)O(-p_4)\rangle_{FB}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X1}$$
$$+ (p_1^b p_{1\nu}\langle J^a(p_1)J_\alpha(-p_1)\rangle_{FB}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{odd} + (1\leftrightarrow 2) + (1\leftrightarrow 3))\Big) \tag{3.65}$$

$O(\tilde{\lambda}^3_{qb})$:

$$\Big(a_1 p_{1\mu}p_{1\nu}p_{1\rho}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{odd} + a_2 p_{1\mu}\langle T_{\nu\alpha}(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{Y1+Y3}$$
$$- a_3 g_{\mu\alpha}p_{1\nu}p_1^2\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{odd} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big)$$
$$+ b_1 p_{4\mu}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X3} =$$
$$- 32i\epsilon_{\mu ab}\Big(p_4^b p_{4\nu}\langle O(p_4)O(-p_4)\rangle_{FB}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X2}$$
$$+ (p_1^b p_{1\nu}\langle J^a(p_1)J_\alpha(-p_1)\rangle_{FB}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{FB+CF} + (1\leftrightarrow 2) + (1\leftrightarrow 3))\Big) \tag{3.66}$$

$O(\tilde{\lambda}^4_{qb})$:

$$\Big(a_1 p_{1\mu}p_{1\nu}p_{1\rho}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{CF} + a_2 p_{1\mu}\langle T_{\nu\alpha}(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{Y2}$$
$$- a_3 g_{\mu\alpha}p_{1\nu}p_1^2\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{CF} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big)$$
$$+ b_1 p_{4\mu}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\nu(p_4)\rangle_{X4} = \tag{3.67}$$
$$- 32i\epsilon_{\mu ab}\Big(p_4^b p_{4\nu}\langle O(p_4)O(-p_4)\rangle_{FB}\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^a(p_4)\rangle_{X3}$$
$$+ (p_1^b p_{1\nu}\langle J^a(p_1)J_\alpha(-p_1)\rangle_{FB}\langle O(p_1)J_\beta(p_2)J_\gamma(p_3)O(p_4)\rangle_{odd} + (1\leftrightarrow 2) + (1\leftrightarrow 3))\Big)$$

X0 and X4 part of the $\langle JJJJ\rangle$ correlator are the free bosonic and critical fermionic correlator and from the pole equation we got that $\langle JJJJ\rangle_{X2} = \langle JJJJ\rangle_{X0} + \langle JJJJ\rangle_{X4}$. Now the unknown part of $\langle JJJJ\rangle_{QB}$ that remains is X1. To get the expression for X1, we use the solution of pole equation $\langle JJJJ\rangle_{X1} = \langle JJJJ\rangle_{X3}$. Then we consider the expression for $O(\tilde{\lambda}^3_{qb}) - 2O(\tilde{\lambda}_{qb})$ equation. Since the equation is (μ,ν) symmetrised, we dot with projector $\pi^{\nu\rho}$ to kills one of the terms from the symmetrisation. Then we dot with $p_{4\mu}$. Doing this we

can get an expression for $\langle JJJJ\rangle_{X1}$ to be

$$
\begin{aligned}
&\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\rho(p_4)\rangle_{X1} = \\
&\left(\frac{p_1 p_1^\rho \epsilon^{\{\alpha,p_1,p_4\}}}{2p_4^2} - \frac{p_1 p_1\cdot p_4 p_4^\rho \epsilon^{\{\alpha,p_1,p_4\}}}{p_4^4} - \frac{p_1 p_1\cdot p_4 \epsilon^{\{\alpha,\rho,p_1\}}}{2p_4^2}\right) \\
&\left(\langle o\left(p_1\right) o\left(p_4\right) J_\beta\left(p_2\right) J_\gamma\left(p_3\right)\rangle_{\mathrm{CF}} - \langle o\left(p_1\right) o\left(p_4\right) J_\beta\left(p_2\right) J_\gamma\left(p_3\right)\rangle_{\mathrm{FB}}\right) \\
&\left(-\frac{p_1^2 p_1\cdot p_4(-\mathrm{a3})\delta_{\alpha,\rho}}{2p_4^2} - \frac{\left(p_1\cdot p_4\right)^2(-\mathrm{a1})p_1^\alpha p_4^\rho}{p_4^4} + \frac{p_1\cdot p_4(-\mathrm{a1})p_1^\alpha p_1^\rho}{p_4^2}\right. \\
&\left. -\frac{p_1^2(-\mathrm{a3})p_4^\alpha p_1^\rho}{2p_4^2} + \frac{p_1^2 p_1\cdot p_4(-\mathrm{a3})p_4^\alpha p_4^\rho}{p_4^4}\right)\langle o\left(p_1\right) o\left(p_4\right) J_\beta\left(p_2\right) J_\gamma\left(p_3\right)\rangle_{\mathrm{odd}} \\
&+\left(\frac{\mathrm{a2}p_1\cdot p_4\delta_{\nu,\rho}}{2p_4^2} + \frac{\mathrm{a2}p_4^\nu p_1^\rho}{2p_4^2} - \frac{\mathrm{a2}p_1\cdot p_4 p_4^\nu p_4^\rho}{p_4^4}\right)\langle o\left(p_4\right) J_\beta\left(p_2\right) J_\gamma\left(p_3\right) T_{\alpha\nu}\left(p_1\right)\rangle_{\mathrm{Y3}} \\
&+\left(-\frac{\mathrm{a2}p_1\cdot p_4\delta_{\nu,\rho}}{2p_4^2} - \frac{\mathrm{a2}p_4^\nu p_1^\rho}{2p_4^2} + \frac{\mathrm{a2}p_1\cdot p_4 p_4^\nu p_4^\rho}{p_4^4}\right)\langle o\left(p_4\right) J_\beta\left(p_2\right) J_\gamma\left(p_3\right) T_{\alpha\nu}\left(p_1\right)\rangle_{\mathrm{Y1}} \\
&-\frac{p_4^\tau \mathrm{X0}_{\alpha\beta\gamma\sigma 8}\epsilon^{\{\sigma 8,\tau,\rho\}}}{p_4} + \frac{p_4^\tau \mathrm{X4}_{\alpha\beta\gamma\sigma 8}\epsilon^{\{\sigma 8,\tau,\rho\}}}{p_4}
\end{aligned}
\tag{3.68}
$$

From the pole equation of $\langle JJTO\rangle$, we know that $\langle JJTO\rangle_{Y1} = \langle JJTO\rangle_{Y3}$. Thus the $\langle JJTO\rangle$ contribution to the expression of X1 cancels out since the coefficients of Y1 and Y3 are exactly negative of each other. In the above expression a_1, a_2, a_3 are the algebra coefficients of the Q_3 charge. Similarly from the pole equation of $\langle JJOO\rangle$, we know that

$$\langle J_\alpha J_\beta OO\rangle_{odd} = e_{\beta ab}p_2^b(\langle J_\alpha J^a OO\rangle_{FB} - \langle J_\alpha J^a OO\rangle_{CF}) \tag{3.69}$$

We plug this in the above expression for X1. If we set $a3 = a1 = -1$, and dot with the null polarization tensors $z_{1\alpha}z_{2\beta}z_{3\gamma}z_{4\rho}$, we get the expression in spinor helicity variables, it can be shown that the $\langle JJOO\rangle$ terms cancel out among each other. Thus, the expression for X1 that we are left with is:

$$\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J_\rho(p_4)\rangle_{X1} = \tag{3.70}$$

$$-\frac{\epsilon_{\sigma\tau\rho}p_4^\tau}{p_4}(\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^\sigma(p_4)\rangle_{X0} - \langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^\sigma(p_4)\rangle_{X4}) \tag{3.71}$$

Thus, we can write the QB correlator as

$$\begin{aligned}
\langle J_\alpha J_\beta J_\gamma J_\rho\rangle_{QB} = \frac{1}{(1+\tilde{\lambda}_{qb}^2)^2}\Bigg(&\langle J_\alpha J_\beta J_\gamma J_\rho\rangle_{FB} \\
&+ \tilde{\lambda}_{qb}(-\frac{\epsilon_{\sigma\tau\rho}p_4^\tau}{p_4}(\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^\sigma(p_4)\rangle_{FB} - \langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^\sigma(p_4)\rangle_{CF})) \\
&+ \tilde{\lambda}_{qb}^2(\langle J_\alpha J_\beta J_\gamma J_\rho\rangle_{FB} + \langle J_\alpha J_\beta J_\gamma J_\rho\rangle_{CF}) \\
&+ \tilde{\lambda}_{qb}^3(-\frac{\epsilon_{\sigma\tau\rho}p_4^\tau}{p_4}(\langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^\sigma(p_4)\rangle_{FB} - \langle J_\alpha(p_1)J_\beta(p_2)J_\gamma(p_3)J^\sigma(p_4)\rangle_{CF})) \\
&+ \tilde{\lambda}_{qb}^4\langle J_\alpha J_\beta J_\gamma J_\rho\rangle_{CF}\Bigg)
\end{aligned} \tag{3.72}$$

However, to cancel the $\langle JJOO\rangle$ terms from the X1 expression we dotted the expression with null polarization tensors, $z_1^\alpha z_2^\beta z_3^\gamma z_4^\rho$ to convert to spinor helicity. Thus, the answer in spinor helicity variables would be

$$\boxed{\begin{aligned}
\langle J_\pm J_\pm J_\pm J_\pm\rangle_{QB} = \frac{1}{(1+\tilde{\lambda}_{qb}^2)^2}\Bigg(&\langle J_\pm J_\pm J_\pm J_\pm\rangle_{FB} \\
&- i\tilde{\lambda}_{qb}(\langle J_\pm J_\pm J_\pm J_\pm\rangle_{FB} - \langle J_\pm J_\pm J_\pm J_\pm\rangle_{CF}) + \tilde{\lambda}_{qb}^2(\langle J_\pm J_\pm J_\pm J_\pm\rangle_{FB} + \langle J_\pm J_\pm J_\pm J_\pm\rangle_{CF}) \\
&- i\tilde{\lambda}_{qb}^3(\langle J_\pm J_\pm J_\pm J_\pm\rangle_{FB} - \langle J_\pm J_\pm J_\pm J_\pm\rangle_{CF}) + \tilde{\lambda}_{qb}^4\langle J_\pm J_\pm J_\pm J_\pm\rangle_{CF}\Bigg)
\end{aligned}} \tag{3.73}$$

It is to be noted that the $\pm$ here denote plus minus helicity and not the lightcone coordinates. To get the above expression, we have used that, in spinor helicty, the epsilon transform of a expression just becomes i times the expression.

3.2.3 $\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{QB}$

In this section we do a rough analysis on constraining a general 4-point correlator, $\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{QB}$. To do this we start by acting Q_4 on $\langle J_{s_1} J_{s_2} J_{s_3} O\rangle_{QB}$. Before moving to the writing the higher spin equation explicitly, we need to consider the general higher spin algebra for Q_4 acting on a general spinning current J_s

$$\begin{aligned}
[Q_4, J_s] &= c_{s,s-2}\partial^5 J_{s-2} + c_{s,s}\partial^3 J_s + c_{s,s+2}\partial J_{s+2} \\
[Q_4, O] &= \partial^3 O + \partial T
\end{aligned} \tag{3.74}$$

Using the above algebra we act Q_4 on $\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{QB}$ and can write the LHS of the HSE. To write the RHS, we again look for the divergence of current J_4 from Eq. 2.23

$$\Big(c_{s_1,s_1-2}\langle\partial^5 J_{s_1-2} J_{s_2} J_{s_3} O\rangle + c_{s_1,s_1}\langle\partial^3 J_{s_1} J_{s_2} J_{s_3} O\rangle + c_{s_1,s_1+2}\langle\partial J_{s_1+2} J_{s_2} J_{s_3} O\rangle$$

$$\{1 \leftrightarrow 2\} + \{1 \leftrightarrow 3\}\Big) + \langle J_{s_1} J_{s_2} J_{s_3} \partial^3 O\rangle + \langle J_{s_1} J_{s_2} J_{s_3} \partial T\rangle \tag{3.75}$$

$$= \frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} r_0 \int \langle \epsilon_{-ab} T_{a-}(x) \partial_b \partial_- O(x) J_{s_1} J_{s_2} J_{s_3} O\rangle \tag{3.76}$$

Then, we factorise the six point correlator in the RHS for the large N limit and looking at the leading order into product of a 2-point and a 4-point correlators. Then, we fourier transform the factorised correlators in the RHS to get

$$\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} r_0 \epsilon_{-ab} p_{3b} p_{3-} \langle OO\rangle \langle J_{s_1} J_{s_2} J_{s_3} T_{a-}(x)\rangle \tag{3.77}$$

Then, we fourier trasnform the LHS to get the corresponding higher spin equation in momentum space as:

$$\Big[c_{s_1,s_1-2} p_{1-}^5 \langle J_{s_1-2} J_{s_2} J_{s_3} O\rangle_{QB} + c_{s_1,s_1} p_{1-}^3 \langle J_{s_1} J_{s_2} J_{s_3} O\rangle_{QB} + c_{s_1,s_1+2} p_{1-} \langle J_{s_1+2} J_{s_2} J_{s_3} O\rangle_{QB}$$

$$+ (1 \leftrightarrow 2) + (1 \leftrightarrow 3)\Big]$$

$$+ p_{4-}^3 \langle J_{s_1} J_{s_2} J_{s_3} O\rangle_{QB} + p_{4-} \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{QB} =$$

$$\frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} r_0 \epsilon_{-ab} p_4^b p_{4-} \langle O(p_4) O(-p_4)\rangle_{QB} \langle J_{s_1}(p_1) J_{s_2}(p_2) J_{s_3}(p_3) T_{a-}(p_4)\rangle_{QB} \tag{3.78}$$

Now again we use the following expansions for the quasi bosonic correlators at various orders in the coupling $\tilde{\lambda}_{qb}$

$$
\begin{aligned}
&\langle OO\rangle_{\rm QB} = (1+\tilde{\lambda}^2)\langle OO\rangle_{\rm FB} \\
&\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\rm QB} = \frac{1}{1+\tilde{\lambda}^2}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\rm FB} + \tilde{\lambda}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\rm W1} + \tilde{\lambda}^2\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\rm W2} \\
&+ \tilde{\lambda}^3\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{\rm W3} \\
&\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{\rm QB} = \frac{1}{(1+\tilde{\lambda}^2)^2}\big(\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{\rm V0} + \tilde{\lambda}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{\rm V1} + \tilde{\lambda}^2\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{\rm V2} \\
&+ \tilde{\lambda}^3\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{\rm V3} + \tilde{\lambda}^4\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{\rm V4}\big)
\end{aligned}
\tag{3.79}
$$

Then, we plug these expansions back in the higher spin equation and then collect terms at various orders in $\tilde{\lambda}$ we get the following equations

O(1):-

$$
\begin{aligned}
&\Big[p^5_{1-}c_{s_1,s_1-2}\langle J_{s_1-2}J_{s_2}J_{s_3}O\rangle_{FB} + p^3_{1-}c_{s_1,s_1}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{FB} \\
&+ p_{1-}c_{s_1,s_1+2}\langle J_{s_1+2}J_{s_2}J_{s_3}O\rangle_{FB} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big] + p^3_{4-}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{FB} \\
&+ p_{4-}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V0} = 0
\end{aligned}
\tag{3.80}
$$

This is the lowest order equation in $\tilde{\lambda}_{qb}$ and this would correspond to the free boson higher spin equation. Thus, we can identify:

$$
\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V0} = \langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{FB} \tag{3.81}
$$

O($\tilde{\lambda}$):-

$$
\begin{aligned}
&\Big[p^5_{1-}c_{s_1,s_1-2}\langle J_{s_1-2}J_{s_2}J_{s_3}O\rangle_{W1} + p^3_{1-}c_{s_1,s_1}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{W1} \\
&+ p_{1-}c_{s_1,s_1+2}\langle J_{s_1+2}J_{s_2}J_{s_3}O\rangle_{W1} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big] + p^3_{4-}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{W1} \\
&+ p_{4-}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V1} = r_0\epsilon_{-ab}p^b_4 p_{4-}\langle OO\rangle_{FB}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{FB}
\end{aligned}
\tag{3.82}
$$

$\mathrm{O}(\tilde{\lambda}^2)$:-

$$\begin{aligned}&\Big[p_{1-}^5 c_{s_1,s_1-2}\langle J_{s_1-2}J_{s_2}J_{s_3}O\rangle_{FB+W2} + p_{1-}^3 c_{s_1,s_1}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{FB+W2}\\&+ p_{1-}c_{s_1,s_1+2}\langle J_{s_1+2}J_{s_2}J_{s_3}O\rangle_{FB+W2} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big] + p_{4-}^3\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{FB+W2}\\&+ p_{4-}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V2} = r_0\epsilon_{-ab}p_4^b p_{4-}\langle OO\rangle_{FB}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V1}\end{aligned} \tag{3.83}$$

$\mathrm{O}(\tilde{\lambda}^3)$:-

$$\begin{aligned}&\Big[p_{1-}^5 c_{s_1,s_1-2}\langle J_{s_1-2}J_{s_2}J_{s_3}O\rangle_{W1+W3} + p_{1-}^3 c_{s_1,s_1}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{W1+W3}\\&+ p_{1-}c_{s_1,s_1+2}\langle J_{s_1+2}J_{s_2}J_{s_3}O\rangle_{W1+W3} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big] + p_{4-}^3\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{W1+W3}\\&+ p_{4-}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V3} = r_0\epsilon_{-ab}p_4^b p_{4-}\langle OO\rangle_{FB}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V2}\end{aligned} \tag{3.84}$$

$\mathrm{O}(\tilde{\lambda}^4)$:-

$$\begin{aligned}&\Big[p_{1-}^5 c_{s_1,s_1-2}\langle J_{s_1-2}J_{s_2}J_{s_3}O\rangle_{W2} + p_{1-}^3 c_{s_1,s_1}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{W2}\\&+ p_{1-}c_{s_1,s_1+2}\langle J_{s_1+2}J_{s_2}J_{s_3}O\rangle_{W2} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big] + p_{4-}^3\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{W2}\\&+ p_{4-}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V4} = r_0\epsilon_{-ab}p_4^b p_{4-}\langle OO\rangle_{FB}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V3}\end{aligned} \tag{3.85}$$

$\mathrm{O}(\tilde{\lambda}^5)$:-

$$\begin{aligned}&\Big[p_{1-}^5 c_{s_1,s_1-2}\langle J_{s_1-2}J_{s_2}J_{s_3}O\rangle_{CF} + p_{1-}^3 c_{s_1,s_1}\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{CF}\\&+ p_{1-}c_{s_1,s_1+2}\langle J_{s_1+2}J_{s_2}J_{s_3}O\rangle_{CF} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\Big] + p_{4-}^3\langle J_{s_1}J_{s_2}J_{s_3}O\rangle_{CF}\\&= r_0\epsilon_{-ab}p_4^b p_{4-}\langle OO\rangle_{FB}\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V4}\end{aligned} \tag{3.86}$$

The $\mathrm{O}(\tilde{\lambda}_{qb}^5)$ is the highest order equation of the HSE. Thus this would correspond to the critical fermionic equation, thus we identify

$$\langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{V4} = \langle J_{s_1}J_{s_2}J_{s_3}T\rangle_{CF} \tag{3.87}$$

To get the pole equations of the higher spin equation, we do a series expansion of the HSE around $\tilde{\lambda}_{qb} = i$ which gives us the two following pole equations

$$b(\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm FB} - \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V2} + \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm CF}) \tag{3.88}$$
$$+ r_0 \epsilon_{-ab} p_{4b} (\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V3} - \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V1}) \langle OO\rangle_{\rm FB} = 0$$
$$b(\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V3} - \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V1}) \tag{3.89}$$
$$- r_0 \epsilon_{-ab} p_{4b} \langle OO\rangle_{\rm FB} (\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm FB} - \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V2} + \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V4}) = 0 \tag{3.90}$$

One possible solution for the above pole equations is

$$\begin{aligned} \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V2} &= \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm FB} + \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm CF} \\ \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V3} &= \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{\rm V1} \end{aligned} \tag{3.91}$$

Consider the equation O($\tilde{\lambda}_{qb}^3$)-2O($\tilde{\lambda}_{qb}$). Identifying W3=W1, V3=V1, V2=V0+V4, we get the expresison for V1 as

$$\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{V1} = r_0 \epsilon_{-ab} p_4^b p_{4-} p_4 (\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{V4} - \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{V0}) \tag{3.92}$$

The expression for the QB correlator is

$$\boxed{\begin{aligned} &\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{QB} = \frac{1}{(1+\tilde{\lambda}_{qb}^2)^2} \Bigg(\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{FB} \\ &- \tilde{\lambda}_{qb} \frac{r_0}{8} \epsilon_{-ab} p_4^b p_{4-} p_4 (\langle J_{s_1} J_{s_2} J_{s_3} T_-^a\rangle_{FB} - \langle J_{s_1} J_{s_2} J_{s_3} T_-^a\rangle_{CF}) \\ &+ \tilde{\lambda}_{qb}^2 (\langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{FB} + \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{CF}) \\ &- \tilde{\lambda}_{qb}^3 \frac{r_0}{8} \epsilon_{-ab} p_4^b p_{4-} p_4 (\langle J_{s_1} J_{s_2} J_{s_3} T_-^a\rangle_{FB} - \langle J_{s_1} J_{s_2} J_{s_3} T_-^a\rangle_{CF}) + \tilde{\lambda}_{qb}^4 \langle J_{s_1} J_{s_2} J_{s_3} T\rangle_{CF} \Bigg) \end{aligned}} \tag{3.93}$$

It is to be noted that in the above expression, wherever the indices on T is not mentioned, it is -(lightcone coordinates).

Chapter 4

Conclusion and Discussion

In general computing 4-point correlators for interacting theories in momentum space is difficult. As expalined at the end of section 1.4, usual arguments of imposing conformal invariance to compute them is substantially difficult compared to 3-point correlators. Conceptually it is because for 4-point even in position space conformal invariance can fix the correlator only upto a arbitrary function of cross ratios. In momentum space it becomes even harder since the analogs of cross ratios in momentum space are not as well defined. On a technical level, the difficulty arises because in 4-point correlators, we can chose 3 independent momenta. This complicates the possible tensor decompositions, action of conformal generators on the correlators are more involved. For general spinning correlators, because of many possible independent tensor structures, solving the conformal Ward identities for the form factors is even computationally a hard problem.
However, for interacting theories with slightly broken higher spin theories, which are important classes of theories, we can make use of higher spin symmetry. Using higher spin algebra and non conservation current equations, we can get higher spin equations. As described in Section 2 and 3, using higher spin equations, we can map the **correlators in interacting theories with slightly broken higher spin symmetry to the free theory correlators** with relative ease compared to the tedious process of solving conformal Ward identities.

Being able to map the correlators in interacting theories with slightly broken higher spin symmetry in terms of free theory correlators opens up interesting new directions. It would

be interesting to do extend the same analysis for higher point correlation functions. It is also exciting to see the impact of these results in the context of Vasiliev theory. Also, the solutions that we have presented are one class of solutions and it would be nice explore if there are more general classes of solutions. In future, to also cross check the obtained results, we need to do an explicit computation in Chern Simons matter theory. However, doing a general calculation will be tedious and it might be possible to do it for a specific kinematic regime. We can also try the same using ususal conformal bootstrap methods to calculate them.

However there are more general questions one can ask. It includes bootstrapping general theories in momentum space beyond the ones with slightly broken higher spin theories. We have seen that compared to position space bootstrap, momentum space bootstrap using higher spin symmetry gives us better insight. This gives us motivation that direct momentum space bootstrap would be more powerful as well. However, ther are challenges because 4-point functions of spinning operators and the notions of cross ratios are not very well understood in momentum space. We would like to start to start from there and it would be interesting to explore how to do general momentum space bootstrapping. As we have seen that spinor helicities play an important role, we believe that spinor helicity basis is appropriate for dealing with such analysis in 3-dimensions. We would like to incorporate that in the future. These analyses wil have important implications in the context of cosmological correlation functions. Also, even though the connection between CFT correlators and scattering amplitudes is studies for 3-point correlators, it hasn't been as developed for 4-point case. It would be interesting to analyse and establish this connection. As a foundation it might be easier to start studying this connection at lower dimensions first. Say for example making the connection between CFT_1 correlators and dS_2 scattering amplitudes.

Chapter 5

Appendix

5.1 Spinor Helicity variables

To use spinor helicity we begin by embedding the thrre momentum vector, $\vec{p}$ into a null 3+1 vector, $p_\mu = (p, \vec{p})$. Taking this vector, we define a corresponding matrix,

$$p_{\rho\dot{\rho}} = p_\mu \sigma^\mu_{\rho\dot{\rho}} \tag{5.1}$$

where, σ is the Pauli matrix.
For massless particles, $det(p) = 0$. So, p is a rank one matrix and can be written as outer product of two spinors

$$p_{\rho\dot{\rho}} = \lambda_\rho \tilde{\lambda}_{\dot{\rho}} \tag{5.2}$$

Here dotted and undotted spinors are $SL(2, C)$ transform. But we can define contractions between them. Consider a vector $\tau = (1, \vec{0})$. In spinor notation this becomes $\tau_{\rho\dot{\rho}} = -\mathbb{I}_{\rho\dot{\rho}}$. To related dotted and undotted indices we introduce, $\tau^{\dot{\sigma}}_\sigma = -\epsilon^{\dot{\sigma}\dot{\rho}}\mathbb{I}_{\dot{\rho}\sigma}$. Using this tensor, we introduce barred tensors related to tilde tensors by

$$\overline{\lambda}_\sigma = \tilde{\lambda}_{\dot{\sigma}} \tau^{\dot{\sigma}}_\sigma \tag{5.3}$$

We can now associate the 3-momenta vector to spinor as

$$\lambda_\sigma \overline{\lambda}_\rho = p_i (\hat{\sigma}^i)_{\sigma\rho} + p\epsilon_{\sigma\rho} \tag{5.4}$$

$$p^i = \frac{1}{2}(\sigma)^\sigma_\rho \lambda_\sigma \overline{\lambda}^\rho \tag{5.5}$$

where $\hat{\sigma}_{\sigma\rho} = \sigma_{\sigma\dot{\sigma}}\tau^{\dot{\sigma}}_\rho$. We define the dot product of the spinors as

$$\langle ab \rangle = \epsilon^{\sigma\rho}\lambda^a_\sigma \lambda^b_\rho \tag{5.6}$$

$$\langle \overline{a}\overline{b} \rangle = \epsilon^{\sigma\rho}\overline{\lambda}^a_\sigma \overline{\lambda}^b_\rho \tag{5.7}$$

$$\langle a\overline{b} \rangle = \epsilon^{\sigma\rho}\lambda^a_\sigma \overline{\lambda}^b_\rho \tag{5.8}$$

Levi civita tensor is also used to rise and lower the indices of the spinors

$$\lambda_\rho = \epsilon_{\sigma\rho}\lambda^\sigma \tag{5.9}$$

A important object in this formalism are the polarization tensors, which when dotted with momenta space expression give the corresponding spinor helicity expression

$$z^-_{\sigma\rho} = \frac{\lambda_\sigma \lambda_\rho}{2p} \qquad z^+_{\sigma\rho} = \frac{\overline{\lambda}_\sigma \overline{\lambda}_\rho}{2p} \tag{5.10}$$

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